ON THE CLASS OF LOCALLY CONVEX SPACES WHOSE FINAL LIMITS CONTAIN ALL BANACH SPACES

by

Antonio Gilioli

IME-USP

Abstract

Valdivia(8) showed that, if a Banach space E is infinite-dimensional and has a weak-star separable dual, then every Banach space is a final loc. convex limit of spaces isomorphic to E. Bellenot(1) removed the restriction that E has a weak-star separable dual. Utilizing the same basic ideas of Bellenot, we study in § 2 the class of loc. convex spaces E such that every Banach space is a final limit of spaces isomorphic to E, giving a complete characterization of such spaces E when they are bornological. In § 1, we prove that for any set I with cardinal smaller than the smallest strongly inaccessible cardinal, C^I is a final limit of spaces isomorphic to C^c, where c is the cardinal of the real line. In both §§, some consequences to the general theory of the closed-graph theorem are given. It is also shown that not all quotients of dual nuclear spaces are dual nuclear, thus solving negatively a problem left by Pietsch(5).

Introduction

We denote by $\mathbb{K}$ either the field of real or of complex numbers. If I is a set, $|I|$ denotes its cardinal number, $\aleph_0 = |\mathbb{N}|$ and $c = |\mathbb{K}|$. By <u>lcs</u> we mean a locally convex Hausdorff vector space, <u>lc</u> stands for locally convex, while <u>ac</u> for absolutely convex. If E is a lcs and I is a set, E^I denotes the product of I copies of E, while $E^{(I)}$ denotes the lc sum of I copies of E, and E^* the (continuous) dual of E. We denote $\mathbb{K}^{\aleph_0}$ by ω. Given a family of lcs E_i, $i \in I$, another lcs F and a family of linear maps $u_i : E_i \longrightarrow F$

we say that F is the final limit (or more precisely, the final lc limit) of the spaces E_i, determined by u_i, if the topology of F is the finest lc one which makes all the u_i continuous. If A is a subset of a vector space E, [A] will denote the vector subspace of E generated by A, and ΓA the ac hull of A. If V is an ac neighborhood of 0 in a lcs E, we put $\ker V = \bigcap_{\lambda>0} \lambda V$. If E is a lcs and B an ac bounded subset of E, E_B will denote the vector space [B] endowed with the norm which is the gauge of B. A lcs E is dual nuclear if its strong dual is nuclear.

If $\mathcal{F}$ is a class of lcs, then $\mathcal{C}^{\ell}(\mathcal{F})$ denotes the class of all lcs E such that every linear map u with closed graph from E to any F belonging to $\mathcal{F}$ is continuous. If $\mathcal{E}$ is a class of lcs, then $\mathcal{C}^{n}(\mathcal{E})$ will denote the class of the lcs F such that for each $E \in \mathcal{E}$, every linear map with closed graph from E to F is continuous. We recall that, if $\mathcal{B}$ (resp $\mathcal{U}$) denotes the class of all Banach (resp ultrabornological) spaces, then $\mathcal{C}^{\ell}(\mathcal{B}) = \mathcal{C}^{n}(\mathcal{U})$ and the elements of this class are called infra-(u) spaces.

1. K^I are final limits of K^c (if $|I|$ is smaller than the smallest strongly inaccessible cardinal)

The essential idea used in (1), as well as in both §§ of this work, is the following simple lemma, whose proof is left to the reader.

Lemma 1.1. Let E and F be two lcs. a) Then, F is the final lc limit of some family of lcs all of them isomorphic to E, if and only if the final lc topology determined on F by the set L(E,F) of all linear continuous maps from E to F coincides with the original topology of F. b) Let us suppose further that F is a Mackey space. Then, F is the final lc limit of some family of lcs all of them isomorphic to E, if and only if, for each discontinuous linear functional y' on F, there is some $u \in L(E,F)$ such that the linear functional $y' \circ u$ on E is discontinuous.

Theorem 1.2. Let J be a set with $|J| \geq c$, and I an arbitrary set with $|I|$ smaller than the smallest strongly inaccessible cardinal. Then, K^I

is the final lc limit of a family of spaces isomorphic to $\mathbb{K}^J$ (hence, $\mathbb{K}^I$ is isomorphic to a quotient of $(\mathbb{K}^J)^{(L)}$, where L is the index set of the family).

<u>Proof</u>. Since $\mathbb{K}^I$ is a Mackey space, it is enough, by Lemma 1.1b, to prove that, if $y': \mathbb{K}^I \to \mathbb{K}$ is linear but not continuous, then there is $u: \mathbb{K}^J \to \mathbb{K}^I$ linear continuous, such that $y' \circ u$ is not continuous.

Let then y' be a linear discontinuous functional on $\mathbb{K}^I$. By the hypothesis on $|I|$, the space $\mathbb{K}^I$ is bornological. Therefore, since y' is discontinuous, there is a bounded sequence $B=\{x^n\}$ in $\mathbb{K}^I$ such that y' is not bounded on B. Let us call E the closed subspace of $\mathbb{K}^I$ generated by B. Then, E is a minimal lcs, hence isomorphic to $\mathbb{K}^L$, for some set L, and E is separable, hence $|L| \leq c \leq |J|$, so that E is isomorphic to a quotient M of $\mathbb{K}^J$. Let $q: \mathbb{K}^J \to M$ be the quotient map, $v: M \to E$ the isomorphism and $i: E \to \mathbb{K}^I$ the canonical inclusion. Since the restriction $y' \circ i$ of y' to E is not continuous, $y' \circ i \circ v$ is not continuous, hence $y' \circ u$ is not continuous, if we call $u = i \circ v \circ q$, although $u: \mathbb{K}^J \to \mathbb{K}^I$ is continuous. ∎

<u>Remark 1.3</u>. Even in the case $\{x^n\}$ is a sequence convergent to zero (even in the Mackey sense), it may happen that the space E in the proof of Theorem 1.2 is isomorphic to $\mathbb{K}^c$ and not to ω. In fact, since $|\mathbb{R}| = c$, if q_n is an enumeration of the rational numbers, let us consider for each sequence $J_1, \dots, J_k$ of disjoint closed intervals with rational end points, and for each sequence $n_1, \dots, n_k$ in N^*, the point $y = y(J_1, \dots, J_k; n_1, \dots, n_k)$ of $\mathbb{K}^{\mathbb{R}}$ defined by: $y_\alpha = q_{n_i}$ if $\alpha \in J_i$, $i=1, \dots, k$; $y_\alpha = q_1$ if $\alpha \notin \bigcup_{i=1}^{k} J_i$. The set D of such points is countable, and it is well known that it is dense in $\mathbb{K}^{\mathbb{R}}$ (see (9), pg 109-110). Let y_n be an enumeration of D. Since each y_n has its coordinates assuming only a finite number of values, there is $\lambda_n > 0$ such that $\lambda_n y_n \in I^{\mathbb{R}}$, where $I = [-1,1]$. Let $\mu_n > 0$, with $\lim_{n\to\infty} \mu_n = 0$, and define $x_n = \mu_n \lambda_n y_n$. Then, the sequence x_n converges to 0 (even in the Mackey sense), and the closed subspace generated by $\{x_n\}$ coincides with the one generated by $\{y_n\}$, that is, with $\mathbb{K}^{\mathbb{R}}$. This shows

that there are difficulties in generalizing Theorem 1.2 to the case $|J| = \aleph_0$, if such generalization is true.

We recall the following fact (see (3)):

(*) if $\mathcal{F}$ is a class of lcs, if $E_i \in \mathcal{C}^l(\mathcal{F})$ for every $i \in I$, and if $\mathbb{K}^I \in \mathcal{C}^l(\mathcal{F})$ then $\prod_{i \in I} E_i \in \mathcal{C}^l(\mathcal{F})$. In other words, if $\mathbb{K}^I \in \mathcal{C}^l(\mathcal{F})$ then $\mathcal{C}^l(\mathcal{F})$ is stable by products of $|I|$ of its elements.

We have then immediately the following corollary:

Corollary 1.3. Let $\mathcal{F}$ be a class of lcs. If $\mathbb{K}^c \in \mathcal{C}^l(\mathcal{F})$, then $\mathcal{C}^l(\mathcal{F})$ is stable by arbitrary products with cardinal set smaller than the smallest strongly inaccessible cardinal number (i e, if E_i, $i \in I$ is a family of elements of $\mathcal{C}^l(\mathcal{F})$ and $|I|$ is smaller than the smallest strongly inaccessible cardinal, then $\prod_{i \in I} E_i \in \mathcal{C}^l(\mathcal{F})$).

Proof. It is well known that $\mathcal{C}^l(\mathcal{F})$ is stable by final lc limits, hence Corollary 1.3 follows from (*) and from Theorem 1.2. ∎

We leave unanswered the following questions:

Q(1.4) Is $\mathbb{K}^c$ the final lc limit of a family of spaces isomorphic to ω?

Q(1.5) If $\mathcal{F}$ is a class of lcs and $\omega \in \mathcal{C}^l(\mathcal{F})$, does it follow that $\mathbb{K}^c \in \mathcal{C}^l(\mathcal{F})$?

Q(1.6) If $\mathcal{F}$ is a class of lcs and $\mathbb{K}^c \in \mathcal{C}^l(\mathcal{F})$, does it follow that $\mathcal{C}^l(\mathcal{F})$ is stable by arbitrary products?

Of course, a positive answer to to Q(1.4) would give a positive answer to Q(1.5).

Remark 1.4. In (5) (pg 86) the following problem was left unanswered:

(P) Is every Hausdorff quotient of a dual nuclear space also dual nuclear?

We may point here a relationship between this problem and Q(1.4). The spaces $\omega^{(I)}$ are dual nuclear for every set I; on the other hand, $\mathbb{K}^c$ is not dual nuclear. If $\mathbb{K}^c$ were the final lc limit of spaces isomorphic to ω, then $\mathbb{K}^c$ would be isomorphic to a quotient of $\omega^{(I)}$ for some I. Hence:

i) a positive answer to (P) would give a negative answer to Q(1.4).

ii) a positive answer to Q(1.4) would give a negative answer to (P).

We will show in Corollary 2.6 that (P) has in fact a negative answer. Hence, there is still some possibility of a positive answer to Q(1.4).

2. Spaces whose final limits contain all Banach spaces

We need the following fact (which is an adaptation of the statement called Fact 2 in (1)).

<u>Fact 1</u>. Let G be a normed space, y' a linear discontinuous functional on G, and $a_n > 0$ an arbitrary sequence of positive real numbers. Then, there is a bounded sequence y_n in G such that $\sum_{n=1}^{\infty} \|y_n\| < \infty$ and $\lim_{n\to\infty} |a_n y'(y_n)| = \infty$.

<u>Proof</u>. Since $\|y'\| = \infty$, there exists $\bar{y}_n \in G$ with $\|\bar{y}_n\| \leq 1$ and $|y'(\bar{y}_n)| \geq 4^n/a_n$. The Fact follows, taking $y_n = \bar{y}_n/2^n$. ∎

Next theorem is one of the basic results of this §, and a generalization of (1). It utilizes the following condition on a lcs E:

(2.1) there is a bounded sequence x_n in E and an equicontinuous sequence f_n in E^*, such that $f_n(x_m) = 0$ for $n \neq m$ and $f_n(x_n) = a_n > 0$ for every n.

<u>Theorem 2.1</u>. Let E be a lcs satisfying (2.1) and G an arbitrary Banach space. Then, G is the final lc limit of a family of spaces isomorphic to E (hence, there is a set I such that G is isomorphic to a quotient of $E^{(I)}$).

<u>Proof</u>. Let y' be a linear discontinuous functional on G. Let x_n, f_n and a_n be as given by (2.1). By Fact 1, there is a bounded sequence y_n in G such that $\sum \|y_n\| < \infty$ and $|a_n y'(y_n)| \to \infty$. Let $D = \{\lambda \in K \mid |\lambda| \leq 1\}$. Since $\{f_n\}$ is an equicontinuous sequence in E^*, there is a neighborhood V of 0 in E such that $f_n(V) \subset D$ for every n. For $x \in V$ we have then $\|f_n(x) y_n\| \leq \|y_n\|$, and since $\sum \|y_n\| < \infty$, the series $\sum f_n(x) y_n$ is absolutely convergent, hence also convergent in G (because G is complete). The same of course happens for every x in E, thus we may define the linear map $u: E \to G$ by the formula $u(x) = \sum_{n=1}^{\infty} f_n(x) y_n$. For $x \in V$ we have $\|u(x)\| \leq \sum \|y_n\| < \infty$, hence

u is continuous. Now, $(y'\circ u)(x_n)=y'(a_ny_n)=a_ny'(y_n)$, which is not bounded, hence $y'\circ u$ is discontinuous. The result follows from Lemma 1.1. ∎

For a better understanding of condition (2.1), and to see how far it may be from being necessary to get the thesis of Theorem 2.1, we give another theorem. Three more facts are needed. Fact 2 below has a great resemblance to Fact 1 of (1).

<u>Fact 2</u>. If L is an infinite dimensional normed space and B is an ac bounded absorbent subset of L, then there are $y_n \in B$ and $g_n \in L^*$, with $\|g_n\|=1$, $g_n(y_m)=0$ for $n\neq m$ and $g_n(y_n)>0$ for every n.

<u>Proof</u>. Start by taking any $y_1 \in B$ with $y_1\neq 0$, and some $g_1 \in L^*$ with $\|g_1\|=1$ and $g_1(y_1)>0$. Suppose $y_1,\dots,y_{k-1}$ in B and $g_1,\dots,g_{k-1}$ in L^* were already chosen, with $\|g_j\|=1$, $j:1,\dots,k-1$; $g_j(y_j)>0$, $j:1,\dots,k-1$ and $g_j(y_i)=0$ for $i\neq j$, $i,j:1,\dots,k-1$. Then, $y_1,\dots,y_{k-1}$ are linearly independent. Set $N=\bigcap_{j:1}^{k-1}\ker g_j$. Since $\dim L=\infty$ and $\dim (L/N)=k-1$, we have $N\neq\{0\}$, hence there is $z_k\in N$, with $z_k\neq 0$, and since B is absorbent, there is $\lambda_k>0$ such that $y_k=\lambda_k z_k\in B$. Of course, $y_1,\dots,y_k$ are linearly independent. Define the linear functional h_k on $[y_1,\dots,y_k]$ by $h_k(y_j)=0$ for $j:1,\dots,k-1$ and $h_k(y_k)=1$. Since h_k is continuous, it has a linear continuous extension $\tilde{h}_k$ to L. It is then enough to choose $g_k=\tilde{h}_k/\|\tilde{h}_k\|$. By induction, the whole sequences y_n and g_n are obtained. ∎

<u>Fact 3</u>. If a lcs F is the final lc limit of a family E_i, $i\in I$ of lcs determined by linear maps $u_i:E_i\to F$, and $\tilde{u}_i:E_i/(\ker u_i)\longrightarrow F$ are the quotient maps of u_i, then F is also the final limit of the family $E_i/(\ker u_i)$ determined by $\tilde{u}_i$ (the same happens if we take $N_i\subset\ker u_i$ and $\approx u_i:E_i/N_i\longrightarrow F$).

<u>Proof</u>. It is enough to remark that a lc topology on F makes all the maps u_i continuous if and only if it makes all the maps $\tilde{u}_i$ continuous (if and only if it makes all $\approx u_i$ continuous). ∎

<u>Fact 4</u>. (in (2) it is said that this fact is stated in (6), and that it is due to S. Dierolf). For each Hausdorff topological vector space

(E,t), there is a complete Hausdorff topological vector space (Y,s) such that: i) (E,t) is isomorphic to a quotient of (Y,s), ii) every bounded subset of (Y,s) is finite dimensional, iii) the dual of (Y,s) separates points of Y. In the case (E,t) is a lcs, (Y,s) may be chosen to be a lcs.

Theorem 2.2. Let E be a lcs and consider the following conditions on E:

(2.1) (already stated);

(2.2) there is a bounded ac subset B of E and an ac neighborhood V of 0 in E such that, calling $M = \ker V$, the vector space $E_B/(E_B \cap M)$ is of infinite dimension;

(2.3) there is a closed subspace H of E such that E/H has a continuous norm and the bornological space associated to E/H does not have the finest lc topology (remark that the last sentence is equivalent to: E/H has a bounded subset B such that $[B]$ is infinite dimensional; remark also that it implies that E/H is infinite dimensional);

(2.4) every Banach space G is the final lc limit of a family of spaces each of them isomorphic to E;

(2.5) every ultrabornological space is the final lc limit of a family of spaces isomorphic to E;

(2.6) there is some Banach space G_0 of infinite dimension which is the final lc limit of a family of spaces isomorphic to E.

Then, a) $(2.1) \Longleftrightarrow (2.2) \Longrightarrow (2.3) \Longrightarrow (2.4) \Longleftrightarrow (2.5) \Longleftrightarrow (2.6)$.

b) If E is a bornological space that does not satisfy (2.2), if V is an ac neighborhood of 0 in E and $M = \ker V$, then E/M has the finest lc topology.

c) If E is a lcs such that, for every ac neighborhood V of 0 and $M = \ker V$, E/M has the finest lc topology (which is the case, by b), if E is bornological) and if G is a lcs which has a continuous norm and which is the final limit of a family of spaces isomorphic to E, then G must have the finest lc topology. In particular, if E is bornological, all the conditions (2.1) to (2.6) are equivalent.

d) In the general case, the implication (2.3) $\Longrightarrow$ (2.2) (and a fortiori also (2.4) $\Longrightarrow$ (2.2)) is false.

Proof. a) (2.1) $\Longrightarrow$ (2.2) Let B be the ac hull of $\{x_n\}$ and V the polar of $\{f_n\}$: then B is bounded and V is an ac neighborhood of 0. Since $f_n(V) \subset D$, it follows that $f_n(M)=\{0\}$, for every n. Let $\bar{f}_n$ be the restriction of f_n to E_B, g_n the quotient map of $\bar{f}_n$, defined on $E_B/(E_B \cap M)$, and $y_n=q(x_n)$, where $q: E_B \longrightarrow E_B/(E_B \cap M)$ is the canonical map. Then, $g_n(y_m)=g_n(q(x_m))=\bar{f}_n(x_m)$ for every n and m, so that $g_n(y_n)=a_n \neq 0$ but $g_n(y_m)=0$ for $n \neq m$. Therefore, the sequence y_n is linearly independent, hence the dimension of $E_B/(E_B \cap M)$ is infinite.

(2.2) $\Longrightarrow$ (2.1) and (2.2) $\Longrightarrow$ (2.3) Let p be the gauge of V. Then p is a continuous seminorm with $\ker p = M$ and $E_V = (E,p)/M$ is a normed space (and we call $\bar{p}$ the norm on E_V associated to p). The canonical map $q_V: E \rightarrow E_V$ is continuous, hence $\bar{B}=q_V(B)$ is a bounded subset of E_V. The subspace L of E_V generated by $\bar{B}$ is $[\bar{B}]=q_V(E_B)=q_V(E_B+M)=(E_B+M)/M$. Since $(E_B+M)/M$ is algebraically isomorphic to $E_B/(E_B \cap M)$, it follows that L has infinite dimension. Hence, if we endow L with the topology induced by E_V, L is an infinite dimensional normed space and $\bar{B}$ is a bounded absorbent ac subset of L.

By Fact 2, there are $y_n \in \bar{B}$ and $g_n \in L^*$ with $\|g_n\|=1$, $g_n(y_m)=0$ for $n \neq m$ and $g_n(y_n) > 0$ for every n. By Hahn-Banach, there is an extension h_n of g_n to E_V, with $\|h_n\|=1$, for every n. Let us define $f_n=h_n \circ q_V$. Since $\|h_n\|=1$, it follows that $f_n(V) \subset D$ for every n, hence $\{f_n\}$ is equicontinuous. As $y_n \in \bar{B} = q_V(B)$, we may take $x_n \in B$ such that $q_V(x_n)=y_n$. Hence, $\{x_n\}$ is bounded and $f_n(x_m)=h_n(q_V(x_m))=h_n(y_m)=g_n(y_m)$ for every n and m, so that $f_n(x_m)=0$ for $n \neq m$ and $f_n(x_n) > 0$ for every n. That shows that (2.2) implies (2.1).

On the other hand, if we choose H=M, as $q_V: E \rightarrow E_V$ is continuous, so must be continuous the identity map $i: E/M \longrightarrow E_V$. Now E_V is a normed space with norm $\tilde{p}$, hence $\tilde{p}$ is a continuous norm on E/M. Since $q: E \to E/M$ is continuous, the set $\tilde{B}=q(B)$ is bounded also in E/M. As we already saw that $[\tilde{B}]=L$ and L is infinite dimensional, we finished the proof that (2.2) implies (2.3).

(2.3) ⟹ (2.4) First we remark that, if (2.3) holds for E, then E/H satisfies (2.2). In fact, since E/H has a continuous norm, the unit ball for that norm is an ac neighb. V of 0 such that M=ker V reduces to $\{0\}$, hence (2.2) for E/H reduces to the existence of an infinite dimensional ac bounded subset B of E/H and that follows from (2.3) for E.

Since E/H satisfies (2.2), which is equivalent to (2.1), and it was already proved in Theorem 2.1 that (2.1) implies (2.4), it follows that E/H satisfies (2.4), that is, every Banach space is the final limit of a family of spaces isomorphic to E/H. Since E/H is a final limit of E, it follows by the transitivity of final lc limits that G is also a final limit of spaces isomorphic to E.

(2.4) ⟹ (2.5) Since the ultrabornological spaces are the final limits of Banach spaces, the result follows by the transitivity of final limits.

(2.5) ⟹ (2.4) and (2.5) ⟹ (2.6) Trivial.

(2.6) ⟹ (2.4) If G_o is a Banach space of infinite dimension, G_o clearly satisfies (2.3). As (2.3) implies (2.4), it follows that every Banach space G is the final limit of a family of spaces isomorphic to G_o. Since by hypothesis G_o is the final limit of a family of spaces isomorphic to E, the same happens by transitivity to every Banach space G.

b) Since E does not satisfy (2.2), the spaces $E_B/(E_B \cap M)$ are finite dimensional for every ac bounded subset B of E. Since E is bornological, E is the final limit of the spaces E_B, determined by the inclusion maps $i_B: E_B \to E$, hence E/M is the final limit of the spaces E_B, determined by the maps $q \circ i_B: E_B \to E/M$, where $q: E \to E/M$ is the quotient map. Remarking that $\ker (q \circ i_B) = E_B \cap M$, it follows from Fact 3, that E/M is the final limit of the spaces $E_B/(E_B \cap M)$, determined by the maps $\widetilde{q \circ i_B}: E_B/(E_B \cap M) \longrightarrow E/M$, quotient of $q \circ i_B$. Since all the spaces $E_B/(E_B \cap M)$ are finite dimensional, it follows that E/M has the finest lc topology.

c) Let I be a set, $E_i = E$ for every $i \in I$ and G the final limit of the family E_i, $i \in I$, determined by a family of linear maps $u_i: E_i \longrightarrow G$. Let B_o be the unit ball of some continuous norm on G. Since B_o is an ac neighb. of 0 in G, the ac sets $V_i = u_i^{-1}(B_o)$ are neighb. of 0 in E_i, with $M_i = \ker V_i$ equal to $\ker u_i$. If $\tilde{u}_i: E_i/M_i \longrightarrow G$ is the quotient map of u_i, then by Fact 3, G is the final limit of the spaces E_i/M_i. Since by hypothesis all the E_i/M_i have the finest lc topology, G also must have the finest lc topology.

We show now that if E is bornological, then (2.6) $\Longrightarrow$ (2.2). In fact, if E where bornological without property (2.2) and a normed space G_o were the final limit of a family of spaces isomorphic to E, then by the first part of c), G_o should have the finest lc topology, which is impossible for an infinite dimensional Banach space.

d) Let F be an infinite dimensional Banach space. By Fact 4, there is a lcs E and a closed subspace H of E such that E/H is isomorphic to F (hence E has property (2.3)), and such that every bounded subset of E is finite dimensional, hence E_B (and a fortiori $E_B/(E_B \cap M)$) is finite dimensional for every ac bounded subset B of E, so that E does not have property (2.2). Therefore, E shows that (2.3) $\not\Longrightarrow$ (2.2). ∎

<u>Corollary 2.3</u>. If E is a lcs that has some continuous norm and some bounded subset of infinite dimension, then E has all the properties of Theorem 2.2. b) If E is a lcs with the finest lc topology and F is a lcs with a weak topology, then E,F and ExF do not have any of the properties of Theorem 2.2. c) If E is a Frechet space of infinite dimension, then E has the properties of Theorem 2.2 if and only if E is not isomorphic to ω. d) The properties (2.4) to (2.6) are equivalent to the following:

(2.7) there is some Frechet space C_o of infinite dimension, not isomorphic to ω, such that G_o is the final limit of a family of spaces isomorphic to E.

Proof. a) Let V be the unit ball of some continuous norm on E. Then the set M=ker V reduces to $\{0\}$. If B_o is a bounded subset with $[B_o]$ of infinite dimension, and $B=\Gamma B_o$, then $E_B/(E_B \cap M) = E_B = [B]$ has infinite dimension. Hence property (2.2) (and a fortiori all the others) is satisfied.

b) It is not enough to show that they do not have property (2.2), since (2.4) does not imply (2.2). Let V be an ac neighb. of 0 in ExF. Then, there are ac neighb. V_1 and V_2 of 0 in E and F, respect., such that $V_1 x V_2 \subset V$. Hence, $M = \ker V \supset \ker(V_1 x V_2) = (\ker V_1)x(\ker V_2) = M_1 x M_2$. Thus, (ExF)/M is a quotient of $(ExF)/(M_1 x M_2)$, which is isomorphic to $(E/M_1)x(F/M_2) = H$. Since F has a weak topology, F/M_2 is finite dimensional, and as E has the finest lc topology, the space H (and a fortiori (ExF)/M) also has the finest lc topology. By Theorem 2.2.c, ExF doesnot have property (2.4), and the same happens to E and F.

c) Let E be a Frechet space of infinite dimension. If its topology is the weak one, then E is isomorphic to ω, and by part b of this corollary, E does not have property (2.4). If its topology is not the weak one, then E is not isomorphic to ω, and there is some ac neighb. V of 0 in E, such that M=ker V has infinite codimension, that is, E/M has infinite dimension. Then, E/M is Frechet (hence bornological) with a continuous norm and with infinite dimension, hence its topology is not the finest lc one. This shows that E satisfies (2.3) (hence also all the others, by Th. 2.2.c).

d) Follows immediately from part c, by the transitivity of the final lc limits. ∎

Remark 2.4. a) Corollary 2.3.a shows that the class of lcs satisfying (2.2) is very large indeed, while Corollary 2.3.b shows that for a lcs to have property (2.2) or even (2.4), topologies that are too fine or too weak in some sense, as well as combinations of both, must be avoided.
b) In (2) it is stated that the following fact is proved in (7):

(**) If G is a bornological lcs, then G is a quotient of some complete and semi-Montel space.

As a consequence of Corollary 2.3, we get the following corollary which may be considered in some sort as the result analogous to (**) for the ultrabornological case.

Corollary 2.5. If G is an ultrabornological lcs, then G is a quotient of a lcs F which is complete, ultrabornological and dual nuclear (hence also Montel).

Proof. Take as E any nuclear Frechet (hence also dual nuclear) space of infinite dimension, not isomorphic to ω (for instance, E may be chosen as the space $\mathcal{s}$ of rapidly decreasing sequences). By Corollary 2.3.c, G is the final limit of a family of spaces isomorphic to E, hence G is isomorphic to a quotient of $E^{(I)}$, for some set I. It is enough to take $F=E^{(I)}$. ∎

Corollary 2.6. There exists a dual nuclear lcs (which may be also chosen as being complete and ultrabornological) which has a quotient (by a closed subspace) which is not dual nuclear. (This solves negatively the problem in (5), pg 86).

Proof. In fact, K^c is an ultrabornological space, but is not dual nuclear. By Corollary 2.5, K^c is a quotient of a dual nuclear space (which may be chosen as $\mathcal{s}^{(I)}$, for some I). ∎

We show next the implications of Theorem 2.2 to the closed graph theorem.

Corollary 2.7. a) Let $\mathcal{F}$ be a class of lcs. If $\mathcal{C}^1(\mathcal{F})$ contains some infinite dimensional Frechet space not isomorphic to ω (or more generally, if $\mathcal{C}^1(\mathcal{F})$ contains some lcs with property (2.2) or (2.3)), then $\mathcal{C}^1(\mathcal{F})$ contains the class $\mathcal{U}$ of all ultrabornological spaces. Hence, conversely, if $\mathcal{C}^1(\mathcal{F})$ does not contain some ultrabornological space, then $\mathcal{C}^1(\mathcal{F})$ does not contain any lcs with property (2.2) or (2.3) (and is, therefore, a very small class of lcs). b) Let $\mathcal{E}$ be a class of lcs. If $\mathcal{E}$

contains some space with property (2.2) or (2.3), then $\mathcal{C}^r(\mathcal{E})$ is contained in the class of all infra-(u) spaces. Hence, conversely, if $\mathcal{C}^r(\mathcal{E})$ is not contained in the class of infra-(u) spaces, then $\mathcal{E}$ does not contain any lcs with property (2.2) or (2.3). c) If $\mathcal{F}$ contains some lcs which is not infra-(u), then $\mathcal{C}^l(\mathcal{F})$ is contained in the class of lcs which do not satisfy (2.3).

Proof. a) The result follows from the fact that $\mathcal{C}^l(\mathcal{F})$ is stable by final limits and from Corollary 2.3.c (or Theorem 2.2).

b) Since $\mathcal{E} \subset \mathcal{C}^l(\mathcal{C}^r(\mathcal{E}))$, it follows from part a that $\mathcal{C}^l(\mathcal{C}^r(\mathcal{E}))$ contains $\mathcal{U}$, hence $\mathcal{C}^r(\mathcal{C}^l(\mathcal{C}^r(\mathcal{E})))$, which is equal to $\mathcal{C}^r(\mathcal{E})$, must be contained in the class of all infra-(u) spaces.

c) Since $\mathcal{C}^r(\mathcal{C}^l(\mathcal{F})) \supset \mathcal{F}$, it follows from the hypothesis that $\mathcal{C}^r(\mathcal{C}^l(\mathcal{F}))$ is not contained in the class of infra-(u) spaces, hence by part b, it follows that $\mathcal{C}^l(\mathcal{F})$ is contained in the class of spaces which satisfy (not 2.3). ∎

Remark 2.8. The negation of property (2.3) is the following:

(not 2.3) every quotient E/H of E such that E/H has a continuous norm, is such that the bornological space associated to it (to E/H) has the finest lc topology.

In the case E is bornological, (not 2.3) reduces to:

(***) every quotient of E with a continuous norm has the finest lc topology.

It should be noted that it follows from Theorem 2.2.c that (***) implies (not 2.4) even without the assumption that E is bornological.

Remark 2.9. Property (***) also appears in (4) (where it is called Quotientenbedingung). If $\mathcal{N}$ is the class of all normed spaces, the elements of $\mathcal{C}^l(\mathcal{N})$ are called GN-spaces in (4), where it is proved that a lcs is GN if and only if it is barrelled and has property (***).

We leave unanswered the following question:

Q(2.9) Is the implication (2.4) $\Rightarrow$ (2.3) (hence the equivalence (2.4) $\Leftrightarrow$ (2.3)) always true? If answer is no, characterize the lcs that have (2.4).

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