ORTHOGONAL GEODESIC CHORDS, BRAKE ORBITS AND HOMOCLINIC ORBITS IN RIEMANNIAN MANIFOLDS

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Abstract. The study of solutions with fixed energy of certain classes of Lagrangian (or Hamiltonian) systems is reduced, via the classical Maupertuis–Jacobi variational principle, to the study of geodesics in Riemannian manifolds. We are interested in investigating the problem of existence of brake orbits and homoclinics, in which case the Maupertuis–Jacobi principle produces a Riemannian manifold with boundary and with metric degenerating in a nontrivial way on the boundary. In this paper we use the classical Maupertuis–Jacobi principle to show how to remove the degeneration of the metric on the boundary, and we prove in full generality how the brake orbit and the homoclinic multiplicity problem can be reduced to the study of multiplicity of orthogonal geodesic chords in a manifold with *regular* and *strongly concave* boundary.

1. Introduction

The study of periodic and homoclinic orbits of Lagrangian and Hamiltonian systems is an extremely active research field in classical and modern mathematics, having a huge number of applications in the physical sciences. One of the peculiarities of the problem is that, although already very popular among classical analysts and geometers, it has never been out of fashion, and it has been studied for a lonc time with techniques of an increasing level of sophistication. Indeed, the study of solutions of Hamiltonian systems has motivated many recent developments of several mathematical theories, including calculus of variations, symplectic geometry and Morse theory, among others, and the vaste literature on the topic witnesses the leading role of the subject in modern mathematics.

The central interest of the present paper is the study of solutions of an autonomous Lagrangian (or Hamiltonian) system, having prescribed energy, in a manifold M and belonging to two special classes of solutions: the homoclinic orbits and the brake orbits. Homoclinic orbits are solutions $x : \mathbb{R} \to M$ of the system for which $\lim_{t \to +\infty} x(t) = \lim_{t \to -\infty} x(t) = x_0$, and $\lim_{t \to \pm\infty} \dot{x}(t) = 0$. The point x_0 must then be a critical point of the potential function of the system. Brake orbits are a special class of periodic solutions that have an oscillating character, i.e., periodic solutions $x : \mathbb{R} \to M$ having period $2T$, with $x(T + t) = x(T - t)$ and $\dot{x}(T + t) = -\dot{x}(T - t)$ for all $t \in \mathbb{R}$. Clearly, $\dot{x}(kT) = 0$ for all $k \in \mathbb{Z}$.

By a classical variational principle, known as the Maupertuis–Jacobi principle, solutions of autonomous Lagrangian or Hamiltonian systems having a fixed value of the energy correspond to geodesics relative to a Riemannian metric, called the Jacobi metric. When dealing with homoclinic orbits issuing from a critical point of the potential function, or with brake orbits, then the classical formulation of the Maupertuis–Jacobi principle fails, due to the fact that such solutions pass through a region where the Jacobi metric degenerates in a nontrivial way. An accurate analysis of the geodesic behavior near such degeneracies, that occur at the level of the prescribed energy, has led many authors to obtain existence results by perturbation techniques. More specifically, following an original idea by Seifert [11], some authors (see [6]) have been able to perform a geometrical construction consisting of attaching a smooth, *convex* and sufficiently small collar (see Figure 1) to the degenerate region, in such a way that the geodesics in the resulting manifold could be counted by standard techniques in convex Riemannian geometry ([2, 8]). Then, a limit argument was used to obtain existence results for geodesics in the original degenerate metric by letting the size of the collar go to zero. The same idea cannot be used if one wants to obtain multiplicity results, due to the fact that such a limit procedure does not guarantee that possibly distinct geodesics in the perturbed metric converge to geometrically distinct geodesics in the original Jacobi metric, unless one poses *ad hoc* "nonresonance" assumptions (see [6]). Here, by geometrically distinct, we mean geodesics having different images; the nonresonance assumptions mentioned above guarantee that we may avoid the situation in which distinct geodesics in the perturbed metric tend to the same periodic geodesic travelled a different number of times.

The starting point of this paper is the idea that, if one wants to preserve the number of distinct geodesics, then one has to perform a geometrical construction that avoids degenerate regions and limits procedure. Such a
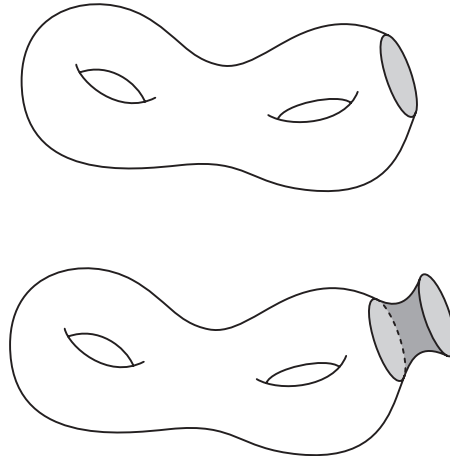
FIGURE 1. Gluing a collar with convex boundary to a concave boundary.

construction would obviously be based on a careful investigation of the geodesic behavior near the boundary of the level set of the prescribed energy. Working in this direction has lead to the quite remarkable observation that a noncritical level set of the potential function, or a small ball around a nondegenerate maximum point of the potential, are near certain hypersurfaces that are *strongly concave* relative to the Jacobi metric, and that have the property that orthogonal geodesic chords arriving on one of these hypersurfaces can be uniquely extended to geodesic chords up to the degenerate boundary. The presence of concave hypersurfaces near the degenerate boundary can be interpreted as an indication that Seifert's technique of gluing a convex collar would be somewhat unnatural in order to study the multiplicity problem in full generality.

The main results of this paper are contained in Theorem 5.9, relating the brake orbits problem to the orthogonal geodesic chords problem, and Theorem 5.19, that deals with the homoclinics problem.

The issue of concavity, as opposed to the *convexity* property used in the classical literature, is the key point to develop a multiplicity theory for brake orbits and homoclinic orbits under purely topological assumptions on the underlying manifolds. These multiplicity results constitute the topic of two of the papers by the authors ([4, 5]).

2. Geodesics and Concavity

Let (M, g) be a smooth (i.e., of class C^2) Riemannian manifold with $\dim(M) = m \geq 2$, let dist denote the distance function on M induced by g; the symbol ∇ will denote the covariant derivative of the Levi-Civita connection of g, as well as the gradient differential operator for smooth maps on M. The Hessian $\mathrm{H}^f(q)$ of a smooth map $f : M \to \mathbb{R}$ at a point $q \in M$ is the symmetric bilinear form $\mathrm{H}^f(q)(v, w) = g\big((\nabla_v \nabla f)(q), w\big)$ for all $v, w \in T_x M$; equivalently, $\mathrm{H}^f(q)(v, v) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\big|_{s=0} f(\gamma(s))$, where $\gamma : \,]-\varepsilon, \varepsilon[\,\to M$ is the unique (affinely parameterized) geodesic in M with $\gamma(0) = q$ and $\dot{\gamma}(0) = v$. We will denote by $\frac{\mathrm{D}}{\mathrm{d}t}$ the covariant derivative along a curve, in such a way that $\frac{\mathrm{D}}{\mathrm{d}t}\dot{x} = 0$ is the equation of the geodesics. A basic reference on the background material for Riemannian geometry is [3].

Let $\Omega \subset M$ be an open subset; $\overline{\Omega} = \Omega \bigcup \partial\Omega$ will denote its closure. There are several notions of convexity and concavity in Riemannian geometry, extending the usual ones for subsets of the Euclidean space $\mathbb{R}^m$. In this paper we will use a somewhat concavity condition for compact subsets of M, that we will refer to as "strong concavity" below, and which is stable by C^2-small perturbations of the boundary. Let us first recall the following:

Definition 2.1. $\overline{\Omega}$ *is said to be* convex *if every geodesic* $\gamma : [a, b] \to \overline{\Omega}$ *whose endpoints* $\gamma(a)$ *and* $\gamma(b)$ *are in* Ω *has image entirely contained in* Ω. *Likewise,* $\overline{\Omega}$ *is said to be* concave *if its complement* $M \setminus \overline{\Omega}$ *is convex.*

If $\partial\Omega$ is a smooth embedded submanifold of M, let $\mathbb{I}_{\mathfrak{n}}(x) : T_x(\partial\Omega) \times T_x(\partial\Omega) \to \mathbb{R}$ denote the *second fundamental form of* $\partial\Omega$ *in the normal direction* $\mathfrak{n} \in T_x(\partial\Omega)^{\perp}$. Recall that $\mathbb{I}_{\mathfrak{n}}(x)$ is a symmetric bilinear form on $T_x(\partial\Omega)$ defined by:

$$\mathbb{I}_{\mathfrak{n}}(x)(v, w) = g(\nabla_v W, \mathfrak{n}), \qquad v, w \in T_x(\partial\Omega),$$

where W is any local extension of w to a smooth vector field along $\partial\Omega$.

Remark 2.2. Assume that a smooth function $\phi : M \to \mathbb{R}$ is given with the property that $\Omega = \phi^{-1}\big((-\infty, 0)\big)$ and $\partial\Omega = \phi^{-1}(0)$, with $\mathrm{d}\phi \neq 0$ on $\partial\Omega$. [1] The following equality between the Hessian H^{ϕ} and the second fundamental

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[1]For example one can choose ϕ such that $|\phi(q)| = \mathrm{dist}(q, \partial\Omega)$ for all q in a (closed) neighborhood of $\partial\Omega$.

form[2] of $\partial\Omega$ holds:

$$\mathrm{H}^{\phi}(x)(v,v) = -\mathbb{II}_{\nabla\phi(x)}(x)(v,v), \quad x \in \partial\Omega, \; v \in T_x(\partial\Omega); \qquad (2.1)$$

namely, if $x \in \partial\Omega$, $v \in T_x(\partial\Omega)$ and V is a local extension around x of v to a vector field which is tangent to $\partial\Omega$, then $v\big(g(\nabla\phi, V)\big) = 0$ on $\partial\Omega$, and thus:

$$\mathrm{H}^{\phi}(x)(v,v) = v\big(g(\nabla\phi, V)\big) - g(\nabla\phi, \nabla_v V) = -\mathbb{II}_{\nabla\phi(x)}(x)(v,v).$$

Note that the second fundamental form is defined intrinsically, while there is in general no natural choice for a function ϕ describing the boundary of Ω as above.

Definition 2.3. *We will say that that $\overline{\Omega}$ is* strongly concave *if $\mathbb{II}_{\mathfrak{n}}(x)$ is positive definite for all $x \in \partial\Omega$ and all inward pointing normal directions $\mathfrak{n}$.*

Remark 2.4. Strong concavity is evidently a C^2-*open condition.* It should also be emphasized that if $\overline{\Omega}$ is strongly concave, then for *any* smooth map $\phi : M \to \mathbb{R}$ as in Remark 2.2, and for all $q \in \partial\Omega$, the Hessian $\mathrm{H}^{\phi}(q)$ is negative definite on $T_q\big(\partial\Omega\big)$. From this observation, it follows immediately that geodesics starting tangentially to $\partial\Omega$ move inside Ω.

The main objects of our study are geodesics in M having image in $\overline{\Omega}$ and with endpoints orthogonal to $\partial\Omega$. We distinguish a special class of such geodesics, called "weak", whose relevance will not be emphasized in the present paper, but it will be used in a substantial way in the proof of the multiplicity results in [4].

Definition 2.5. *A geodesic $\gamma : [a, b] \to M$ is called a* geodesic chord *in $\overline{\Omega}$ if $\gamma\big(\,]a, b[\,\big) \subset \Omega$ and $\gamma(a), \gamma(b) \in \partial\Omega$; by a* weak geodesic chord *we will mean a geodesic $\gamma : [a, b] \to M$ with image in $\overline{\Omega}$ and endpoints $\gamma(a), \gamma(b) \in \partial\Omega$. A (weak) geodesic chord is called* orthogonal *if $\dot\gamma(a^+) \in (T_{\gamma(a)}\partial\Omega)^{\perp}$ and $\dot\gamma(b^-) \in (T_{\gamma(b)}\partial\Omega)^{\perp}$, where $\dot\gamma(\,\cdot\,^{\pm})$ denote the lateral derivatives. An orthogonal geodesic chord in $\overline{\Omega}$ whose endpoints belong to distinct connected components of $\partial\Omega$ will be called a* crossing orthogonal geodesic chord *in $\overline{\Omega}$ (see Figure 2).*

For shortness, we will write **OGC** for "orthogonal geodesic chord" and **WOGC** for "weak orthogonal geodesic chord".

For the proof of the multiplicity results in [4], we will use a geometrical construction that will work in a situation where one can exclude *a priori* the

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[2]Observe that, with our definition of ϕ, then $\nabla\phi$ is a normal vector to $\partial\Omega$ pointing *outwards* from Ω.
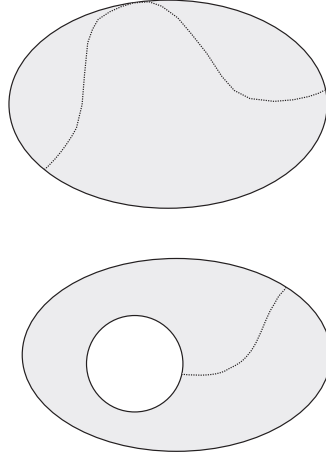
FIGURE 2. A weak orthogonal geodesic chord (WOGC) in $\overline{\Omega}$ (above), and a crossing OGC (below).

existence in $\overline{\Omega}$ of (crossing) weak orthogonal geodesic chords in $\partial\Omega$. We will now show that one does not lose generality in assuming that there are no such WOGC's in $\overline{\Omega}$ by proving the following:

Proposition 2.6. *Let $\Omega \subset M$ be an open set whose boundary $\partial\Omega$ is smooth and compact and with $\overline{\Omega}$ strongly concave. Assume that there are only a finite number of (crossing) orthogonal geodesic chords in $\overline{\Omega}$. Then, there exists an open subset $\Omega' \subset \Omega$ with the following properties:*

(1) *$\overline{\Omega'}$ is diffeomorphic to $\overline{\Omega}$ and it has smooth boundary;*
(2) *$\overline{\Omega'}$ is strongly concave;*
(3) *the number of (crossing) OGC's in $\overline{\Omega'}$ is less than or equal to the number of (crossing) OGC's in $\overline{\Omega}$;*
(4) *every (crossing) WOGC in $\overline{\Omega'}$ is a (crossing) OGC in $\overline{\Omega'}$.*

Proof. The desired set Ω' will be taken of the form: $\Omega' = \phi^{-1}\big((-\infty, -\delta)\big)$, with $\delta > 0$ small, and with ϕ a smooth map as in Remark 2.2 such that $|\phi(q)| = \mathrm{dist}(q, \partial\Omega)$ for q near $\partial\Omega$. Observe that if δ is small enough, then by continuity $\mathrm{d}\phi \neq 0$ on $\phi^{-1}([-\delta, 0])$, which implies that $\partial\Omega'$ is smooth and that $\overline{\Omega'}$ is diffeomorphic to $\overline{\Omega}$, as we see using the integral curves of $\nabla\phi$. Since strong concavity is an open condition in the C^2 topology, if $\delta > 0$ is small enough then $\overline{\Omega'}$ is strongly concave, proving (2).

Moreover, δ must be chosen small enough so that the exponential map gives a diffeomorphism from an open neighborhood of the zero section of the normal bundle of $\partial\Omega$ to the set $\phi^{-1}\big((-2\delta, 2\delta)\big)$; the existence of such δ is guaranteed by our compactness assumption on $\partial\Omega$. Since $\phi(q) = -\mathrm{dist}(q, \partial\Omega)$ near $\partial\Omega$, then every (crossing) geodesic in Ω' that arrives orthogonally at $\partial\Omega'$ can be smoothly extended to a (crossing) geodesic in $\overline{\Omega}$ that arrives orthogonally at $\partial\Omega$; observe that any such extended geodesic only touches $\partial\Omega$ at the endpoints; i.e., it is a (crossing) OGC in $\overline{\Omega}$. This proves part (3).

We claim that there exists $\delta > 0$ arbitrarily small such that every (crossing) WOGC is a (crossing) OGC in $\phi^{-1}\big((-\infty, -\delta)\big)$. Assume on the contrary that there exists a sequence $\delta_n > 0$ with $\delta_n \to 0$ as $n \to \infty$, a sequence $0 < s_n < 1$ and a sequence of (crossing) geodesics $\gamma_n : [0, 1] \to \Omega$ with $\phi(\gamma_n(0)) = \phi(\gamma_n(s_n)) = \phi(\gamma_n(1)) = -\delta_n$, $\dot{\gamma}_n(0)$ and $\dot{\gamma}_n(1)$ orthogonal to $\phi^{-1}(-\delta_n)$ and $\phi(\gamma_n(s)) \leq -\delta_n$ for all $s \in [0, 1]$ and all $n \in \mathbb{N}$. As we have observed, for n large each geodesic γ_n can be smoothly extended to a (crossing) OGC in $\overline{\Omega}$, and clearly all such extensions cannot make a *finite* set of geometrically distinct (crossing) OGC's in $\overline{\Omega}$. Namely, each γ_n is tangent to the surface $\phi^{-1}(-\delta_n)$, and to *no other* surface of the form $\phi^{-1}(-\delta)$ with $\delta < \delta_n$. This says that the extensions of the γ_n are all geometrically distinct, which contradicts the fact that there is only a finite number of (crossing) OGC's in $\overline{\Omega}$ and proves part (4). $\qquad\square$

3. Brake and Homoclinic Orbits of Hamiltonian Systems

Let $p = (p_i)$, $q = (q^i)$ be coordinates on $\mathbb{R}^{2m}$, and let us consider a *natural* Hamiltonian function $H \in C^2\big(\mathbb{R}^{2m}, \mathbb{R}\big)$, i.e., a function of the form

$$H(p, q) = \frac{1}{2} \sum_{i,j=1}^{m} a^{ij}(q) p_i p_j + V(q), \tag{3.1}$$

where $V \in C^2\big(\mathbb{R}^m, \mathbb{R}\big)$ and $A(q) = \big(a^{ij}(q)\big)$ is a positive-definite quadratic form on $\mathbb{R}^m$:

$$\sum_{i,j=1}^{m} a^{ij}(q) p_i p_j \geq \nu(q) |q|^2$$

for some continuous function $\nu : \mathbb{R}^m \to \mathbb{R}^+$ and for all $(p, q) \in \mathbb{R}^{2m}$.

The corresponding Hamiltonian system is:

$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial q} \\ \dot{q} = \frac{\partial H}{\partial p}, \end{cases} \tag{3.2}$$

where the dot denotes differentiation with respect to time.

For all $q \in \mathbb{R}^m$, denote by $\mathcal{L}(q) : \mathbb{R}^m \to \mathbb{R}^m$ the linear isomorphism whose matrix with respect to the canonical basis is $(a_{ij}(q))$, the inverse of $(a^{ij}(q))$; it is easily seen that, if (p, q) is a solution of class C^1 of (3.2), then q is actually a map of class C^2 and

$$p = \mathcal{L}(q)\dot{q}. \tag{3.3}$$

With a slight abuse of language, we will say that a C^2 map $q : I \to \mathbb{R}^m$ is a solution of (3.2) if (p, q) is a solution of (3.2) where p is given by (3.3). Since the system (3.2) is autonomous, i.e., time independent, then the function H is constant along each solution, and it represents the total energy of the solution of the dynamical system. There exists a large amount of literature concerning the study of periodic solutions of autonomous Hamiltonian systems having energy H prescribed (see for instance [7] and the references therein).

We will be concerned with a special kind of periodic solutions of (3.2), called *brake orbits*. A brake orbit for the system (3.2) is a nonconstant periodic solution $\mathbb{R} \ni t \mapsto (p(t), q(t)) \in \mathbb{R}^{2m}$ of class C^2 with the property that $p(0) = p(T) = 0$ for some $T > 0$. Since H is even in the variable p, a brake orbit (p, q) is $2T$ periodic, with p odd and q even about $t = 0$ and about $t = T$. Clearly, if E is the energy of a brake orbit (p, q), then $V(q(0)) = V(q(T)) = E$.

The link between solutions of brake orbits and orthogonal geodesic chords is obtained in Theorem 5.9 (used in [4] to obtain a multiplicity result for brake orbits). Its proof is based on a well-known variational principle, that relates solutions of (3.2) having prescribed energy E with curves in the open subset $\Omega_E \subset \mathbb{R}^m$:

$$\Omega_E = V^{-1}\big((-\infty, E)\big) = \big\{ x \in \mathbb{R}^m : V(x) < E \big\} \tag{3.4}$$

endowed with the *Jacobi metric* (see Proposition 4.1):

$$g_E(x) = \big(E - V(x)\big) \cdot \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(x)\, \mathrm{d}x^i\, \mathrm{d}x^j. \tag{3.5}$$

Let us now consider the problem of homoclinics on a Riemannian manifold (M, g).

Assume that we are given a map $V \in C^2(M, \mathbb{R})$; the corresponding second-order Hamiltonian system is the equation:

$$\tfrac{\mathrm{D}}{\mathrm{d}t}\dot{q} + \nabla V(q) = 0. \tag{3.6}$$

Note that if $M = \mathbb{R}^m$ and g is the Riemannian metric

$$g = \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(x) \, \mathrm{d}x^i \, \mathrm{d}x^j, \tag{3.7}$$

where the coefficients a_{ij} are as above, then equation (3.6) is equivalent to (3.2), in the sense that x is a solution of (3.6) if and only if the pair $q = x$ and $p = \mathcal{L}(x)\dot{x}$ is a solution of (3.2).

Let $x_0 \in M$ be a critical point of V, i.e., such that $\nabla V(x_0) = 0$. We recall that a *homoclinic orbit* for the system (3.6) emanating from x_0 is a solution $q \in C^2(\mathbb{R}, M)$ of (3.6) such that:

$$\lim_{t \to -\infty} q(t) = \lim_{t \to +\infty} q(t) = x_0, \tag{3.8}$$

$$\lim_{t \to -\infty} \dot{q}(t) = \lim_{t \to +\infty} \dot{q}(t) = 0. \tag{3.9}$$

To the authors' knowledge, the only result available in the literature on multiplicity of homoclinics in the autonomous case is due to Ambrosetti and Coti–Zelati [1], to Paturel [9], to Rabinowitz [10] and to Tanaka [12]. A quite general multiplicity result for homoclinics, generalizing those in [1], in [9] and in [12], is given in [4] (and announced in [5]) using the result of Theorem 5.19 that links homoclinics to orthogonal geodesic chords.

It should also be mentioned that very likely all the results in this paper can be extended to the case of Hamiltonian functions H more general than (3.1). As observed by Weinstein in [13], Hamiltonians that are positively homogeneous in the momenta lead to Finsler metrics rather than Riemannian metrics.

4. The Maupertuis Principle

Throughout this section, (M, g) will denote a Riemannian manifold of class C^2; all our constructions will be made in suitable (relatively) compact subsets of M, and for this reason it will not be restrictive to assume, as we will, that (M, g) is complete.

4.1. **The variational framework.** The symbol $H^1([a, b], \mathbb{R}^m)$ will denote the Sobolev space of all absolutely continuous functions $f : [a, b] \to \mathbb{R}^m$ whose weak derivative is square integrable. Similarly, $H^1([a, b], M)$ will denote the infinite-dimensional Hilbert manifold consisting of all absolutely continuous curves $x : [a, b] \to M$ such that $\varphi \circ x|_{[c,d]} \in H^1([c, d], \mathbb{R}^m)$ for all charts $\varphi : U \subset M \to \mathbb{R}^m$ of M such that $x([c, d]) \subset U$. By $H^1_{\mathrm{loc}}((a, b), \mathbb{R}^m)$ we will denote the vector space of all continuous maps $f : (a, b) \to \mathbb{R}^m$ such

that $f|_{[c,d]} \in H^1\big([c,d],\mathbb{R}^m\big)$ for all $[c,d] \subset \,]a,b[$; the set $H^1_{\text{loc}}\big(\,]a,b[\,,M\big)$ is defined similarly. The Hilbert space norm of $H^1\big([a,b],\mathbb{R}^m\big)$ will be denoted by $\|\cdot\|_{a,b}$; for the purposes of this paper it will not be necessary to make the choice among equivalent norms of $H^1\big([a,b],\mathbb{R}^m\big)$.

4.2. **The Maupertuis–Jacobi principle for brake orbits.** Let $V \in C^2\big(M,\mathbb{R}\big)$ and let $E \in \mathbb{R}$. Consider the sublevel Ω_E of V in (3.4) and the *Maupertuis integral* $f_{a,b} : H^1\big([a,b],\Omega_E\big) \to \mathbb{R}$, which is the geodesic action functional relative to the metric g_E (3.5), given by:

$$f_{a,b}(x) = \frac{1}{2} \int_a^b \big(E - V(x)\big)g\big(\dot{x},\dot{x}\big)\,\mathrm{d}t, \tag{4.1}$$

where g is the Riemannian metric (3.7). Observe that the metric g_E degenerates on $\partial\Omega_E$.

The functional $f_{a,b}$ is smooth, and its differential is readily computed as:

$$\mathrm{d}f_{a,b}(x)W = \int_a^b \big(E - V(x)\big)g\big(\dot{x},\tfrac{\mathrm{D}}{\mathrm{d}t}W\big)\,\mathrm{d}t - \frac{1}{2}\int_a^b g\big(\dot{x},\dot{x}\big)g\big(\nabla V(x),W\big)\,\mathrm{d}t, \tag{4.2}$$

where $W \in H^1\big([a,b],\mathbb{R}^m\big)$. The corresponding Euler–Lagrange equation of the critical points of $f_{a,b}$ is

$$\big(E - V(x(s))\big)\tfrac{\mathrm{D}}{\mathrm{d}t}\dot{x}(s) - g\big(\nabla V(x(s)),\dot{x}(s)\big)\dot{x}(s) + \frac{1}{2}g\big(\dot{x}(s),\dot{x}(s)\big)\nabla V(x(s)) = 0, \tag{4.3}$$

for all $s \in (a,b)$.

Solutions of the Hamiltonian system (3.2) having fixed energy E and critical points of the functional $f_{a,b}$ of (4.1) are related by the following variational principle, known in the literature as the *Maupertuis–Jacobi principle*:

Proposition 4.1. *Assume that E is a regular value of the function V.*
Let $x \in C^0\big([a,b],\mathbb{R}^m\big) \cap H^1_{loc}\big((a,b),\mathbb{R}^m\big)$ be a nonconstant curve such that

$$\int_a^b \big(E - V(x)\big)g\big(\dot{x},\tfrac{\mathrm{D}}{\mathrm{d}t}W\big)\,\mathrm{d}t - \frac{1}{2}\int_a^b g\big(\dot{x},\dot{x}\big)g(\nabla V(x),W)\,\mathrm{d}t = 0 \tag{4.4}$$

for all $W \in C_0^\infty\big(\,]a,b[\,,\mathbb{R}^m\big)$, and such that

$$V\big(x(s)\big) < E, \quad \text{for all } s \in (a,b); \tag{4.5}$$

and

$$V\big(x(a)\big), V\big(x(b)\big) \leq E. \tag{4.6}$$

Then, $x \in H^1\big([a, b], \mathbb{R}^m\big)$, and if $V\big(x(a)\big) = V\big(x(b)\big) = E$, then $x(a) \neq x(b)$. Moreover, in the above situation, there exist positive constants c_x and T and a C^1 diffeomorphism $\sigma : [0, T] \to [a, b]$ such that:

$$\big(E - V(x)\big)g\big(\dot{x}, \dot{x}\big) \equiv c_x \quad on \ [a, b], \tag{4.7}$$

and, setting $q = x \circ \sigma : [0, T] \to \mathbb{R}^m$, and $p(s) = \mathcal{L}(q(s))\dot{q}(s)$, the pair $(q, p) : [0, T] \to \mathbb{R}^{2m}$ is a solution of (3.2) having energy E with $q(0) = x(a)$, $q(T) = x(b)$. If $V\big(x(a)\big) = V\big(x(b)\big) = E$ then q can be extended to a $2T$-periodic brake orbit of (3.2).

Proof. Proposition 4.1 is a classical result. For convenience of the reader we give however a sketch of the proof.

Since x satisfies (4.4), standard regularization arguments show that x is of class C^2 on $]a, b[$, while integration by parts gives (4.3) for all $s \in]a, b[$. Equation (4.7) follows by contracting both sides of (4.3) with $\dot{x}$ using g. Now set

$$t(s) = \frac{1}{2} \int_a^s \frac{c_x}{E - V(x(\tau))} \, \mathrm{d}\tau. \tag{4.8}$$

We claim that $T \equiv t(b) < +\infty$. Indeed, fix $s_0 \in]a, b[$ and consider for instance $t_0(s) = \frac{1}{2} \int_{s_0}^s \frac{c_x}{E - V(x(\tau))} \, \mathrm{d}\tau$. Denote by $\sigma_0(t)$ the inverse of t_o and consider $q(t) = x(\sigma_0(t))$.

Since $\sigma_0'(t) = 2(c_x)^{-1}\big(E - V(x(\sigma_0(t)))\big)$, a straightforward computation shows that $\frac{\mathrm{D}}{\mathrm{d}s}\dot{q} = -\nabla V(q)$ and $\frac{1}{2}g\big(\dot{q}, \dot{q}\big) + V(q) \equiv E$. Since $\nabla V \neq 0$ on $V^{-1}(E)$, if $V(x(b)) = E$, studying the second derivative of $E - V(q(t))$ shows that q arrives in $x(b) \in V^{-1}(E)$ in a finite amount of time. Then $t_0(b)$ is finite. Analogously we can study the behavior near a, proving that T is finite.

Now, denote by $\sigma : [0, T] \to [a, b]$ the inverse map of (4.8), and set $q(t) = x(\sigma(t))$. As already observed, a straightforward computation shows that $\frac{\mathrm{D}}{\mathrm{d}s}\dot{q} = -\nabla V(q)$ and $\frac{1}{2}g\big(\dot{q}, \dot{q}\big) + V(q) \equiv E$. Therefore, the pair $(q, \mathcal{L}(q)\dot{q}) : [0, T] \to \mathbb{R}^{2m}$ is a solution of (3.2) with energy E.

Moreover, $q(0) = x(a)$ and $q(T) = x(b)$, and by the uniqueness of the Cauchy problem, if $V(x(a)) = V(x(b)) = E$ it must be that $q(0) \neq q(T)$, and q can be extended to a periodic brake orbit. $\qquad \square$

4.3. **The Maupertuis–Jacobi Principle near a nondegenerate maximum of the potential energy.** The above formulation of the Maupertuis–Jacobi principle is not suited to studying homoclinic orbits issuing from a critical point of the potential function V. Our next goal is to establish an extension of the principle that will be applied in this situation.

Proposition 4.2. *Let (M, g) be a Riemannian manifold, $V \in C^2(M, \mathbb{R})$, let $x_0 \in M$ be a nondegenerate maximum of V, and set $E = V(x_0)$. Assume that x is a curve in the set $C^0([a, b], \overline{\Omega}_E) \bigcap H^1_{loc}([a, b), \overline{\Omega}_E)$ such that:*

$$\int_a^b \left(E - V(x)\right) g\left(\dot{x}, \tfrac{\mathrm{D}}{\mathrm{d}t} W\right) \mathrm{d}t - \frac{1}{2} \int_a^b g\left(\dot{x}, \dot{x}\right) g(\nabla V(x), W) \, \mathrm{d}t = 0 \qquad (4.9)$$

for all $W \in C_0^\infty\left((a, b), \mathbb{R}^m\right)$, and such that

$$V\left(x(s)\right) < E, \ \text{for } s \in [a, b); \qquad (4.10)$$

$$x(b) = x_0. \qquad (4.11)$$

Then, there exists a C^1 diffeomorphism $\sigma : [0, +\infty) \to [a, b)$ such that the curve $q = x \circ \sigma$ is a solution of (3.6) satisfying $q(0) = x(a)$ and $\lim_{t \to +\infty} q(t) = x_0$, $\lim_{t \to +\infty} \dot{q}(t) = 0$.

Proof. Choose $\varrho \in (0, \mathrm{dist}(x(a), x_0))$ and define $\alpha_1 \in \,]a, b[$ as the *first* instant s at which $\mathrm{dist}\left(x(s), x_0\right) = \varrho$. By (4.9), the restriction $x|_{[a, \alpha_1]}$ is a geodesic relative to the metric g_E, since $x\left([a, \alpha_1]\right)$ is contained in a region where $E - V$ is positive. Denote by c_x the constant value of $(E - V(x)) g(\dot{x}, \dot{x})$; for all $s \in [a, \alpha_1]$ set:

$$t(s) = \frac{1}{2} \int_a^s \frac{c_x}{E - V(x(\tau))} \, \mathrm{d}\tau$$

and denote by $\sigma : [0, t(\alpha_1)] \to [a, \alpha_1]$ the inverse function of $s \mapsto t(s)$. Then, a straightforward calculations shows that the map $q = x \circ \sigma$ is a solution of the equation (3.6) with $\frac{1}{2} g(\dot{q}, \dot{q}) + V(q) \equiv E$ on $[0, s(\alpha_1)]$.

Let $\alpha_2 \in (\alpha_1, b)$ be the *first* instant s at which $\mathrm{dist}(x(s), x_0) = \frac{\varrho}{2}$; we can repeat the construction above obtaining a solution q_* of (3.6) defined on an interval $[0, t(\alpha_2)]$. The key observation here is that, in fact, such a function q_* is an extension of q, and therefore it satisfies the same conservation law $\frac{1}{2} g(\dot{q}_*, \dot{q}_*) + V(q_*) \equiv E$ on $[0, t(\alpha_2)]$. An iteration of this construction produces a sequence $a < \alpha_1 < \alpha_2 < \ldots < b$ such that $\mathrm{dist}(x(\alpha_k), x_0) = \frac{\varrho}{2^{k-1}}$, maps of class C^1, $t : [a, L) \to [0, T)$, its inverse $\sigma : [0, T) \to [a, L)$, where:

$$T = \frac{1}{2} \int_a^L \frac{c_x}{E - V(x(\tau))} \, \mathrm{d}\tau \in \,]0, +\infty] \,, \quad L = \lim_{k \to \infty} \alpha_k \in \,]a, b] \,,$$

and a curve of class C^2, $q = x \circ \sigma : [0, T) \to \overline{\Omega}_E$, that satisfies (3.6), and with

$$\frac{1}{2} g(\dot{q}, \dot{q}) + V(q) \equiv E \qquad (4.12)$$

on $[0, T)$; in particular, $g(\dot{q}, \dot{q})$ is bounded.

Let us prove that $T = +\infty$ and that $\lim_{t \to +\infty} q(t) = x_0$. We know that, by construction, $\lim_{k \to \infty} t(\alpha_k) = T$ and $\lim_{k \to \infty} q(t(\alpha_k)) = x_0$; suppose by contradiction that there exists $\bar\rho > 0$, and a sequence β_k such that $\lim_{k \to \infty} \beta_k = L$ and $\mathrm{dist}(q(t(\beta_k)), x_0) \geq \bar\rho$ for all k. Since x_0 is an isolated maximum point, we can assume $\bar\rho$ is small enough so that

$$\inf_{\frac{1}{2}\bar\rho \leq \mathrm{dist}(Q,x_0) \leq \bar\rho} \big(E - V(Q)\big) \equiv \bar e > 0. \tag{4.13}$$

Up to subsequences, we can obviously assume that $\beta_k \in (\alpha_k, \alpha_{k+1}]$ for all k; for k sufficiently large, there exists $\gamma_k \in (\alpha_k, \beta_k)$ which is the first instant $t \in (\alpha_k, \beta_k)$ at which $\mathrm{dist}\big(q(s(t)), x_0\big) = \frac{\bar\rho}{2}$. Since $g(\dot q, \dot q)$ is bounded, there exists $\bar\nu > 0$ such that

$$t(\gamma_k) - t(\alpha_k) \geq \bar\nu, \quad \text{for all } k; \tag{4.14}$$

from (4.13) and (4.14) we get:

$$\int_0^{t(\alpha_{N+1})} \big(E - V(q(\tau))\big)\, \mathrm{d}\tau \geq \sum_{k=1}^N \int_{t(\alpha_k)}^{t(\gamma_k)} \big(E - V(q(\tau))\big)\, \mathrm{d}\tau \geq \sum_{k=1}^N \bar e \bar\nu = N \bar e \bar\nu \longrightarrow +\infty \tag{4.15}$$

as $N \to \infty$. On the other hand, for all $s \in\,]a, L[$,

$$\int_0^{t(s)} \big(E - V(q(\tau))\big)\, \mathrm{d}\tau = \frac{1}{2} \int_a^s c_x\, \mathrm{d}\theta = \frac{(b-a)}{2} c_x,$$

which is obviously inconsistent with (4.15), and therefore proves that

$$\lim_{t \to T^-} q(t) = x_0.$$

Moreover, the conservation law (4.12) implies that $\lim_{t \to T^-} \dot q(t) = 0$.

Finally, the local uniqueness of the solution of an initial-value problem implies immediately that T cannot be finite; for, the only solution q of (3.6) satisfying $q(T) = x_0$ and $\dot q(T) = 0$ is the constant $q \equiv x_0$. $\qquad\square$

5. Orthogonal Geodesic Chords and the Maupertuis Integral

In this section we will prove the main result of the paper, showing how to reduce the brake orbit and the homoclinics multiplicity problem to a multiplicity result for orthogonal geodesic chords.

We will begin with the study of the Jacobi metric near the level surface $V^{-1}(E)$, with E a regular value of V.

5.1. **The Jacobi distance near a regular value of the potential.** Let g be a Riemannian metric, $g_E = (E - V(x))g$, Ω_E as in (3.4); assume $\nabla V(x) \neq 0$ for all $x \in V^{-1}(E)$ and that $\overline{\Omega}_E$ is compact.

Lemma 5.1. *For all $Q \in \Omega_E$, the infimum:*

$$d_E(Q) := \inf \Big\{ \int_0^1 \big((E - V(x))g(\dot{x}, \dot{x})\big)^{1/2} \, dt : x \in H^1\big([0,1], \overline{\Omega}_E\big),$$
$$x(0) = Q, \ x(1) \in \partial\Omega \Big\}$$

is attained on at least one curve $\gamma_Q \in H^1\big([0,1], \overline{\Omega}_E\big)$ such that

$$(E - V(\gamma_Q))g(\dot{\gamma}_Q, \dot{\gamma}_Q)$$

is constant, $\gamma_Q\big([0,1)\big) \subset \Omega$, and γ_Q is a C^2 curve on $[0,1)$. Moreover, such a curve satisfies assumption (4.4) *of Proposition* 4.1 *on the interval $[a, b] = [0, 1]$.*

Proof. For all $k \in \mathbb{N}$ sufficiently large, set $\Omega_k = V^{-1}\big((-\infty, E - \frac{1}{k})\big) \subset \Omega_E$, and consider the problem of minimization of the g_E-length functional:

$$L_E(x) = \int_0^1 \big[(E - V(x))g(\dot{x}, \dot{x})\big]^{\frac{1}{2}} \, ds,$$

in the space $\mathfrak{G}_k$ consisting of curves $x \in H^1\big([0,1], \overline{\Omega}_k\big)$ with $x(0) = Q$ and $x(1) \in \partial\Omega_k$.

It is not hard to prove, by standard arguments, that for all $\Omega_k \neq \emptyset$, the above problem has a solution γ_k which is a g_E geodesic, and with $\gamma_k\big([0,1[\big) \subset \Omega_k$.

Set $q_k = \gamma_k(1) \in \partial\Omega_k$ and $l_k = L_E(\gamma_k)$. Since q_k approaches $\partial\Omega$ as $k \to \infty$, arguing by contradiction we get:

$$\liminf_{k \to \infty} l_k \geq d_E(Q).$$

Now, if by contradiction it was:

$$\liminf_{k \to \infty} l_k > d_E(Q),$$

then we could find a curve $x \in H^1\big([0,1], \overline{\Omega}\big)$ with $x(0) = Q$, $x(1) \in \partial\Omega$, and with $L_E(x) < \liminf_{k \to \infty} l_k$. Then, a suitable reparameterization of x would yield a curve $y \in \mathfrak{G}_k$ with $L_E(y) < l_k$, which contradicts the minimality of l_k and proves that

$$\liminf_{k \to \infty} l_k = d_E(Q). \tag{5.1}$$

Now, arguing as in the proof of Proposition 4.1, we see that the sequence:

$$\int_0^1 \frac{\mathrm{d}t}{E - V(\gamma_k(t))} \tag{5.2}$$

is bounded. Now, $\int_0^1 \big(E - V(\gamma_k)\big)g\big(\dot\gamma_k, \dot\gamma_k\big)\,\mathrm{d}\tau = l_k^2 \equiv \big(E - V(\gamma_k)\big)g\big(\dot\gamma_k, \dot\gamma_k\big)$ is bounded, which implies $\int_0^1 g\big(\dot\gamma_k, \dot\gamma_k\big)\,\mathrm{d}\tau$ is bounded, namely the sequence γ_k is bounded in $H^1\big([0,1], \overline\Omega_E\big)$. Up to subsequences, we have a curve $\gamma_Q \in H^1\big([0,1], \overline\Omega_E\big)$ which is an H^1-weak limit of the γ_k's; in particular, γ_k is uniformly convergent to γ_Q.

We claim that such a curve γ_Q satisfies the required properties. First, $\gamma_Q([0,1)) \subset \Omega_E$. Otherwise, if $b < 1$ is the first instant where $\gamma_Q(b) \in \partial\Omega_E$, by (5.1) and the conservation law of the energy for γ_k one should have

$$(b-1)l_k^2 = \int_b^1 \big(E - V(\gamma_k)\big)g\big(\dot\gamma_k, \dot\gamma_k\big)\,\mathrm{d}\tau \longrightarrow 0,$$

in contradiction with $Q \notin \partial\Omega_E$. Then γ_Q satisfies (4.4) in $[0,1]$ since it is an H^1–weak limit of γ_k, which is a sequence of g_E geodesics.

Clearly, γ_Q is of class C^2 on $[0,1)$, because the convergence on each interval $[0,b]$ is indeed smooth for all $b < 1$.

Finally, since $L_E(z) \le \liminf_{k\to\infty} l_k$, from (5.1) it follows that $L_E(\gamma_Q) = d_E(Q)$, and this concludes the proof. $\qquad\square$

Remark 5.2. It is immediate to see that γ_Q is a minimizer as in Lemma 5.1 if and only if is a minimizer for the functional

$$f_{0,1}(x) = \frac{1}{2} \int_0^1 \big(E - V(x)\big)g\big(\dot x, \dot x\big)\,\mathrm{d}t \tag{5.3}$$

in the space of curves

$$X_Q = \{x \in H^1([0,1], \overline\Omega_E) \,:\, x(0) = Q, x([0,1[) \subset \Omega_E, x(1) \in \partial\Omega_E\}. \tag{5.4}$$

Then, by Lemma 5.1, $f_{0,1}$ has at least one minimizer on X_Q.

Using a simple argument, we also have:

Lemma 5.3. *The map* $d_E : \Omega_E \to [0, +\infty)$ *defined in the statement of Lemma* 5.1 *is continuous, and it admits a continuous extension to* $\overline\Omega_E$ *by setting* $d_E = 0$ *on* $\partial\Omega_E$.

Now we shall study the map

$$\psi(y) = \frac{1}{2}d_E^2(y), \tag{5.5}$$

proving that it is C^2 and satisfies a convex condition when y is nearby $\partial\Omega_E$.

Proposition 5.4. *If Q is sufficiently close to $\partial\Omega_E$, then the minimizer of the functional* (5.3) *in the space X_Q is unique.*

Proof. Let $z = z(t, 0, Q)$ be the solution of the Cauchy problem

$$\begin{cases} \dot{z}(t) = J \cdot D_z H(z(t)) \\ z(0) = (0, Q), \quad Q \in \partial\Omega_E, \end{cases} \tag{5.6}$$

where H is the Hamiltonian function (3.1), and J is the matrix

$$J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$$

and I_m is the $m \times m$ identity matrix. Since V and a_{ij} are C^2, $z = (p, q)$ is of class C^1 with respect to (t, Q), therefore $\dot{z} = \dot{z}(t, Q)$ is of class C^1 with respect to (t, Q) so $\dot{q} = \dot{q}(t, Q)$ is C^1. Since $\dot{q} = \dot{q}(0, Q) = 0$, in a neighborhood of a fixed point $Q_0 \in \partial\Omega_E$ we have

$$\dot{q}(t, Q) = t\ddot{q}(0, Q_0) + \varphi(t, Q) = -t\nabla V(Q_0) + \varphi(t, Q) \tag{5.7}$$

where φ is of class C^1 and $\mathrm{d}\varphi(0, Q_0) = 0$. Moreover,

$$q(t, Q) = Q - \frac{t^2}{2}\nabla V(Q_0) + \varphi_0(t, Q) \tag{5.8}$$

where $\varphi_0(t, Q) = \int_0^t \varphi(s, Q)\,\mathrm{d}s$. Then, if $\{y_1, \ldots, y_{m-1}\}$ is a coordinate system of $V^{-1}(E)$ in a neighborhood of Q_0, by (5.8) we deduce that, setting $\tau = t^2$, the set $\{y_1, \ldots, y_{m-1}, \tau\}$ is a local coordinate system on the manifold with boundary $\partial\Omega_E$ and $(\tau, Q) \mapsto q(\tau, Q)$ defines a local chart.

Then, due to the compactness of $\partial\Omega_E$, and denoting by $\mathrm{dist}(\cdot, \cdot)$ the distance induced by g, there exists $\bar{\rho} > 0$ having the following property:

$\forall y \in \Omega_E$ with $\mathrm{dist}(y, \partial\Omega_E) \leq \bar{\rho}$ *there exists a unique solution* (p_y, q_y)

of (3.2) *with energy E, and a unique $t_y > 0$*

such that $q_y(0) \in \partial\Omega_E$, $q_y(t_y) = y$. \hfill (5.9)

Then, by Proposition 4.1, for all $y \in \Omega_E$ with $\mathrm{dist}(y, \partial\Omega_E) \leq \bar{\rho}$ there exists a unique minimizer γ_y for $f_{0,1}$ on X_y. \hfill $\square$

Remark 5.5. Note that $q_y(t) = q(t, Q_y)$, where Q_y is implicitly defined by $q(t_y, Q_y) = y$. By the variable change used in Proposition 4.1, it turns out that

$$q(t, Q_y) = \gamma_y(1 - \sigma), \quad \text{where } t(\sigma) = \psi(y) \int_0^\sigma \frac{1}{E - V(\gamma_y(\tau))}\,\mathrm{d}\tau. \tag{5.10}$$

In particular, since $\sigma = \sigma(t)$ is the inverse of $t(\sigma)$ we have

$$\psi(y)\dot{q}(t_y, Q_y) = -(E - V(y))\dot{\gamma}_y(0). \qquad (5.11)$$

Note also that $t_y = \sqrt{\tau_y}$ is of class C^1 when $\tau_y > 0$ since (τ, Q) is a local coordinate system.

In the following result we are assuming $\overline{\Omega}_E \subset \mathbb{R}^m$.

Proposition 5.6. *Let $\bar{\rho}$ satisfy property* (5.9). *Whenever $0 < \text{dist}(y, \partial\Omega_E)$ $\leq \bar{\rho}$, ψ is differentiable at y and*

$$d\psi(y)[\xi] = -\big(E - V(y)\big)g\big(\dot{\gamma}_y(0), \xi\big) \qquad \forall \xi \in \mathbb{R}^m. \qquad (5.12)$$

Proof. Given the local nature of the result, it will not be restrictive to assume that M is topologically embedded as an open subset of $\mathbb{R}^m$. Consider

$$v_\xi(s) = (1 - 2s)^+\xi,$$

where $(\cdot)^+$ denotes the positive part. For ε sufficiently small (with respect to ξ) the curve $\gamma_y(s) + \varepsilon v_\xi(s)$ belongs to $X_{y+\varepsilon\xi}$ (see (5.4)). Then, by the definition of ψ as minimum value,

$$\psi(y + \varepsilon\xi) \leq f_{0,1}(\gamma_y + \varepsilon v_\xi)$$

and therefore

$$\psi(y + \varepsilon\xi) - \psi(y) \leq f_{0,1}(\gamma_y + \varepsilon v_\xi) - f_{0,1}(\gamma_y).$$

Now

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(f_{0,1}(\gamma_y + \varepsilon v_\xi) - f_{0,1}(\gamma_y)\right) =$$

$$\int_0^1 \big(E - V(\gamma_y)\big)g\big(\dot{\gamma}_y, \tfrac{\mathrm{D}}{\mathrm{d}t}v_\xi\big) - \frac{1}{2}g\big(\nabla V(\gamma_y), v_\xi\big)g\big(\dot{\gamma}_y, \dot{\gamma}_y\big)\,\mathrm{d}s$$

uniformly as $|\xi| \leq 1$. Moreover, since $v_\xi = 0$ in the interval $[\frac{1}{2}, 1]$, using the differential equation satisfied by γ_y and integrating by parts gives

$$\int_0^1 \big(E - V(\gamma_y)\big)g\big(\dot{\gamma}_y, \tfrac{\mathrm{D}}{\mathrm{d}t}v_\xi\big) - \frac{1}{2}g\big(\nabla V(\gamma_y), v_\xi\big)g\big(\dot{\gamma}_y, \dot{\gamma}_y\big)\,\mathrm{d}s =$$

$$- \big(E - V(\gamma_y(0))\big)g\big(\dot{\gamma}_y(0), v_\xi(0)\big) = -\big(E - V(y)\big)g\big(\dot{\gamma}_y(0), \xi\big).$$

Therefore, uniformly as $|\xi| \leq 1$,

$$\limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\left(\psi(y + \varepsilon v_\xi) - \psi(y)\right) + \big(E - V(y)\big)g\big(\dot{\gamma}_y(0), \xi\big) \leq 0. \qquad (5.13)$$

Moreover, since $\psi(y+\varepsilon\xi) = f_{0,1}(\gamma_{y+\varepsilon\xi})$ and $\psi(y) \leq f_{0,1}(\gamma_{y+\varepsilon\xi} - \varepsilon v_\xi)$ one has

$$\psi(y + \varepsilon\xi) - \psi(y) \geq f_{0,1}(\gamma_{y+\varepsilon\xi}) - f_{0,1}(\gamma_{y+\varepsilon\xi} - \varepsilon v_\xi) =$$

$$\varepsilon\langle f'_{0,1}(\gamma_{y+\epsilon\xi}), v_\xi\rangle_1 - \frac{\varepsilon^2}{2}\langle f''_{0,1}(\gamma_{y+\varepsilon\xi} - \vartheta_\varepsilon\varepsilon v_\xi)[v_\xi], v_\xi\rangle_1, \quad (5.14)$$

for some $\vartheta_\varepsilon \in (0,1)$. Here $\langle\cdot,\cdot\rangle_1$ denotes the standard scalar product in H^1 and f', f'' are respectively gradient and Hessian with respect to $\langle\cdot,\cdot\rangle_1$.

Now, we have $\gamma_{y+\varepsilon\xi}(0) = y + \varepsilon_\xi$ and $y \notin V^{-1}(E)$. Moreover, by the uniqueness of the minimizer it is not difficult to prove that, for all $\delta > 0$ there exists $\varepsilon(\delta) > 0$ such that

$$\text{dist}(\gamma_{y+\varepsilon\xi}(s), \gamma_y(s)) \leq \delta \qquad \text{for any } \varepsilon \in (0, \varepsilon(\delta)], |\xi| \leq 1, s \in [0,1].$$

Then, since γ_y is uniformly far from $V^{-1}(E)$ on the interval $[0, \frac{1}{2}]$, the same holds for $\gamma_{y+\varepsilon\xi}$ whenever ε is small and $|\xi| \leq 1$. Thus, recalling the definition of d_E in Lemma 5.1, the conservation law satisfied by the minimizer $\gamma_{y+\varepsilon\xi}$ is

$$\big(E - V(\gamma_{y+\varepsilon\xi})\big)g\big(\dot\gamma_{y+\varepsilon\xi}, \dot\gamma_{y+\varepsilon\xi}\big) = d_E^2(y + \varepsilon\xi).$$

This implies the existence of a constant $C > 0$ such that

$$\int_0^{1/2} g\big(\dot\gamma_{y+\varepsilon\xi}, \dot\gamma_{y+\varepsilon\xi}\big)\,\mathrm{d}s \leq C$$

for any ε small and $|\xi| \leq 1$.

Therefore $\langle f''_{0,1}(\gamma_{y+\varepsilon\xi} - \vartheta_\varepsilon\varepsilon v_\xi)[v_\xi], v_\xi\rangle_1$ is uniformly bounded with respect to ε small and $|\xi| \leq 1$, due to $v_\xi = 0$ on $[\frac{1}{2}, 1]$, and by (5.14) we get

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big(f_{0,1}(\gamma_{y+\varepsilon\xi}) - f_{0,1}(\gamma_{y+\varepsilon\xi-\varepsilon v_\xi})\big) = \lim_{\varepsilon\to 0}\langle f'_{0,1}(\gamma_{y+\varepsilon\xi}), v_\xi\rangle_1 \qquad (5.15)$$

uniformly as $|\xi| \leq 1$.

Now, using the differential equation (4.3) satisfied by $\gamma_{y+\varepsilon\xi}$ and integrating by parts one obtains

$$\langle f'_{0,1}(\gamma_{y+\varepsilon\xi}), v_\xi\rangle_1 = -\big(E - V(y + \varepsilon\xi)\big)g\big(\dot\gamma_{y+\varepsilon\xi}(0), \xi\big),$$

while by (5.11) and the continuity of $\dot q(t_y, Q_y)$ and $\psi(y)$ we have

$$\lim_{\varepsilon\to 0}\big(E - V(y + \varepsilon\xi)\big)\dot\gamma_{y+\varepsilon\xi}(0) = \big(E - V(y)\big)\dot\gamma_y(0) \qquad (5.16)$$

uniformly as $|\xi| \leq 1$. Therefore, by (5.14)–(5.16) it is

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\big(\psi(y + \varepsilon\xi) - \psi(y)\big) + \big(E - V(y)\big)g\big(\dot\gamma_y(0), \xi\big) \geq 0 \qquad (5.17)$$

uniformly as $|\xi| \leq 1$. Finally, combining (5.13) and (5.17) one has (5.12). $\quad\square$

Remark 5.7. By (5.11) we deduce that $(E - V(y))\dot\gamma_y(0)$ is continuous, therefore by (5.12), ψ is of class C^1. Again by (5.11) and the C^1 regularity of $\dot q_y(t_y, Q_y)$ we deduce that $(E - V(y))\dot\gamma_y(0)$ is of class C^1 whenever $y \notin V^{-1}(E)$, and by (5.12) it turns out that ψ is of class C^2.

In the following proposition we will show that ψ satisfies a strongly convex assumption near $V^{-1}(E)$.

Proposition 5.8. *There exists $\widehat\rho \leq \bar\rho$ with the property that, for any $y \in \Omega_E$ such that $0 < \mathrm{dist}(y, V^{-1}(E)) \leq \widehat\rho$ the Hessian (with respect to the Jacobi metric g_E) of Ψ at y satisfies*

$$\mathrm{H}^\psi(y)[v, v] > 0 \qquad \forall v \, : \, d\psi(y)[v] = 0, \quad v \neq 0. \tag{5.18}$$

Proof. Recall that

$$\mathrm{H}^\psi(y)[v, v] = \frac{\partial^2}{\partial s^2} \left(\psi(\eta(s))\right)_{|s=0},$$

where $\eta(s)$ is a geodesic with respect to the Jacobi metric g_E, namely a solution of the differential equation (4.3) satisfying the initial data conditions

$$\eta(0) = y, \qquad \dot\eta(0) = \xi.$$

Now, by (5.11) and (5.12)

$$d\psi(\eta(s))[\dot\eta(s)] = -\left(E - V(\eta(s))\right)g\left(\dot\gamma_{\eta(s)}(0), \dot\eta(s)\right)$$
$$= \psi(\eta(s))g\left(\dot q(t_{\eta(s)}, Q_{\eta(s)}), \dot\eta(s)\right).$$

Since $\lim_{s\to 0} Q_{\eta(s)} = Q_y$, using (5.7) we can write

$$\dot q(t, Q_{\eta(s)}) = -t\nabla V(y) + \varphi(t, Q_{\eta(s)})$$

as $d\varphi(0, Q_y) = 0$, and

$$\frac{\partial^2}{\partial s^2}\left(\psi(\eta(s))\right) = \psi(\eta(s))\left(g\left(\dot q(t_{\eta(s)}, Q_{\eta(s)}), \dot\eta(s)\right)\right)^2$$
$$+ \psi(\eta(s))g\left(\dot q(t_{\eta(s)}, Q_{\eta(s)}), \tfrac{\mathrm{D}}{\mathrm{ds}}\dot\eta(s)\right) + \psi(\eta(s))g\left(-\mathrm{d}t_{\eta(s)}[\dot\eta(s)]\nabla V(y)\right.$$
$$+ \frac{\partial\varphi}{\partial t}(t_y, Q_{\eta(s)})\mathrm{d}t_{\eta(s)}[\dot\eta(s)] + \frac{\partial\varphi}{\partial Q}\frac{\partial Q}{\partial\eta}[\dot\eta(s)], \dot\eta(s)\Big).$$

Since $\eta(s)$ satisfies (4.3) and $d\varphi(0, Q_y) = 0$, it suffices to show that for any y sufficiently close to $\partial\Omega$,

$$\psi(\eta(s))\left(g\left(\dot q(t_y, Q_y), v\right)\right)^2 + \psi(y)\mathrm{d}t_y[v]g\left(-\nabla V(y), v\right)$$
$$+ \frac{\psi(y)}{E - V(y)}\left(g(\nabla V(y), v)g(\dot q(t_y, Q_y), v) - \frac{1}{2}g(\dot q(t_y, Q_y), \nabla V(y))g(v, v)\right) > 0$$

for any v such that $\mathrm{d}\psi(y)[v] = 0$. This means that $g\big(\dot{q}(t_y, Q_y), v\big) = 0$ so it will suffice to show

$$\sup_{|v|=1} |\mathrm{d}t_y[v]| g\big(\nabla V(y), \nabla V(y)\big)^{1/2} - \frac{1}{2(E - V(y))} g\big(\dot{q}(t_y, Q_y), \nabla V(y)\big) > 0$$
(5.19)

for any y close to $V^{-1}(E)$. Since $q(t_y, Q_y) = y$ we get

$$\mathrm{d}t_y[v]\dot{q}(t_y, Q_y) + \frac{\partial q}{\partial Q} \frac{\partial Q_y}{\partial y}[v] = v.$$

Moreover, $\frac{\partial q}{\partial Q}(t_y, Q_y)$ goes to the identity map as y tends to $\partial\Omega$, while $\frac{\partial Q_y}{\partial y}[v]$ tends to v uniformly as $|v| \leq 1$, since $(0, Q)$ is a coordinate system for $V^{-1}(E)$. Then, as $y \to V^{-1}(E)$, $\mathrm{d}t_y[v]\dot{q}(t_y, Q_y) \to 0$ uniformly in v.

Note that $\frac{1}{2}g(\dot{q}, \dot{q}) = E - V(q)$, therefore

$$g\big(\dot{q}(t_y, Q_y), \dot{q}(t_y, Q_y)\big) = 2\big(E - V(y)\big)$$
(5.20)

so

$$\lim_{y \to \partial\Omega} \sqrt{E - V(y)}\, |\mathrm{d}t_y[v]| = 0$$
(5.21)

uniformly in $|v| \leq 1$.

Finally, by (5.7) we have

$$\lim_{y \to V^{-1}(E)} g\Big(\frac{\dot{q}(t_y, Q_y)}{\sqrt{g\big(\dot{q}(t_y, Q_y), \dot{q}(t_y, Q_y)\big)}}, \frac{\nabla V(y)}{\sqrt{g\big(\nabla V(y), \nabla V(y)\big)}}\Big) = -1$$

therefore by (5.20)

$$\lim_{y \to V^{-1}(E)} \inf \frac{-g\big(\dot{q}(t_y, Q_y), \nabla V(y)\big)}{\sqrt{E - V(y)}} > 0$$
(5.22)

and combining (5.21) with (5.22) one obtains (5.19) and the proof is complete. $\square$

By Proposition 5.6, Remark 5.7, and Proposition 5.8 one immediately obtains the following proposition, which is the main result of the section:

Theorem 5.9. *Let E be a regular value for $V(x)$, and let $d_E : \Omega \to [0, +\infty)$ be the map defined in the statement of Lemma 5.1, and assume that $\overline{\Omega}_E$ is compact. There exists a positive number δ_* such that, setting:*

$$\Omega_* = \big\{x \in \Omega_E \,:\, d_E(x) > \delta_*\big\},$$

the following statements hold:

  (1) $\partial\Omega_*$ *is of class C^2;*

(2) $\overline{\Omega}_*$ is homeomorphic to $\overline{\Omega}_E$;

(3) $\overline{\Omega}_*$ is strongly *concave relative to the Jacobi metric g_E;*

(4) *if $x : [0,1] \to \overline{\Omega}_*$ is an orthogonal geodesic chord in $\overline{\Omega}_*$ relative to the Jacobi metric g_E, then there exists $[\alpha,\beta] \supset [0,1]$ and a unique extension $\widehat{x} : [\alpha,\beta] \to \overline{\Omega}$ of x with $\widehat{x} \in H^1([\alpha,\beta],\overline{\Omega})$ satisfying:*
   - *assumption (4.4) of Proposition 4.1 on the interval $[\alpha,\beta]$;*
   - $\widehat{x}(s) \in d_E^{-1}\big((-\delta_*,0)\big)$ *for all $s \in (\alpha,0) \bigcup (1,\beta)$;*
   - $V\big(\widehat{x}(\alpha)\big) = V\big(\widehat{x}(\beta)\big) = E.$

Remark 5.10. Theorem 5.9 tells us that the study of multiple brake orbits can be reduced to the study of multiple orthogonal geodesic chords in a Riemannian manifold with regular and strongly concave boundary.

5.2. **The Jacobi distance near a nondegenerate maximum point of the potential.** Let us now assume that $x_0 \in M$ is a nondegenerate maximum point of V, with $V(x_0) = E$, and let us make the following assumptions:
   - $V^{-1}\big(]-\infty, E]\big)$ is compact;
   - $V^{-1}(E) \setminus \{x_0\}$ is a regular embedded hypersurface of M.

We will show how to get rid of the singularity of the Jacobi metric at x_0, while the singularity on $V^{-1}(E) \setminus \{x_0\}$ can be removed as in the case of brake orbits, using Theorem 5.9.

First, we need a preparatory result. Let $\delta > 0$ be fixed in such a way that the set:
$$\Big\{p \in M : V(p) > E - \delta\Big\}$$
has precisely two connected components; let Ω_δ denote the connected component of the point x_0.

Lemma 5.11. *Let $Q \in \Omega_\delta \setminus \{x_0\}$ be fixed; then, the infimum:*

$$d_E(Q) := \inf \Big\{ \Big[\int_0^1 (E - V(x))g(\dot{x}, \dot{x})\, dt \Big]^{1/2} :$$

$$x \in C^0\big([0,1], \overline{\Omega_\delta}\big) \cap H^1_{loc}\big([0,1[, \overline{\Omega_\delta}\big),\ x(0) = Q,\ x(1) = x_0 \Big\} \quad (5.23)$$

is attained on some curve γ_Q with the property that $(E - V(\gamma_Q))g(\dot{\gamma}_Q, \dot{\gamma}_Q)$ is constant and $\gamma_Q([0,1[) \subset \overline{\Omega_\delta} \setminus \{x_0\}$. Moreover

$$\lim_{Q \to x_0} d_E(Q) = 0, \quad\quad\quad\quad\quad (5.24)$$

$$\lim_{Q \to x_0} \Big[\sup_{s \in [0,1]} \text{dist}\big(\gamma_Q(s), x_0\big) \Big] = 0. \quad\quad (5.25)$$

In particular, for Q sufficiently close to x_0,

$$\gamma_Q\big([0,1]\big) \subset \Omega_\delta, \tag{5.26}$$

so it is of class C^2 and satisfies assumption (4.9) of Proposition 4.2 on the interval $[a,b] = [0,1]$.

Proof. Let $x_n \in C^0\big([0,1], \overline{\Omega_\delta}\big) \cap H^1\big([0,1), \overline{\Omega_\delta}\big)$ be a minimizing sequence for the length functional $\int_0^1 [(E - V(x))g(\dot{x}, \dot{x})]^{1/2}\, dt$, leaving $(E - V(x))g(\dot{x}, \dot{x})$ constant. Choose $\rho > 0$ such that $\text{dist}(Q, x_0) > \rho$ and, for all $n \in \mathbb{N}$, define $\alpha_1^n \in (0,1)$ to be the first instant s such that $\text{dist}\big(x_n(s), x_0\big) = \rho$.

The sequence α_1^n stays away from 0 and 1, because for all intervals $I \subset x_n^{-1}([\frac{\rho}{2}, \rho])$ the integral $\int_I g(\dot{x}_n, \dot{x}_n)\, ds$ is bounded. We can therefore find a subsequence $\alpha_1^{n_k}$ converging to $\alpha_1 \in (0,1)$.

Furthermore, since $\int_0^{\alpha_1} g(\dot{x}_n, \dot{x}_n)\, ds$ is bounded, taking a subsequences x_n^1 we can assume that x_n^1 is H^1-weakly and uniformly convergent to some $x_1 \in H^1\big([0, \alpha_1], \overline{\Omega_\delta}\big)$; then, $\text{dist}\big(x(\alpha_1), x_0\big) = \rho$. Repeating the construction, we can find $\alpha_2 \in (\alpha_1, 1)$ and a subsequence x_n^2 of x_n^1 which is H^1-weakly and uniformly convergent to a curve $x_2 \in H^1\big([0, \alpha_2], \overline{\Omega_\delta}\big)$ with $\text{dist}\big(x(\alpha_2), x_0\big) = \frac{\rho}{2}$ and $x_{2|[0,\alpha_1]} = x_1$. Iteration of this construction yields a weak-H^1 limit of x_n^n, which is a curve $x \in H^1_{\text{loc}}\big([0, \bar{\alpha}), \overline{\Omega_\delta}\big)$, where $\bar{\alpha} = \lim_k \alpha_k$, and $\text{dist}\big(x(\alpha_k), x_0\big) = \frac{\rho}{2^k}$. Now, for all $k \geq 1$:

$$\int_0^{\alpha_k} \big((E - V(x))g(\dot{x}, \dot{x})\big)^{1/2}\, ds \leq \liminf_{n \to \infty} \int_0^{\alpha_k} \big((E - V(x_n))g(\dot{x}_n, \dot{x}_n)\big)^{1/2}\, ds$$

$$\leq \liminf_{n \to \infty} \int_0^1 \big((E - V(x_n))g(\dot{x}_n, \dot{x}_n)\big)^{1/2}\, ds = d_E(Q),$$

hence,

$$\int_0^{\bar{\alpha}} \big((E - V(x))g(\dot{x}, \dot{x})\big)^{\frac{1}{2}}\, ds = \lim_{k \to \infty} \int_0^{\alpha_k} \big((E - V(x))g(\dot{x}, \dot{x})\big)^{\frac{1}{2}}\, ds \leq d_E(Q),$$

and we can assume, as usual, $(E - V(x))g(\dot{x}, \dot{x})$ is constant (and positive since $Q \neq x_0$). The curve x can be extended continuously to $\overline{\alpha}$ by setting $x(\overline{\alpha}) = x_0$. Indeed, if by contradiction there exists a sequence $\beta_n < \alpha_n < \overline{\alpha}$ such that $\lim_k \beta_k = \overline{\alpha}$ and a positive number $\overline{\nu}$ such that $\text{dist}(x(\beta_k), x_0) \geq \overline{\nu}$, there exists $\beta_k^1 \in]\beta_k, \alpha_k[$ such that $\text{dist}(x(\beta_k^1), x_0) = \frac{\overline{\nu}}{2}$ and $\text{dist}(x(s), x_0) \geq \frac{\overline{\nu}}{2}$, for all $s \in [\beta_k^1, \beta_k]$. But $E - V(x(s))$ is far from zero in $[\beta_k^1, \beta_k]$ therefore

$g(\dot{x}, \dot{x}) \leq K \in \mathbb{R}^+$ on $[\beta_k^1, \beta_k]$ for some K, and then

$$\frac{\overline{\nu}}{2} \leq \text{dist}(x(\beta_k^1), x(\beta_k)) \leq \int_{\beta_k^1}^{\beta_k} g(\dot{x}, \dot{x}) \, dt \leq K(\beta_k - \beta_k^1) \longrightarrow 0$$

which is a contradiction.

Clearly, up to reparameterizations on x we can assume $\overline{\alpha} = 1$ and $x([0, 1)) \subset \overline{\Omega_\delta} \setminus \{x_0\}$. Taking $\gamma_Q = x$ we have the existence of a minimizer satisfying the conservation law $(E - V(\gamma_Q))g(\dot{\gamma}_Q, \dot{\gamma}_Q)$ constant.

Now, taking a chord C_Q joining Q and x_0 we have that $l(C_Q) \to 0$ as $Q \to x_0$, and since $d_E(Q) \leq l(C_Q)$ we obtain (5.24).

Moreover, if by contradiction (5.25) does not hold for any Q sufficiently close to x_0, there exists s_Q such that

$$\text{dist}(\gamma_Q(s_Q), x_0) \geq \overline{\nu} > 0.$$

Let $t_Q > s_Q$ be such that $\text{dist}(\gamma_Q(t_Q), x_0) = \frac{\overline{\nu}}{2}$ and $\text{dist}(\gamma_Q(s), x_0) \geq \frac{\overline{\nu}}{2}$ for all $s \in [s_Q, t_Q]$. Since $g(\dot{\gamma}_Q, \dot{\gamma}_Q)$ is bounded in $[s_Q, t_Q]$ we must have $t_Q - s_Q$ far from zero as $Q \to x_0$. But also $E - V(\gamma_Q)$ and $g(\dot{\gamma}_Q, \dot{\gamma}_Q)$ are far from zero in $[s_Q, t_Q]$ so we deduce that

$$\int_{s_Q}^{t_Q} \Big(\int_0^1 (E - V(x))g(\dot{\gamma}_Q, \dot{\gamma}_Q) \, dt \Big)^{1/2} \text{ is far from zero}$$

which is in contradiction with (5.24).

Note that (5.25) immediately implies (5.26) and since γ_Q is a minimizer satisfying $(E - V(\gamma_Q))g(\dot{\gamma}_Q, \dot{\gamma}_Q)$ constant, we immediately see that (4.9) is satisfied in the interval $[0, 1]$. $\qquad\square$

As for Lemma 5.3 a simple argument shows

Lemma 5.12. *The map $d_E : \Omega_\delta \to [0, +\infty)$ defined in the statement of Lemma 5.11 is continuous.*

For any y sufficiently close to x_0, let q_y be the reparameterization of γ_y given by Proposition 4.2. We have

$$\begin{cases} \frac{D}{ds}\dot{q}_y + \nabla Y(q_y) = 0, & q_y(0) = y, \\ \lim_{t \to +\infty} q_y(t) = x_0, & \lim_{t \to +\infty} \dot{q}_y(t) = 0. \end{cases} \tag{5.27}$$

The following estimate holds:

Proposition 5.13. *Let q_y be as above. Then there exist $\bar{\rho}$ and a constant $\alpha > 0$ such that*

$$\text{dist}(q_y(t), x_0) \leq \text{dist}(y, x_0)e^{-\alpha t} \tag{5.28}$$

for any y such that $\mathrm{dist}(y, x_0) \leq \bar{\rho}$.

To obtain the above result we need the following maximum principle in $\mathbb{R}$.

Lemma 5.14. *Let* $\varphi : [0, +\infty) \to \mathbb{R}$ *be a* C^2 *map with* $\lim_{t \to +\infty} \varphi(t) = 0$. *Let* $\nu > 0$ *such that* $\varphi''(t) \geq \nu \varphi(t)$, *for all* $t \geq 0$. *Then* $\varphi \leq \varphi(0) e^{-\sqrt{\nu} t}$.

Proof. Consider the map $\psi = \varphi - \varphi_0$ where $\varphi_0(t) = \varphi(0) e^{-\sqrt{\nu} t}$. Clearly $\psi(0) = \lim_{t \to +\infty} \psi(t) = 0$ and so ψ has a global maximum at some $\bar{t} \in [0, +\infty)$. If $\bar{t} > 0$, then $\psi(\bar{t}) \leq \frac{1}{\nu} \psi''(\bar{t}) \leq 0$. $\qquad \square$

Remark 5.15. Clearly, an analogous result as in the above Lemma 5.14 holds, reversing all inequalities.

Proof of Proposition 5.13. Let q be a solution of (5.27) (with $q(0) = y$), and let $\varphi(t) = \frac{1}{2} \mathrm{dist}(q(t), x_0)^2$. By (5.25) we can choose $\bar{\rho}$ sufficiently small so that

$$\mathrm{dist}(q(t), x_0) < \rho_0, \quad \text{for any } t \geq 0,$$

where ρ_0 is chosen so that the function $\mathfrak{d}(z) = \frac{1}{2} \mathrm{dist}(z, x_0)^2$, in the open ball $B(x_0, \rho_0)$ of center x_0 and radius ρ_0, it is of class C^2, strictly convex and, calling x_z the unique minimal geodesic with respect to g such that $x_z(0) = x_0$, $x_z(1) = z$ (see [3]), one has

$$\nabla \mathfrak{d}(z) = \dot{x}_z(1).$$

Now $\varphi'(t) = g\big(\nabla \mathfrak{d}(q(t)), \dot{q}(t)\big)$ and

$$\varphi''(t) = \mathrm{H}^{\mathfrak{d}}(q(t))[\dot{q}(t), \dot{q}(t)] + g\big(\nabla \mathfrak{d}(q(t)), \tfrac{\mathrm{D}}{\mathrm{d}t}\dot{q}(t)\big) \geq g\big(\nabla \mathfrak{d}(q(t)), \nabla V(q(t))\big).$$

Now, take z in $B(x_0, \rho_0)$, consider the minimal geodesic x_z as above, and define the map

$$\rho(s) := g\big(\nabla \mathfrak{d}(x_z(s)), -\nabla V(x_z(s))\big).$$

By the choice of x_z we have $\nabla \mathfrak{d}(x_z(s)) = s\, \dot{x}_z(s)$, so

$$\dot{\rho}(s) = g\big(\dot{x}_z(s), -\nabla V(x_z(s))\big) - s\, \mathrm{H}^V(x_z(s))[\dot{x}_z(s), \dot{x}_z(s)]$$
$$\geq g\big(\dot{x}_z(s), -\nabla V(x_z(s))\big) + s\nu g\big(\dot{x}_z(s), \dot{x}_z(s)\big)$$

for a suitable choice of ν (x_0 is a nondegenerate maximum point). Since $\rho(0) = 0$ then

$$g\big(\nabla \mathfrak{d}(z), -\nabla V(z)\big) = \varphi(1) = \int_0^1 \dot{\rho}(s)\, \mathrm{d}s$$
$$\geq \int_0^1 g\big(\dot{x}_z(s), -\nabla V(x_z(s))\big) + s\nu g\big(\dot{x}_z(s), \dot{x}_z(s)\big)\, \mathrm{d}s$$

$$= -V(x_z(s))\big|_{s=0}^{s=1} + \nu\mathrm{dist}(z,x_0)^2 \int_0^1 s\,\mathrm{d}s$$

$$= (E - V(z)) + \frac{\nu}{2}\mathrm{dist}(z,x_0)^2 \geq \frac{\nu}{2}\mathrm{dist}(z,x_0)^2,$$

where $V(x_0) = E$ has also been used. Therefore $\varphi''(t) \geq \frac{\nu}{2}\mathrm{dist}(q(t),x_0)^2 = \nu\,q(t)$, and by Lemma 5.14

$$\mathrm{dist}(q(t),x_0)^2 \leq \mathrm{dist}(q(0),x_0)e^{-\sqrt{\nu}t},$$

and (5.28) follows taking the square root of both members above. $\square$

The regularity of the distance function from x_0 with respect to the Jacobi metric is based on the following proposition.

Proposition 5.16. *For any y close to x_0 there exists a unique q_y satisfying (5.27). Moreover, the map*

$$q \longmapsto \dot{q}_y(0) \tag{5.29}$$

is of class C^1 and its differential satisfies $d\dot{q}_y(0)[v] = \dot{\xi}(0)$, where $\xi(t)$ is the unique solution of

$$\begin{cases} \frac{\mathrm{D}^2}{\mathrm{d}t^2}\xi(t) + R(\dot{q}_y,\xi(t))\dot{q}_y + \mathrm{L}^V(q_y)\xi(t) = 0, \quad \xi(0) = 0, \\ \lim_{t\to+\infty}\xi(t) = \lim_{t\to+\infty}\dot{\xi}(t) = 0 \end{cases} \tag{5.30}$$

where $\frac{\mathrm{D}^2}{\mathrm{d}t^2}\xi$ is the second covariant derivative and $R(\cdot,\cdot)$ the Riemann tensor with respect to g, and $\mathrm{L}^V(x)[v] \in T_xM$ is the vector defined through $g\big(\mathrm{L}^V(x)[v],w\big) = H^V(x)[v,w]$ for all $w \in T_xM$.

Proof. Consider the ball $B(x_0,\rho)$, with $\rho > 0$ small, and the spaces

$$X_2 = \{q \in C^2(\mathbb{R}^+, \overline{B(x_0,\rho)}) : \lim_{t\to+\infty}q(t) = x_0, \lim_{t\to+\infty}\dot{q}(t) = \lim_{t\to+\infty}\ddot{q}(t) = 0\}$$

with the norm (we can assume we are working in a local chart)

$$\|q_2 - q_1\| := \sup_{t\in\mathbb{R}^+}|q_2(t) - q_1(t)| + \sup_{t\in\mathbb{R}^+}|\dot{q}_2(t) - \dot{q}_1(t)| + \sup_{t\in\mathbb{R}^+}|\ddot{q}_2(t) - \ddot{q}_1(t)| \tag{5.31}$$

and

$$X_0 = \{q \in C^0(\mathbb{R}^+, \mathbb{R}^m) : \lim_{t\to+\infty}|q(t)| = 0\}$$

with the norm

$$\|q_2 - q_1\| := \sup_{t\in\mathbb{R}^+}|q_2(t) - q_1(t)|,$$

that are clearly Banach spaces. Now, consider the open set

$$A_2 = \{q \in X_2 \; : \; \sup_{t \in \mathbb{R}^+} \text{dist}(q(t), x_0) < \rho\} \subset X_2$$

and the map $F : A_2 \times B(x_0, \rho) \longrightarrow X_0 \times \mathbb{R}^m$ given by

$$F(q, y) = (\tfrac{\mathrm{D}}{\mathrm{d}t}\dot{q} + \nabla V(q), q(0) - y).$$

Thanks to the behaviour at infinity, we can use the same standard arguments exploited in finite intervals to prove that F is differentiable and (see [3])

$$\mathrm{d}F(q, y)[\xi, v] = \big(\tfrac{\mathrm{D}^2}{\mathrm{d}t^2}\xi + R(\dot{q}, \xi)\dot{q} + \mathrm{L}^V(q)[\xi], \xi(0) - v\big).$$

Moreover, thanks again to the behaviour at infinity, it is a straightforward check to verify that $\mathrm{d}F(q, y)$ is continuous (recall that g and V are of class C^2).

Now consider $\frac{\partial F}{\partial q}(x_0, 0)[\xi] = \big(\ddot{\xi} + \mathrm{L}^V(0)[\xi], \xi(0)\big)$ where x_0 denotes the constant curve with image x_0. We claim that

$$\frac{\partial F}{\partial q}(x_0, 0) : X_2 \longmapsto X_0 \times \mathbb{R}^m \tag{5.32}$$

is an isomorphism.

Recalling the definition of L^V, and since $\mathrm{H}^V(0)$ is symmetric and negative definite, using a base consisting of eigenvectors for $\mathrm{H}^V(0)$, it is sufficient to show that for any function $h \in C^0(\mathbb{R}^+, \mathbb{R})$ such that $\lim_{t \to +\infty} h(t) = 0$ and for any $\theta \in \mathbb{R}$, the solution of

$$\begin{cases} \ddot{x} - \alpha^2 x = h, \quad x(0) = \theta, \\ \lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} \dot{x}(t) = 0 \end{cases} \tag{5.33}$$

exists and is unique (where $x : \mathbb{R}^+ \to \mathbb{R}$).

The general solution of the differential equation above is

$$x(t) = \Big(a + \frac{1}{2\alpha}\int_0^t h(s)e^{-\alpha s}\,\mathrm{d}s\Big)e^{\alpha t} + \Big(b - \frac{1}{2\alpha}\int_0^t h(s)e^{\alpha s}\,\mathrm{d}s\Big)e^{-\alpha t}.$$

Since $\lim_{t \to +\infty} h(t) = 0$ we have

$$\lim_{t \to +\infty} \frac{1}{2\alpha}\Big(\int_0^t h(s)e^{\alpha s}\,\mathrm{d}s\Big)e^{-\alpha t} = 0;$$

then $\lim_{t \to +\infty} x(t) = 0$ only if we choose

$$a = -\frac{1}{2\alpha}\int_0^{+\infty} h(s)e^{-\alpha s}\,\mathrm{d}s.$$

With such a choice indeed $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \dot{x}(t) = 0$, while $x(0) = \theta$ for

$$b = \theta - a = \theta + \frac{1}{2\alpha} \int_0^{+\infty} h(s) e^{-\alpha s} \, ds,$$

proving that the solution of (5.33) exists and is unique, and therefore the map defined in (5.32) is an isomorphism.

Then, by the implicit function theorem and Proposition 5.13 we have the uniqueness of q_y for any y close to x_0 and its C^1 differentiability in X_2. In particular the map (5.29) is of class C^2. Denoting by ξ the differential $dq_y[v]$, and differentiating the expression $F(q_y, y) \equiv 0$, in particular we obtain that ξ solves (5.30). Since, as we have already seen, the solution exists and is unique for $y = x_0$, Proposition 5.13 ensures that this remains true for y close to x_0 also.

Finally, C^1 regularity of q_y with respect to the norm (5.31) immediately implies that

$$d\dot{q}_y(0)[v] = \dot{\xi}(0),$$

where ξ is the solution of (5.30), and then $dq_y[v](t) = \xi(t)$. $\qquad\square$

Now set

$$\psi(y) = \frac{1}{2} d_E(y)^2 \tag{5.34}$$

where l is the map defined in (5.23) of Lemma 5.11. Thanks to the above proposition we can repeat the proof of Proposition 5.6 to get its counterpart in the case of a nondegenerate maximum point.

Proposition 5.17. *There exists $\bar{\rho} > 0$ such that for any y with $\mathrm{dist}(y, x_0) \leq \bar{\rho}$ the map ψ defined in (5.34) is of class C^2 and its differential is given by*

$$d\psi(y)[v] = -(E - V(y))g\big(\dot{\gamma}_y(0), v\big) = -\psi(y)g\big(\dot{q}_y(0), v\big). \tag{5.35}$$

Note that the variable change used in the proof of Proposition 4.2 yields $q_y(t) = \gamma_y(\sigma)$ where $t(\sigma) = \psi(y) \int_0^\sigma \frac{1}{E - V(\gamma_y(\tau))} \, d\tau$.

We now are going to show the counterpart of Proposition 5.8. We cannot repeat, of course, the same argument as before: indeed, since E is not a regular value for the potential $V(x)$, the curve $q_y(t) = q(t, Q_y)$ (see Remark 5.5) does not reach the boundary $\partial\Omega$ in a finite amount of time and therefore it cannot be reparameterized in a bounded interval.

Proposition 5.18. *There exists $\widehat{\rho} \leq \bar{\rho}$ such that for any y with $\mathrm{dist}(y, x_0) \leq \widehat{\rho}$ we have*

$$\mathrm{H}^\psi(y)[v, v] > 0, \qquad \forall v \, : \, d\psi(y)[v] = 0.$$

Proof. We need to evaluate
$$\frac{\partial^2}{\partial s^2}\big(\psi(\eta(s))\big)_{|s=0},$$
where $\eta(s)$ is the geodesic with respect to the Jacobi metric g_E such that $\eta(0) = y$, $\dot\eta(0) = v$, where $\mathrm{d}\psi(y)[v] = 0$. We also recall that $\eta(s)$ satisfies equation (4.3). By (5.35)

$$
\frac{\partial^2}{\partial s^2}\big(\psi(\eta(s))\big) = \frac{\partial}{\partial s}\big(\mathrm{d}\psi(\eta(s))[\dot\eta(s)]\big) = \frac{\partial}{\partial s}\big(-\psi(\eta(s))g\big(\dot q_{\eta(s)}(0),\dot\eta(s)\big)\big)
$$
$$
= -\mathrm{d}\psi(\eta(s))[\dot\eta(s)]g\big(\dot q_{\eta(s)}(0),\dot\eta(s)\big) - \psi(\eta(s))g\big(\tfrac{\mathrm{D}}{\mathrm{d}s}(\dot q_{\eta(s)}(0)),\dot\eta(s)\big)
$$
$$
- \psi(\eta(s))g\big(\dot q_{\eta(s)}(0),\tfrac{\mathrm{D}}{\mathrm{d}s}\dot\eta(s)\big),
$$

then, using again (5.35), and exploiting (4.3), one gets

$$
\mathrm{H}^\psi(y)[v,v] = \psi(y)g\big(\dot q_y(0),v\big)^2 - \psi(y)g\big(\mathrm{d}\dot q_y(0)[v],v\big)
$$
$$
- \frac{\psi(y)}{E - V(y)}\Big(-\frac{1}{2}g(v,v)g\big(\dot q_y(0),\nabla V(y)\big) + g\big(\nabla V(y),v\big)g\big(\dot q_y(0),v\big)\Big).
$$

Since $g(\dot q_y(0),v) = \mathrm{d}\psi(y)[v] = 0$, it suffices to show the existence of $\nu_0 > 0$ such that

$$
\inf_{|v|=1} g\big(\mathrm{d}\dot q_y(0)[v],v\big) + \frac{g\big(\dot q_y(0),\nabla V(y)\big)}{2\big(E - V(y)\big)} \geq \nu_0 \tag{5.36}
$$

for any y close sufficiently to x_0. Let us consider the map

$$
\mu(t) = g\big(\dot q_y(t),\nabla V(q_y(t))\big).
$$

By (5.27) we have

$$
\mu(t) - \mu(0) = \int_0^t \mu'(\tau)\,\mathrm{d}\tau = \int_0^t \big[g\big(-\nabla V(q_y),\nabla V(q_y)\big) + \mathrm{H}^V(q_y)[\dot q_y,\dot q_y]\big]\,\mathrm{d}\tau;
$$

then, by Proposition 5.13 and nondegeneracy of the maximum point x_0, we see that there exists $\nu > 0$ such that

$$
\mu(t) - \mu(0) \leq -\nu \int_0^t \frac{1}{2}\,|\dot q_y|^2\,\mathrm{d}\tau = -\nu \int_0^t \big(E - V(q_y(\tau))\big)\,\mathrm{d}\tau,
$$

and since $\lim_{t\to+\infty}\mu(t) = 0$ we have

$$
g\big(\dot q_y(0),\nabla V(y)\big) = \mu(0) \geq \nu \int_0^{+\infty} \big(E - V(q_y(\tau))\big)\,\mathrm{d}\tau.
$$

Now, consider the map $\kappa(t) = E - V(q_y(t))$: we have

$$
\kappa''(t) = -\mathrm{H}^V(q_y)[\dot q_y,\dot q_y] + g\big(\nabla V(q_y),\nabla V(q_y)\big).
$$

Again, by nondegeneracy of x_0 as the maximum point and Proposition 5.13 there exists $A > 0$ such that

$$g\big(\nabla V(q_y(t)), \nabla V(q_y(t))\big) \leq A(E - V(q_y(t)))$$

while the conservation law of the energy for q_y gives $\frac{1}{2}g\big(\dot{q}_y, \dot{q}_y\big) = E - V(q_y)$. Then there exists $B > 0$ such that $\kappa''(t) \leq B\kappa(t)$ for $t \geq 0$, and by Remark 5.15

$$E - V(q_y(t)) \geq \big(E - V(y)\big)e^{-\sqrt{B}t}.$$

Then

$$g\big(\dot{q}_y(0), \nabla V(y)\big) \geq \nu\big(E - V(y)\big) \int_0^{+\infty} e^{-\sqrt{B}\tau}\,\mathrm{d}\tau.$$

Finally, by Proposition 5.16, $\mathrm{d}\dot{q}_y(0) \to \mathrm{d}\dot{q}_{x_0}(0)$ while $\mathrm{d}\dot{q}_{x_0}(0)[v] = \dot{\xi}_0(0)$ where $\xi_0(t)$ is the unique solution of

$$\begin{cases} \ddot{\xi}_0 + \mathrm{L}^V(x_0)[\xi_0] = 0, \quad \xi_0(0) = v, \\ \lim_{t \to +\infty} \xi_0(t) = \lim_{t \to +\infty} \dot{\xi}_0(t) = 0. \end{cases}$$

But, denoting by $\mathbf{e_i}$ a basis of eigenvectors for $\mathrm{L}^V(x_0)$ and by $\lambda_i < 0$ the corresponding eigenvalues we have

$$\xi_0(t) = \sum_{i=1}^m v_i e^{-\sqrt{\lambda_i}t}\mathbf{e_i}.$$

Since $\mathrm{d}\dot{q}_{x_0}(0)[v] = \dot{\xi}_0(0)$ and $-\mathrm{H}^V(x_0)$ is positive definite, there exists $\mu_0 > 0$ such that

$$g\big(\mathrm{d}\dot{q}_0(0)[v], v\big) \geq \mu_0\,g\big(v, v\big),$$

and (5.36) is completely proved. $\qquad\square$

Finally, we give the result needed to prove our multiplicity result for homoclinics in [4]. To this aim, take $y \in \{x : V(x) < E\}$ and consider

$$d(y) = \mathrm{dist}_E(y, V^{-1}(E)) \tag{5.37}$$

where dist_E is the distance with respect to the Jacobi metric. Combining the results of Theorem 5.9, Lemma 5.11, and Propositions 5.17–5.18 and using the function (5.37) gives us the following:

Theorem 5.19. *Assume that:*

    (a) $V^{-1}\big(\,]-\infty, E[\,\big) \bigcup \{x_0\}$ *is homeomorphic to an open ball of $\mathbb{R}^m$;*

    (b) $\mathrm{d}V(x) \neq 0$ *for all* $x \in V^{-1}(E) \setminus \{x_0\}$;

moreover, let d be as in (5.37). Then, there exists a positive number δ_* such that, setting $\Omega_* = \{x \in \mathbb{R}^M : d(x) > \delta_*\}$ and denoting by D_0 the connected component of $\partial\Omega_*$ close to x_0 and by D_1 the connected component of $\partial\Omega_*$ near $V^{-1}(E) \setminus \{x_0\}$, the following results hold:

(1) *$\partial\Omega_*$ is of class C^2;*

(2) *$\overline{\Omega_*}$ is homomorphic to an annulus;*

(3) *$\overline{\Omega_*}$ is strongly concave with respect to the Jacobi metric g_E;*

(4) *if $x : [0,1] \to \overline{\Omega_*}$ is an orthogonal geodesic chord in $\overline{\Omega_*}$ relative to the Jacobi metric g_E such that $x(0) \in D_0$ and $x(1) \in D_1$, then there exists $]\alpha, \beta[\supset [0,1]$ and a unique extension $\widehat{x} : [\alpha, \beta] \to \overline{\Omega}$, $x \in C^0 \cap H^1_{loc}([\alpha, \beta], \overline{\Omega}_E)$ satisfying*
   - *$\widehat{x}$ is a geodesic with respect to the Jacobi metric;*
   - *$\widehat{x}(s) \in d^{-1}(]-\delta_*, 0[)$ for all $s \in]\alpha, 0[\bigcup]1, \beta[$;*
   - *$\widehat{x}(\alpha) = x_0$, $\widehat{x}(\beta) \in V^{-1}(E) \setminus \{x_0\}$.*

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