



Holographic Transport with Topological Term and Entropy Function

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Abstract

In this paper, we consider the effect of a topological Maxwell term $W(\Phi)F_{\mu\nu}\tilde{F}^{\mu\nu}$ on holographic transport and thermodynamics in $2 + 1$ dimensions, in the case with a dyonic black hole in the gravity dual. We find that for a constant W the modifications to the thermodynamics are easily quantified, and transport is affected only for σ_{xy} . If one considers also the attractor mechanism, and writing the horizon transport in terms of charges, the transport coefficients are affected explicitly. We also introduce the case of radially dependent $W(z)$, in which case, however, analytical calculations become very involved. We also consider the implications of the two models for the S-duality of holographic transport coefficients.

1 Introduction

The AdS/CFT correspondence [1] (see [2, 3] for a review) describes strongly coupled phenomena in field theories using perturbative string theory or, more commonly, its classical (super)gravity limit. Applications to condensed matter via AdS/CMT (see [4] for a review) are (usually) phenomenological in nature: one defines a gravity theory which, via the holographic map that is still presumed to work, has expected properties for the dual strongly coupled field theory. One of the most standard applications is for electric and thermal transport in various materials at nonzero temperature, dual to properties in the background of a black hole (see [4, 5] for a review).

One set of methods was considered, more recently, in [6–12], where one considers an Einstein-Maxwell-dilaton $(g_{\mu\nu}, A_\mu, \phi)$ system in the gravity dual, also coupled to axions (χ_1, χ_2) , that allows for a more general analysis of transport, from the horizon of a black hole, based on the application of the membrane paradigm to AdS/CFT in the form started in [13], relating the horizon to the boundary,

where standard AdS/CFT quantities are obtained, via a radial evolution equation. In [14–16], an extension of the analysis was considered, by adding a topological $W(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu}$ term and considering the resulting S-duality properties. One can also use the attractor mechanism and Sen's entropy function [17] to calculate conductivities from black hole horizons [12], and this method was also used in [15].

But the original analysis of transport (for shear viscosity η of a strongly coupled field theory fluid), defined in [18–20], was used earlier for calculating the thermoelectric transport from a dyonic black hole in the gravity dual in [21, 22], with a more general analysis in [23] (see also, for instance, more recently [24]).

In this paper, we are interested in considering the effect of a $W(\phi)$ topological term on the calculation of thermodynamics and transport in [21, 22], as well as the calculations via the attractor mechanism. The presence of the $W(\phi)$ term is necessary in order to obtain complete S-duality transformations in the gravity dual, and they correspond to $Sl(2, \mathbb{Z})$ transformations on observables in the gauge theory, including particle-vortex duality [14–16], both of which are of great theoretical importance in condensed matter systems. We will first consider the case of a constant $W(\phi)$ and then, in order to obtain more nontrivial results, we try to consider varying $W(\phi)$. However, it turns out that the simplest possible model is to consider directly a fixed $W(\phi(z)) = W(z)$, with z the radial direction, since otherwise solving the equations is very difficult.

The paper is organized as follows. In Section 2, we consider the topological term with constant coefficient W , calculating the thermodynamics, the holographic transport

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coefficients σ_{ab} and α_{ab} from the Kubo formulas at the boundary, and then the attractor mechanism using Sen’s entropy function, to write these in terms of charges and parameters at the horizon. The action of S-duality on the holographic model and the transport coefficients is also explained. In Section 3, we consider a “toy model” for a field dependent $W(\phi)$, where both W and Z (the coefficient of the Maxwell term) are instead explicit functions of the radial coordinate z . We find the solutions for fluctuations, though the holographic transport coefficients are too complicated to write, as are the results of the entropy function formalism. The effect of S-duality is explained, as is the introducing of anisotropy in the model. In Section 4, we conclude, and the Appendices contain some long formulas.

2 Thermoelectric Conductivities from Kubo Formulas, with Topological Term

We want to extend the analysis of thermodynamics and transport from fluctuations found in [21, 22] to the presence of the topological term. As there, we will consider a full solution for a dyonic black hole, but otherwise, like [15, 16], where only the horizon was considered (still with nonzero magnetic field), we will add the topological term.

Consider the action for gravity and a Maxwell field given by

$$I = -\frac{4}{2\kappa_4^2} \int d^4x \sqrt{-g} \left(-\frac{1}{4}R + \frac{L^2}{4}F_{\mu\nu}F^{\mu\nu} - \frac{3}{2}\frac{1}{L^2} + W\frac{L^2}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} \right), \tag{2.1}$$

where $F_{\mu\nu}$ is the field strength of A_μ , $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\tilde{F}_{\mu\nu}$ is its dual,

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{2.2}$$

The normalization of the Newton’s constant κ_4 relative to the radius L for the (negative) cosmological constant is given in terms of the rank N of the dual field theory by

$$\frac{2L^2}{\kappa_4^2} = \frac{\sqrt{2}N^{3/2}}{6\pi}. \tag{2.3}$$

Note that with respect to the more general formulas in [15, 16] we have set

$$\frac{Z}{g_4^2} = \frac{L^2}{4\pi G_N} \equiv \frac{c}{\pi}, \quad W(\phi) = W\frac{L^2}{16\pi G_N}, \tag{2.4}$$

in order to match with the analysis of the dyonic black hole in [21, 22], and to simplify the analysis of the thermodynamics.

The equations of motion for gravity and the Maxwell field are

$$R_{\mu\nu} = 2L^2F_{\mu\sigma}F_{\nu}^{\sigma} - \frac{L^2}{2}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} - \frac{3}{L^2}g_{\mu\nu}, \tag{2.5}$$

$$\nabla_{\mu}(F^{\mu\nu} + W\tilde{F}^{\mu\nu}) = 0. \tag{2.6}$$

However, since

$$\nabla_{\mu}F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g})F^{\mu\nu}, \tag{2.7}$$

the Maxwell equation becomes

$$\frac{1}{\sqrt{-g}}\partial_{\mu}\sqrt{-g}(F^{\mu\nu} + W\tilde{F}^{\mu\nu}) = 0. \tag{2.8}$$

On the other hand, by its topological nature, the W term does not affect the Einstein equation. Therefore, we can use the same solution as in [21, 22],

$$\frac{1}{L^2}ds^2 = -\frac{\alpha^2}{z^2}f(z)dt^2 + \frac{\alpha^2}{z^2}(dx^2 + dy^2) + \frac{1}{z^2}\frac{dz^2}{f(z)}, \tag{2.9}$$

where $f(z)$ is written in terms of (properly normalized) magnetic charges h and electric charges q as

$$f(z) = 1 + (h^2 + q^2)z^4 - (1 + h^2 + q^2)z^3. \tag{2.10}$$

Here the horizon of the black hole is at $z = 1$, and the boundary of the asymptotic AdS space in which it is embedded is at $z \rightarrow 0$. Note that $\sqrt{-g} = L^4\alpha^3/z^4$.

Since W is constant, the solution for the gauge field A_μ is also the same,

$$A = h\alpha^2x dy + q\alpha(z - 1)dt, \tag{2.11}$$

where the gauge choice was such that $A_t \rightarrow 0$ at the horizon $z \rightarrow 1$, as is usual for AdS/CFT. Then, we also have

$$F = h\alpha^2dx \wedge dy + q\alpha dz \wedge dt, \tag{2.12}$$

or, explicitly in components (t, x, y, z) ,

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -q\alpha \\ 0 & 0 & h\alpha^2 & 0 \\ 0 & -h\alpha^2 & 0 & 0 \\ q\alpha & 0 & 0 & 0 \end{pmatrix}, \tag{2.13}$$

$$\tilde{F}^{\mu\nu} = \frac{z^4}{L^4\alpha^3} \begin{pmatrix} 0 & 0 & 0 & h\alpha^2 \\ 0 & 0 & -q\alpha & 0 \\ 0 & q\alpha & 0 & 0 \\ -h\alpha^2 & 0 & 0 & 0 \end{pmatrix}.$$

2.1 Thermodynamics

The temperature of the black hole is, as usual,

$$T = \frac{\alpha f'(1/z)|_{z=1}}{4\pi} = \frac{\alpha(3 - h^2 - q^2)}{4\pi}. \tag{2.14}$$

To compute the thermodynamics, we note that we are in the grand canonical ensemble, since we have fixed charges, so the thermodynamic potential is $\Omega = TI_{\text{ren}}$. However, the action needs to be renormalized by the addition of counterterms,

$$I_{\text{ren}} = I + I_{\text{ct}}, \tag{2.15}$$

and we can use the standard counterterm action from the boundary of AdS,

$$I_{\text{ct}} = -\frac{1}{\kappa_4^2} \int d^3x \sqrt{-\gamma} \theta - \frac{2}{\kappa_4^2} \frac{1}{L} \int d^3x \sqrt{-\gamma}. \tag{2.16}$$

The unit normal to the boundary, satisfying $n^\mu n_\mu = 1$, is

$$n^\mu = z \frac{\sqrt{f(z)}}{L} \times (0, 0, 0, 1). \tag{2.17}$$

The boundary metric γ_{MN} is the one induced by $g_{\mu\nu}$ in the three boundary directions, but since it blows up as $z \rightarrow 0$, it must be defined a bit away from the boundary, at $z = \epsilon$ with $\epsilon \rightarrow 0$, obtaining

$$\gamma_{MN} = -\frac{\alpha^2}{\epsilon^2} f(\epsilon) dt^2 + \frac{\alpha^2}{\epsilon^2} (dx^2 + dy^2). \tag{2.18}$$

Equivalently, we then have $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$.

The θ appearing in the counterterm action is the trace of the extrinsic curvature

$$\theta_{MN} = -\frac{1}{2} (\nabla_\mu n_\nu + \nabla_\nu n_\mu) = \begin{pmatrix} \frac{L\sqrt{f(\epsilon)}(\epsilon f'(\epsilon) - 2f(\epsilon))}{2\epsilon^2} & 0 & 0 \\ 0 & -\frac{L\sqrt{f(\epsilon)}\alpha^2}{\epsilon^2} & 0 \\ 0 & 0 & -\frac{L\sqrt{f(\epsilon)}\alpha^2}{\epsilon^2} \end{pmatrix}, \tag{2.19}$$

with

$$\theta = \gamma^{MN} \theta_{MN} = \frac{\epsilon f'(\epsilon) - 6f(\epsilon)}{2L\sqrt{f(\epsilon)}}. \tag{2.20}$$

Plugging into the counterterm action (2.16), and doing the integral over time t up to the inverse temperature β , we get, in a power series in ϵ ,

$$I_{\text{ct}} = \beta V \frac{\sqrt{2}N^{3/2}}{6\pi} \frac{\alpha^3}{2} \left(\frac{-1}{\epsilon^3} + \frac{1+h^2+q^2}{2} + \mathcal{O}(\epsilon) \right), \tag{2.21}$$

where $V = \int dx dy$.

Integrating z in the action I from ϵ (the regularized boundary) to 1 (the horizon), we obtain

$$I = \beta V \frac{\sqrt{2}N^{3/2}}{6\pi} \frac{\alpha^3}{2} \left(\frac{1}{\epsilon^3} - 1 + h^2 - q^2 - 2Whq + \mathcal{O}(\epsilon) \right), \tag{2.22}$$

so that the renormalized action becomes

$$I_{\text{ren}} = \lim_{\epsilon \rightarrow 0} (I + I_{\text{ct}}) = \beta V \frac{\sqrt{2}N^{3/2}}{6\pi} \frac{\alpha^3}{4} (-1 - q^2 + 3h^2 - 4Whq). \tag{2.23}$$

Multiplying with the temperature gives the grand canonical potential,

$$\Omega = \frac{\sqrt{2}N^{3/2}}{6\pi} \frac{\alpha^3 V}{4} (-1 - q^2 + 3h^2 - 4Whq). \tag{2.24}$$

The magnetic field and the chemical potential are found from the $z \rightarrow 0$ limit of A in (2.11) and its field strength F in (2.12),

$$B = \alpha^2 h, \quad \mu = -\alpha q. \tag{2.25}$$

In terms of these (its natural) parameters, the grand canonical potential is

$$\Omega = \frac{\sqrt{2}N^{3/2}}{6\pi} \frac{\alpha^3 V}{4} \left(-1 - \frac{\mu^2}{\alpha^2} + 3\frac{B^2}{\alpha^4} + 4W\frac{\mu B}{\alpha^3} \right). \tag{2.26}$$

While this potential depends on W , the entropy S and energy E derived from it must not, since they depend only on the geometry of the black hole. Indeed, for the entropy, we first find from the above Ω and the T in (2.14) that

$$dT|_{B,\mu} = \frac{d\alpha}{4\pi} \left(3 + 3\frac{B^2}{\alpha^4} + \frac{\mu^2}{\alpha^2} \right), \tag{2.27}$$

$$d\Omega|_{B,\mu} = \frac{\sqrt{2}N^{3/2}}{6\pi} \frac{3\alpha^2 d\alpha V}{4} \left(-3 - \frac{\mu^2}{\alpha^2} - 3\frac{B^2}{\alpha^4} \right),$$

so that

$$S = -\frac{\partial \Omega}{\partial T}|_{B,\mu} = \frac{\sqrt{2}N^{3/2}}{6\pi} \alpha^2 V = c\alpha^2 V. \tag{2.28}$$

This indeed matches the entropy calculated from the area of the horizon, with our conventions in (2.4).

Then, the energy is

$$E = \Omega + TS + \mu N = \frac{\sqrt{2}N^{3/2}}{6\pi} \frac{\alpha^3 V}{2} (1 + q^2 + h^2), \tag{2.29}$$

where $N = Q$ (the electric charge) is calculated below:

Indeed, from (2.14), we find that when T, B are fixed, we get $\mu = \mu(\alpha)$, as

$$d\alpha \left(3 + \frac{\mu^2}{\alpha^2} + 3\frac{B^2}{\alpha^4} \right) = 2\mu \frac{d\mu}{\alpha}, \tag{2.30}$$

and then

$$d\Omega|_{T,B} = \frac{\sqrt{2}N^{3/2}}{6\pi} \frac{V}{4} \left\{ d\alpha \left[-3\alpha^2 - \mu^2 - 3\frac{B^2}{\alpha^2} \right] + d\mu[-2\alpha\mu + 4BW] \right\}, \tag{2.31}$$

so that

$$Q = -\frac{\partial\Omega}{\partial\mu} \Big|_{T,B} = \frac{\sqrt{2}N^{3/2}}{6\pi} \frac{V}{4} (4\alpha\mu - 4BW). \tag{2.32}$$

Again from (2.14), when T, μ are fixed, we get $\alpha = \alpha(B)$,

$$d\alpha \left(3 + \frac{\mu^2}{\alpha^2} + 3\frac{B^2}{\alpha^4} \right) = 2B \frac{dB}{\alpha^3}, \tag{2.33}$$

and then

$$d\Omega|_{T,\mu} = \frac{\sqrt{2}N^{3/2}}{6\pi} \frac{V}{4} \left\{ d\alpha \left[-3\alpha^2 - \mu^2 - 3\frac{B^2}{\alpha^2} \right] + 6\frac{BdB}{\alpha} + 4W\mu dB \right\}, \tag{2.34}$$

so that

$$M = \frac{\partial\Omega}{\partial B} \Big|_{T,\mu} = \frac{\sqrt{2}N^{3/2}}{6\pi} \frac{V}{4} \left(\frac{4B}{\alpha} + 4W\mu \right). \tag{2.35}$$

So the charge and magnetization do get a W term, while the entropy and energy do not. More precisely, the statement is that: if we keep B (and T) fixed, the relation between Q and μ gets an extra, W -dependent term (the leading term being independent on W) and, if we keep μ (and T) fixed, the relation between M and B gets an extra, W -dependent term (the leading term being independent on W).

2.2 Magnetization and Fluctuations

We want to use the Kubo formulas to calculate transport coefficients, and for that we need to calculate the holographic Green’s functions, from fluctuations in the AdS dyonic black hole background. Therefore, in this subsection, we set up the problem of fluctuations in this background.

We follow the prescription from [21, 22]. But as described there, and found in [25], in the case of nonzero magnetic field, we must subtract the magnetization currents from the standard thermoelectric ones,

$$J_i^{\text{mag}} = \epsilon_{ij}\partial_j M, \tag{2.36}$$

$$T_{ii}^{\text{mag}} = \epsilon_{ij}\partial_j M^E, \tag{2.37}$$

where M and M^E are respectively the magnetization and the energy magnetization densities.

We must consider fluctuations on top of the background that will give the holographic Green’s functions, compatible with the boundary condition in terms of a magnetic field B and a “energy magnetic field B^E ” (sometimes called B_1 , that sources g_{ty} instead of A_y). Since δA_μ^0 couples to a source current J_μ and $\delta g_{\mu\nu}^0$ couples to a source energy-momentum tensor $T_{\mu\nu}$, we must have the boundary condition

$$\begin{aligned} \lim_{z \rightarrow 0} \delta A_y &= xB, \\ \lim_{z \rightarrow 0} \delta G_y &= xB^E, \end{aligned} \tag{2.38}$$

where $G_y = g_{ty}z^2/\alpha$.

The ansatz compatible with these boundary conditions is [22]

$$\begin{aligned} \delta A_y &= x(B - qB^E z), \\ \delta G_y &= xf(z)B^E. \end{aligned} \tag{2.39}$$

From the equations of motion, it follows also that we need (only) δA_t to compensate the above. We then construct the on-shell quadratic action for the fluctuations.

To find it, we introduce a parameter ϵ to keep track of fluctuations to linear order (linear response),

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu}^{\text{background}} + \epsilon\delta g_{\mu\nu}, \\ A_\mu &= A_\mu^{\text{background}} + \epsilon\delta A_\mu, \end{aligned} \tag{2.40}$$

and solve the equations of motion in an expansion in ϵ . The zeroth order term is just the equation of motion for the background, so vanishes, while the Einstein equation

$$0 = 2L^2 F_{\mu\sigma} F_\nu^\sigma - \frac{L^2}{2} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} - \frac{3}{L^2} g_{\mu\nu} - R_{\mu\nu} \tag{2.41}$$

at order ϵ , for $(\mu\nu) = (xx)$, for example, gives

$$2z^2(Bh - B^E qhz + q\alpha\delta A_t'(z))\epsilon + \mathcal{O}(\epsilon^2), \tag{2.42}$$

where prime denotes differentiation with respect to z . Then, the linear order for the equations of motion fixes the δA_t fluctuation,

$$\delta A_t'(z) = -\frac{Bh - B^E qhz}{q\alpha}. \tag{2.43}$$

It then turns out that this solves the Einstein equations at linear order in ϵ for all $(\mu\nu)$.

On the other hand, the Maxwell equation in the absence of W is expanded in ϵ as

$$\nabla_\mu F^{\mu t} = \frac{z^4}{\alpha^3} (B^E h - \alpha\delta A_t''(z))\epsilon + \mathcal{O}(\epsilon^2), \tag{2.44}$$

giving

$$\delta A_t''(z) = \frac{B^E h}{\alpha}. \tag{2.45}$$

The solution of (2.43) and (2.45) is

$$\delta A_t(z) = \frac{hB^E}{2\alpha}(z^2 - 1) - \frac{hB}{q\alpha}(z - 1), \tag{2.46}$$

where the integration constants were fixed such that $\delta A_t \rightarrow 0$ at the horizon $z = 1$.

In the presence of the topological term with W , since W is a constant, the extra term vanishes since

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0, \tag{2.47}$$

so for a modification of the action for fluctuations we would need to consider the case of a W that depends on z , which will be done later.

2.3 Transport Coefficients from Quadratic Action at Boundary

Here we find the thermoelectric coefficients from the quadratic boundary action.

The (retarded) Green's function is found from the on-shell quadratic action [19]. Consider as an example a scalar field ϕ with a kinetic function $K(z)$ at the $2 + 1$ dimensional boundary,

$$S_{cl} = \frac{1}{2} \int dz d^2x K(z) (\partial_\mu \phi)^2, \tag{2.48}$$

where we have solved the linearized field equations in terms of the boundary value ϕ_0 in frequency space ω ,

$$\phi(z) = H_\omega(z) \phi_0, \tag{2.49}$$

such that $H_\omega(z) = 1$, and impose incoming wave boundary conditions at the horizon.

Then, the retarded Green's function is

$$G^R = \lim_{z \rightarrow 0} K(z) H_{-\omega}(z) \partial_z H_\omega. \tag{2.50}$$

A more precise recipe is to go to momentum space \vec{k} in all boundary directions, and write the on-shell action as

$$S_{\text{on-shell}} = - \int \frac{d^3k}{(2\pi)^3} \phi_0(-\vec{k}) \frac{1}{2} \mathcal{F}(\vec{k}, z) \phi_0(\vec{k}) \Big|_{z=0}^{z=1}, \tag{2.51}$$

which identifies \mathcal{F} as G^R .

Either way, in our case, we consider frequency-space fluctuations

$$\delta A_i = A_i(z) e^{i\omega t}, \quad \delta G_i = G_i(z) e^{i\omega t}, \tag{2.52}$$

for $i = x, y$ and $\alpha G_i = g_{ii} z^2$, and compute the linearized Maxwell equations, giving

$$\begin{aligned} f(fA_x')' + \bar{\omega} A_x + i\bar{\omega} h G_y + qfG_x' &= 0 \\ f(fA_y')' + \bar{\omega} A_y + i\bar{\omega} h G_x + qfG_y' &= 0, \end{aligned} \tag{2.53}$$

where $\bar{\omega} = \omega/\alpha$.

The Einstein equations give equations that imply the above Maxwell ones, so are consistent with them, but for our purposes it is enough to consider these Maxwell ones.

Since near the horizon $f(z) \sim 1 - z$, and near the horizon we obtain the solutions $A_i(z) \sim (1 - z)^\nu$ and $G_i(z) \sim (1 - z)^{\nu+1}$, we can write the solutions that are incoming at the horizon as

$$\begin{aligned} A_i(z) &= f(z)^\nu a_i(z) \\ G_i(z) &= f(z)^{(1+\nu)} g_i(z), \end{aligned} \tag{2.54}$$

where $\nu = i\bar{\omega}/(h^2 + q^2 - 3)$ and the functions $a_i(z)$ and $g_i(z)$ are required to be regular at the horizon $z = 1$.

Note that we also have solutions with $\nu \rightarrow -\nu$ at the horizon, as well as the constant solution

$$G_y = \frac{i\bar{\omega}}{h} A_x, \quad G_x = -\frac{i\bar{\omega}}{h} A_y. \tag{2.55}$$

For the *static* transport coefficients, we are interested in the hydrodynamic limit $\bar{\omega} \rightarrow 0$. For this, we expand the fluctuations at constant dyonic charges h and q ,

$$\begin{aligned} a_i(z) &= a^{(0)} + \bar{\omega} a_i^{(1)}(z) + \dots, \\ g_i(z) &= g^{(0)} + \bar{\omega} g_i^{(1)}(z) + \dots \end{aligned} \tag{2.56}$$

and solve (2.53) to first order, obtaining

$$\begin{aligned} G_x(z) &= -\frac{i\bar{\omega}}{h} \delta_y + f(z)^{1+\nu} \left(G_x^0 + \frac{i\bar{\omega}}{h} \delta_y - i\bar{\omega} G_x^0 \int_0^z \frac{du}{\psi^2(u)} P_5(u) \right), \\ A_x(z) &= \delta_x + f(z)^\nu \left(A_x^0 - \delta_x - \left(G_x^0 + \frac{i\bar{\omega}}{h} \right) qz \right. \\ &\quad \left. + i\bar{\omega} q G_x^0 \int_0^z \frac{du(z-u)}{\psi^2(u)} P_5(u) - i\bar{\omega} \int_0^z \frac{du}{f(u)} (G_x^0 Q_4(u) + G_y^0 Q_3(u)) \right). \end{aligned} \tag{2.57}$$

Here $\psi(z) = f(z)/z$, the functions $P_5(u)$, $Q_3(u)$, and $Q_4(u)$ are polynomials found in Appendix 1 of [21], and δ_i are constants that depend on the boundary values of the fields,

$$\delta_x = A_x^0 + \frac{G_y^0 h(h^2 + q^2 - 3) - 3G_x^0 q(1 + h^2 + q^2)}{4(h^2 + q^2)}, \tag{2.58}$$

and similarly for δ_y . Note that G_x has a δ_y component, so a A_y^0 component, and similarly G_y has a A_x^0 component (thus mixing of x with y components), while A_x has only an A_x^0 component (no mixing).

For the quadratic action for fluctuations, we derive first the gauge field terms (with A_x, A_y), which come from the Maxwell and topological terms in the action,

$$\frac{2}{\kappa_4^2} \int d^4x \sqrt{-g} \left(\frac{L^2}{4} F_{\mu\nu} F^{\mu\nu} + W \frac{L^2}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \rightarrow \frac{\alpha L^2}{\kappa_4^2} \int d^4x \left(\frac{\bar{\omega}}{f(z)} (A_x^2 + A_y^2) + f(z) ((A_x')^2 + (A_y')^2) + 2iW\bar{\omega}(A_x' A_y - A_x A_y') \right). \tag{2.59}$$

Integrating by parts and neglecting terms calculated at the horizon ($z = 1$), and focusing on the lowest order in ω in each of the two terms (Maxwell and topological) separately, we obtain

$$\frac{2}{\kappa_4^2} \int d^4x \sqrt{-g} \left(\frac{L^2}{4} F_{\mu\nu} F^{\mu\nu} + W \frac{L^2}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \rightarrow \frac{\alpha L^2}{\kappa_4^2} \int d^3x \lim_{z \rightarrow 0} \left[f(z) (A_x' A_x + A_y' A_y) + 2i\bar{\omega} W \epsilon^{ab} A_a' A_b \right]. \tag{2.60}$$

Indeed, we will see that the Maxwell term contribution is only from a term without $\bar{\omega}$, while the topological term contribution is from the term linear in $\bar{\omega}$.

Now considering also the metric perturbations, we obtain

$$I \rightarrow \frac{2\alpha L^2}{\kappa_4^2} \int d^3x \lim_{z \rightarrow 0} \left[\left(\frac{f^{1/2} - 1}{2z^3 f^{1/2}} (G_x G_x + G_y G_y) + \frac{q}{2} (A_x G_x + A_y G_y) + \frac{1}{8z^2} (G_x G_x' + G_y G_y') + \frac{f}{2} (A_x A_x' + A_y A_y') \right) + 2i\bar{\omega} W \epsilon^{ab} A_a' A_b \right], \tag{2.61}$$

which, except for the new extra term with W , matches the result in [21].

Expanding in frequency Fourier modes for the fluctuations, for example

$$A_x^0(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A_x^0(\omega) e^{-i\omega t}, \tag{2.62}$$

and considering the on-shell linearized fluctuations (2.57), we obtain

$$I_{\text{quadratic}} = \frac{i}{2} \int \frac{d\omega}{2\pi} dx^2 \omega (A_x^0(\omega) A_y^0(-\omega) - A_y^0(\omega) A_x^0(-\omega)) \left(\frac{q}{h} - W \right) + \frac{-3i(1 + h^2 + q^2)}{4h} \int \frac{d\omega}{2\pi} dx^2 \omega (A_x^0(\omega) G_y^0(-\omega) - A_y^0(\omega) G_x^0(-\omega)) + \frac{9iq(1 + h^2 + q^2)^2}{32(h^2 + q^2)} \int \frac{d\omega}{2\pi} dx^2 \omega (G_x^0(\omega) G_y^0(-\omega) - G_y^0(\omega) G_x^0(-\omega)) + \frac{i(-3 + h^2 + q^2)^2}{32(h^2 + q^2)} \int \frac{d\omega}{2\pi} dx^2 \omega (G_x^0(\omega) G_x^0(-\omega) - G_y^0(\omega) G_y^0(-\omega)), \tag{2.63}$$

which is now in the form where we can use the recipe for calculating holographic retarded Green's functions. Note that the leading terms of order $\mathcal{O}(\bar{\omega}^0)$ have cancelled, and the only contributions are from the mixing of terms alluded before (G_x containing A_y^0 , etc.) for the Maxwell and gravity terms, leading to an extra $\bar{\omega}$, but from the leading term in the topological one, and seeing as we already had a $\bar{\omega}$, this is of the same order.

Green's functions for the current-current, current-energy-momentum tensor and two energy-momentum tensors are thus

$$G_{J_a J_b}^R = -i\omega \epsilon_{a,b} \frac{q}{h} = -i\omega \epsilon_{a,b} \left(\frac{\rho}{B} - W \right), \\ G_{J_a T_{tb}}^R = -i\omega \epsilon_{a,b} \frac{3(1 + h^2 + q^2)}{2h} = -i\omega \epsilon_{a,b} \frac{\epsilon}{2B}, \\ G_{T_{ta} T_{tb}}^R = -i\omega \epsilon_{ab} \frac{9q(1 + h^2 + q^2)^2}{32(h^2 + q^2)} + i\omega \delta_{ab} \frac{(-3 + h^2 + q^2)^2}{32(h^2 + q^2)}, \tag{2.64}$$

where we have written the results in terms of the magnetic field $B = h\alpha^2$, the charge density $\rho = q\alpha^2$, and the energy density $\epsilon = (1 + h^2 + q^2)$ of the boundary theory.

Then finally, using the Kubo formulas, we find the static electric conductivity as

$$\sigma_{ab} = -\lim_{\omega \rightarrow 0} \frac{\text{Im} G_{J_a J_b}^R}{\omega} = \epsilon_{ab} \left(\frac{\rho}{B} - W \right), \tag{2.65}$$

while for the static thermoelectric and heat conductivities we must take into account the magnetization term, as already noted, obtaining

$$\alpha_{ab} = -\lim_{\omega \rightarrow 0} \frac{\text{Im} G_{T_{ta} J_b}^R}{\omega} + \frac{M}{T} \epsilon_{ab} = \frac{s}{B} \epsilon_{ab}, \\ \bar{\kappa}_{ab} = -\lim_{\omega \rightarrow 0} \frac{\text{Im} G_{T_{ta} T_{tb}}^R}{\omega} + \frac{2(M^E - \mu M)}{T} \epsilon_{ab}. \tag{2.66}$$

We see then that the effect of the W term is only in σ^{ab} , α_{ab} , and $\bar{\kappa}_{ab}$ remaining the same.

2.4 Anisotropy

In order to break isotropy, we consider two parameters (k_x, k_y) multiplying the boundary space coordinates, giving a metric

$$\frac{1}{L^2} ds^2 = -\frac{\alpha^2}{z^2} f(z) dt^2 + \frac{\alpha^2}{z^2} (k_x dx^2 + k_y dy^2) + \frac{1}{z^2} \frac{dz^2}{f(z)}. \tag{2.67}$$

Note that then we obtain $\sqrt{-g} = \sqrt{k_x k_y} L^4 \alpha^3 / z^4$.

From the equations of motion for gravity, we obtain the condition

$$k_x = 1/k_y. \tag{2.68}$$

Nothing new is obtained from this moment on, and the analysis proceeds as before, with no new physics. We have shown this, however, since in the next section, when we have varying $W(z)$, there will be new results.

2.5 Conductivity from Entropy Function via Attractor Mechanism and Membrane Paradigm

The conductivity can also be obtained using the attractor mechanism, for calculations at the horizon, via Sen’s entropy function [17]. The application of Sen’s entropy function formalism to holography was considered in [26] (see also [27]), while the application to the calculation of conductivity was done in [12], and was also used in [15] for the case of metrics with charge density and magnetic field, defined only near the horizon. The fact that we can calculate the conductivity at the horizon, as well as at its natural AdS/CFT location, the boundary, is related to the application of the membrane paradigm to the AdS black holes, as argued initially by Iqbal and Liu [13].

The power of the attractor mechanism is, therefore, that one can both calculate the conductivities only at the horizon and nevertheless obtain them as functions of only the charges of the black hole (defined at infinity), related to charges in the field theory. In a top-down (completely known) solution, that is perhaps not too impressive, nevertheless, we show that this works as expected, since it confirms its use in cases in which the solution is only known near the horizon, yet we can still write the conductivities as functions of the charges defined at infinity (or in the field theory).

The near-horizon geometry of an *extremal* 4-dimensional planar black hole (such as obtained in the attractor mechanism) is $AdS_2 \times \mathbb{R}^2$, written as

$$ds^2 = -vr^2 dt^2 + w(dx^2 + dy^2) + \frac{v}{r^2} dr^2, \tag{2.69}$$

where $r = 1/z$. Note that this is for a planar black hole, but the formalism works as well as for the spherical (S^2 instead of \mathbb{R}^2) black hole case [17]. Also, sometimes v and w are denoted by v_1 and v_2 , in order to emphasize their similarities.

The values of the magnetic and electric fields at the horizon are $F_{xy}^A = B^A$ (sometimes denoted p^A) and $F_{zt}^A = e_A$, and for the scalars we have the horizon values $\phi_s = u_s$.

Defining the function f as the integral over the horizon of the Lagrangian density,

$$f(u_s, v, w, B^A, e_A) = \int dx dy \sqrt{-g} \mathcal{L}, \tag{2.70}$$

as shown by Sen, the Einstein equations imply that the u_s, v and w are extrema of f , while the charges Q_A can be defined as its variations with respect to the electric fields, so

$$\frac{\partial f}{\partial u_s} = 0, \quad \frac{\partial f}{\partial v} = 0, \quad \frac{\partial f}{\partial w} = 0, \quad \frac{\partial f}{\partial e_A} = Q^A. \tag{2.71}$$

Then, Sen’s entropy function is

$$\mathcal{E}(u_s, v, w, e_A, B^A; Q^A) = 2\pi(e_A Q^A - f(u_s, v, w, e_A, B^A)), \tag{2.72}$$

and its value *at its extremum*, defined by the *attractor equations*

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0, \quad \frac{\partial \mathcal{E}}{\partial v} = 0, \quad \frac{\partial \mathcal{E}}{\partial w} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_A} = 0. \tag{2.73}$$

In our case, we do not have a scalar, and we have a single electromagnetic field, with electric field e , so we only have three attractor equations. The function f in our case is then (with $4\pi G_N = 1$)

$$\begin{aligned} -f &= \int dx dy \sqrt{-g} \left(-\frac{1}{4}R + \frac{L^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{3}{2} \frac{1}{L^2} + W \frac{L^2}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \\ &= \frac{1}{2} \int dx dy \left(\frac{B^2 L^2 v}{w} + w \left(1 - \frac{3v}{L^2} - \frac{e^2 L^2}{v} \right) + 2BeL^2 W \right), \end{aligned} \tag{2.74}$$

and the entropy function is

$$\mathcal{E} = 2\pi e(\tilde{Q} - BL^2 W) + \pi \frac{B^2 L^2 v}{w} + \pi w \left(1 - \frac{3v}{L^2} - \frac{e^2 L^2}{v} \right), \tag{2.75}$$

where $\tilde{Q} = Q/\text{Vol}$. The attractor equations are then

$$\begin{aligned} w^2 \left(3 - \frac{e^2 L^4}{v^2} \right) - B^2 L^4 &= 0 \\ L^2 \left(\frac{B^2 v}{w^2} + \frac{e^2}{v} \right) + \frac{3v}{L^2} - 1 &= 0 \\ \frac{L^2 (ew - BvW)}{v} - \tilde{Q} &= 0. \end{aligned} \tag{2.76}$$

We use the last equation to solve for e in terms of \tilde{Q} (inverting it), and the other two to solve for v and w . Then, the parameters v , w , and e of the ansatz are written in terms of the charge density \tilde{Q} and the magnetic field B , as well as the parameters L and W in the action, as

$$\begin{aligned}
 v &= \frac{L^2}{6}, \\
 w &= \frac{\sqrt{B^2L^4(W^2 + 1) - 2BL^2\tilde{Q}W + \tilde{Q}^2}}{\sqrt{3}}, \\
 e &= \frac{BL^2W - \tilde{Q}}{2\sqrt{3}z^2\sqrt{B^2L^4(W^2 + 1) - 2BL^2\tilde{Q}W + \tilde{Q}^2}}.
 \end{aligned}
 \tag{2.77}$$

Since the charge density of the field theory, ρ , is dual (coupled) to A_t , we have

$$\rho = \frac{\delta S}{\delta A_t}, \tag{2.78}$$

which gives, on the solution, the same result as the charge density parameter of the black hole,

$$\rho = \sqrt{-g}(F^{tz} + W\tilde{F}^{tz}) = \frac{\tilde{Q}}{L^2}. \tag{2.79}$$

For the entropy density, the extremum of the entropy function gives the same as the Hawking formula,

$$s = \frac{\sqrt{B^2L^4(W^2 + 1) - 2BL^2\tilde{Q}W + \tilde{Q}^2}}{4\sqrt{3}G_N} = \frac{w}{4G_N}, \tag{2.80}$$

and this is identified with the entropy density in the field theory.

We can now use these formulas for the entropy density s and the charge density ρ of the field theory in the formulas already derived, from Kubo formulas, for the electric and thermoelectric conductivities, to find these as functions of the charges of the black hole. We find

$$\begin{aligned}
 \sigma_{ab} &= \epsilon_{ab} \left(\frac{\rho}{B} - W \right) = \epsilon_{ab} \left(\frac{\tilde{Q}}{BL^2} - W \right), \\
 \alpha_{ab} &= \epsilon_{ab} \frac{\sqrt{B^2L^4(W^2 + 1) - 2BL^2\tilde{Q}W + \tilde{Q}^2}}{4\sqrt{3}BG_N}.
 \end{aligned}
 \tag{2.81}$$

We note that now, with the holographic transport coefficients written in terms of \tilde{Q} , B and W via the attractor mechanism, the explicit W dependence becomes more complicated.

2.6 S-duality

As observed in [15], and made more precise in [16], the general formulas for transport coefficients obtained from fluctuations around a solution near the horizon of a black hole have an action of S-duality ($Sl(2;\mathbb{Z})$) on them, coming from the S-duality invariance of the gravitational action. For a general $Z(\phi)$ (term in front of the Maxwell action), the invariance is under

$$\begin{aligned}
 F_{\mu\nu} &\rightarrow Z\tilde{F}_{\mu\nu} - WF_{\mu\nu} \\
 Z &\rightarrow -\frac{Z}{Z^2 + W^2} \\
 W &\rightarrow \frac{W}{Z^2 + W^2}.
 \end{aligned}
 \tag{2.82}$$

This led to the duality relation on the complex conductivity $\sigma \equiv \sigma_{xy} + i\sigma_{xx}$,

$$\sigma' = -\frac{1}{\sigma}. \tag{2.83}$$

In our case, this is less obvious, since we have effectively fixed Z to 1, but the duality is there.

In the case of the dyonic black hole of [21, 22], which gives a subset of the formulas in [15], the action of S-duality was hard to understand, as the only relevant limit is the one that takes $\rho \rightarrow 0$, followed by $s \rightarrow 0$ in the transport coefficients, and previously there was nothing left.

With the introduction of the W term in this dyonic black hole calculation, and the associated entropy function result, it becomes relevant to take the limit $\rho \rightarrow 0$, and obtain

$$\sigma_{xx} = 0, \quad \sigma_{xy} = -W, \quad \alpha_{xx} = 0, \quad \alpha_{xy} = \frac{s}{B}. \tag{2.84}$$

S-duality then acts nontrivially, as

$$W \rightarrow \frac{1}{W} \Rightarrow \sigma_{xy} \rightarrow -\frac{1}{\sigma_{xy}}. \tag{2.85}$$

The limit $\rho \rightarrow 0$ is understood, from the point of view of the dyonic black hole, as the limit when the electric charge goes to zero, keeping the magnetic charge finite. Considering also the entropy function calculation leading to (2.80), we obtain that in this limit,

$$\alpha_{xy} = \frac{s}{B} = \frac{c}{\sqrt{3}}\sqrt{W^2 + 1}, \tag{2.86}$$

where

$$c = \frac{L^2}{4G_N} = \frac{\pi Z}{g_4^2} \tag{2.87}$$

is the central charge of the dual field theory, with the second form being due to our fixing Z/g_4^2 .

But, moreover, if we would keep Z free, we would obtain $\sqrt{W^2 + Z^2}$ in (2.86), which is an S-duality invariant, see (2.82).

3 Radially Varying Topological Term $W(z)$

In this section, we consider a more general model, with a field-dependent topological term, so an Einstein-Maxwell-dilaton model with a nontrivial dilaton, as considered for instance in [15, 16],

$$I = \frac{2}{\kappa_4^2} \int d^4x \sqrt{-g} \left(-\frac{1}{4}R - \frac{1}{2} [(\partial\phi)^2 + \Phi(\phi)((\partial\chi_1)^2 + (\partial\chi_2)^2)] - V(\phi) + Z(\phi) \frac{L^2}{4} F_{\mu\nu} F^{\mu\nu} + W(\phi) \frac{L^2}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \right). \tag{3.1}$$

The reason for introducing $W(\phi)$ is to have a nontrivial contribution to the Einstein equation (and the Maxwell equation); otherwise, as we saw in the previous section, this is a topological term.

Here the axions χ_i are introduced to have a breaking of translational invariance of the theory, via a linear axion ansatz

$$\begin{aligned} \chi_1 &= k_1 x \\ \chi_2 &= k_2 y. \end{aligned} \tag{3.2}$$

We will go back to the isotropic case by considering $k_1 = k_2 = k$.

In order for the ansatz above to be consistent with the axion equations of motion,

$$\Phi(\phi) \partial_\mu \partial^\mu \chi_{(1,2)} + \Phi'(\phi) \partial_\mu \chi_{(1,2)} \partial^\mu \phi = 0, \tag{3.3}$$

we will assume that the dilaton is static and only depends on the radial direction, so $\phi = \phi(z)$.

The equation of motion for the dilaton is

$$\begin{aligned} \partial_\mu \partial^\mu \phi - V'(\phi) - \frac{1}{2} \Phi'(\phi) [(\partial\chi_1)^2 + (\partial\chi_2)^2] \\ + \frac{L^2}{4} (Z'(\phi) F_{\mu\nu} + W'(\phi) F_{\mu\nu} \tilde{F}^{\mu\nu}) = 0. \end{aligned} \tag{3.4}$$

To guarantee that we have a solution, we must impose that

$$\begin{aligned} V(0) &= \frac{-6}{L^2}, \\ V'(0) &= 0. \end{aligned} \tag{3.5}$$

3.1 Set-Up

However, the ansatz considered so far is still too complicated to solve, so instead of the arbitrary functions of the dilaton $W(\phi)$, $\Phi(\phi)$, $V(\phi)$, and $Z(\phi)$, on top of the radially varying dilaton $\phi(z)$, we will simplify further and directly consider independent functions of the radial coordinate z , so $W(z)$, $\Phi(z)$, $V(z)$, and $Z(z)$. At this time, we could not find a way to solve the more interesting case of arbitrary (given functions) $W(\phi)$, $\Phi(\phi)$, $V(\phi)$, and $Z(\phi)$ and solving for $\phi(z)$, $F_{\mu\nu}$, and $g_{\mu\nu}$, so we are restricting to this simple case. The resulting reduced Einstein-Maxwell-dilaton model is

$$I = \frac{2}{\kappa_4^2} \int d^4x \sqrt{-g} \left(-\frac{1}{4}R - \frac{1}{2} [(\partial\phi)^2 + \Phi(z)((\partial\chi_1)^2 + (\partial\chi_2)^2)] - V(z) + Z(z) \frac{L^2}{4} F_{\mu\nu} F^{\mu\nu} + W(z) \frac{L^2}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \right). \tag{3.6}$$

Now, the gauge field equation of motion becomes

$$\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} (Z(z) F^{\mu\nu} + W(z) \tilde{F}^{\mu\nu}) = 0. \tag{3.7}$$

We take the same ansatz for the metric as in the previous section,

$$\frac{1}{L^2} ds^2 = -\frac{\alpha^2}{z^2} f(z) dt^2 + \frac{\alpha^2}{z^2} (dx^2 + dy^2) + \frac{1}{z^2} \frac{dz^2}{f(z)}, \tag{3.8}$$

where $f(z)$ is the same function,

$$f(z) = 1 + (h^2 + q^2)z^4 - (1 + h^2 + q^2)z^3, \tag{3.9}$$

and the ansatz for the field strength is also unchanged,

$$F = h\alpha^2 dx \wedge dy + q\alpha dz \wedge dt \Rightarrow F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -q\alpha \\ 0 & 0 & h\alpha^2 & 0 \\ 0 & -h\alpha^2 & 0 & 0 \\ q\alpha & 0 & 0 & 0 \end{pmatrix}. \tag{3.10}$$

Then, the Maxwell equation of motion gives

$$hW'(z) - qZ'(z) = 0, \tag{3.11}$$

solved by

$$Z(z) = \frac{hW(z)}{q} + Z_0, \tag{3.12}$$

where Z_0 is a constant.

The Einstein equations,

$$R_{\mu\nu} = K_{\mu\nu} + \frac{1}{2} g_{\mu\nu} V(z) + Z(z) \left(2L^2 F_{\mu\sigma} F_{\nu}^{\sigma} - \frac{L^2}{2} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} \right), \tag{3.13}$$

give three independent equations, which can be taken to be the xx , zz , and xy components, for instance, and can be used to fix $K_{zz}(z)$, $K_{xx}(z) = K_{yy}(z)$, and $V(z)$, where

$$K_{zz} = \frac{1}{2}(\partial_z \phi)^2, \quad K_{xx} = \frac{1}{2}\Phi(z)k_1^2, \quad K_{yy} = \frac{1}{2}\Phi(z)k_2^2. \tag{3.14}$$

The solution of the equations of motion is then

$$\begin{aligned} K_{zz}(z) &= 0, \\ K_{xx}(z) &= -\frac{2\alpha^2 z^2 (h^2 + q^2)(hW(z) + q(Z_0 - 1))}{q}, \\ V(z) &= \frac{2hz^4 (h^2 + q^2)W(z) + 2q(z^4(Z_0 - 1)(h^2 + q^2) - 3)}{L^2 q}. \end{aligned} \tag{3.15}$$

Note that, strictly speaking, the above solution means that ϕ is constant, so Φ , V , Z , and W should have been constant as well. Except, of course, in a correct solution, we should have varied $\Phi(\phi)$, $V(\phi)$, $Z(\phi)$, and $W(\phi)$ with the chain rule to obtain the correct dilaton equation of motion, which was not done here.

So the above must be thought of as a simple toy model for the correct case. We have fixed $Z(z)$, $\Phi(z)$, and $V(z)$ in terms of the independent $W(z)$ from the equations of motion, considered as the only variable, set by hand.

3.2 Fluctuations

To calculate the quadratic action for fluctuations around the background solution, and find the holographic Green's functions and the transport coefficients through Kubo formulas, we proceed as in the previous section.

We add fluctuations to the off-diagonal metric and gauge field in the spatial boundary directions,

$$\begin{aligned} g_{tx} &= \frac{\alpha \epsilon G_x(z) e^{-it\omega}}{z^2}, \\ g_{ty} &= \frac{\alpha \epsilon G_y(z) e^{-it\omega}}{z^2}, \\ \delta A_x &= \epsilon A_x(z) e^{-it\omega}, \\ \delta A_y &= \epsilon A_y(z) e^{-it\omega}, \end{aligned} \tag{3.16}$$

and consider the same ansatz with infalling boundary conditions at the horizon,

$$\begin{aligned} A_x(z) &= f(z)^\nu a_x(z), \\ G_x(z) &= f(z)^{(1+\nu)} g_x(z). \end{aligned} \tag{3.17}$$

At linear level, we now obtain the equations of motion

$$\begin{aligned} i\bar{\omega} G'_x(z) &= -\frac{4z^2(hW(z) + qZ_0)}{q} [-iq\bar{\omega}A_x(z) \\ &\quad + h(z-1)A'_y(z)\{z[z(h^2 + q^2) - 1] - 1\} + hqG_y(z)], \\ &\quad q(z-1)\{z[z(h^2 + q^2) - 1] - 1\} \times \\ &\quad \times [4z^3 A'_x(z)(hW(z) + qZ_0) - 2G'_x(z) + zG''_x(z)] \\ &\quad = 4ih\bar{\omega}z^3 A_y(z)(hW(z) + qZ_0) + 4qz^3 G_x(z)(h^2 - hqW(z) - q^2(Z_0 - 1)). \end{aligned} \tag{3.18}$$

We expand again the solutions in orders of $\bar{\omega}$:

$$\begin{aligned} a_x(z) &= a_x^0(z) + \bar{\omega}a_x^1(z) + \dots \\ g_x(z) &= g_x^0(z) + \bar{\omega}g_x^1(z) + \dots \end{aligned} \tag{3.19}$$

We solve the equations of motion order by order in ϵ and in $\bar{\omega}$, as before, focusing on the linearized perturbations (first order in ϵ). At zeroth order in $\bar{\omega}$, we obtain

$$\begin{aligned} a_x^{0r}(z) &= -qg_x^0(z), \\ g_x^{0r}(z) &= -\frac{2g_x^{0r}(z)(3h^2z^4 - 2h^2z^3 + 3q^2z^4 - 2q^2z^3 - 2z^3 - 1)}{(z-1)z(h^2z^3 + q^2z^3 - z^2 - z - 1)} \\ &= -2\frac{\psi'(z)}{\psi(z)}g_x^{0r}(z), \end{aligned} \tag{3.20}$$

which matches what we got in the previous section for constant W , as expected. So the same solution as before is also valid now,

$$a_x^0(z) = \alpha_x - qz\gamma_x g_x^0(z) = \gamma_x, \tag{3.21}$$

where γ_x and α_x are constants.

At first order in $\bar{\omega}$, we obtain the equation for a_x

$$a_x^{1r}(z) = \frac{\theta(z) + \gamma(z) + \psi(z)}{4h(z-1)z^2(z(z(h^2 + q^2) - 1) - 1)(hW(z) + qZ_0)}, \tag{3.22}$$

where we have defined the functions

$$\begin{aligned} \theta(z) &= -4qh(z-1)z^2g_x^1(z)(z(z(h^2 + q^2) - 1) - 1)(hW(z) + qZ_0), \\ \gamma(z) &= q\frac{4iz^2(hW(z) + qZ_0)}{q(h^2 + q^2 - 3)}(h^3z^2(4z-3)(\gamma_x qz - \alpha_x) + h^2q(\gamma_y qz - \alpha_y) + \\ &\quad + hz^2(q^2(4z-3) - 3)(\gamma_x qz - \alpha_x) + q(q^2 - 3)(\gamma_y qz - \alpha_y)), \\ \psi(z) &= -qi\gamma_y z^2(4z(h^2 + q^2) - 3(h^2 + q^2 + 1)). \end{aligned} \tag{3.23}$$

There is also a similarly complicated equation for g_x^1 . But it turns out that we can write both these equations (for a_x and g_x^1) in the same form as we did in the previous section,

$$a_x^1(z) + qg_x^1(z) = \mathcal{A}_x^0(z),$$

$$g_x^{1''}(z) + 2\frac{\psi'(z)}{\psi(z)}g_x^1(z) = \mathcal{G}_x^0(z), \tag{3.24}$$

except with different \mathcal{G}_x^0 and $\mathcal{A}_x^0(z)$. Solving the equations in the same way (see [21]), we obtain

$$g_x^1(z) = \frac{c_2}{\psi(z)^2} + \frac{1}{\psi(z)^2} \int_0^z \mathcal{G}_x^0(u)\psi(u)^2 du. \tag{3.25}$$

This again blows up at $z = 1$, so we must impose regularity by putting $c_2 = 0$, and

$$\int_0^1 \mathcal{G}_x^0(u)\psi(u)^2 du = 0, \tag{3.26}$$

which relates γ_x to the other integration constants,

$$\gamma_x = -\frac{(h^2 + q^2 - 3)(\gamma_y q(2Z_0(h^2 + q^2) + h^2 + q^2 + 3) - 4\alpha_y Z_0(h^2 + q^2))}{3h(h^2 + q^2 + 1)} + \frac{i(h^2 + q^2 - 3)}{3(h^2 + q^2 + 1)} \int_0^1 -\frac{4i(h^2 + q^2)W(z)(\gamma_y qz - \alpha_y)}{q} dz, \tag{3.27}$$

$$V(z) = \frac{2(hz^4 W(z)(h^2 + k_x k_y q^2) + q\sqrt{k_x k_y}(h^2 z^4 Z_0 - k_x k_y(z^4(h^2 - q^2 Z_0 + q^2) + 3)))}{L^2 q(k_x k_y)^{3/2}}, \tag{3.32}$$

$$K_{xx}(z) = -\frac{2\alpha^2 k_x z^2 (W(z)(h^3 + hk_x k_y q^2) + q\sqrt{k_x k_y}(h^2(Z_0 - k_x k_y) + k_x k_y q^2(Z_0 - 1)))}{q(k_x k_y)^{3/2}},$$

$$K_{yy} = -\frac{2\alpha^2 k_y z^2 (W(z)(h^3 + hk_x k_y q^2) + q\sqrt{k_x k_y}(h^2(Z_0 - k_x k_y) + k_x k_y q^2(Z_0 - 1)))}{q(k_x k_y)^{3/2}}.$$

and similarly for γ_y . Therefore, we have the solution

$$a_x^1(z) = \alpha_x - q \int_0^z g_x^1(u) du - i \int_0^z \mathcal{A}_x^0(u) du,$$

$$g_x^1(z) = \gamma_x - i \int_0^z \frac{1}{\psi(u)^2} \gamma_x \mathcal{W}(u) du, \tag{3.28}$$

where $\mathcal{W}(u)$ is a function depending on the parameters of the solution, which we write, together with $\mathcal{A}_x^0(u)$, in Appendix 1.

Next, one would need to calculate the quadratic action as we did in (2.61) and (2.63), and extract the transport coefficients, but it is now too involved (one could do numerics for it, but we leave that for further work).

Note that, as before, besides the solutions with ansatz (3.17), with $\pm v$, we also have a constant solution,

$$A_x = \delta_x, \quad G_x = -\frac{i\bar{\omega}}{h}\delta_x,$$

$$A_y = \delta_y, \quad G_y = \frac{i\bar{\omega}}{h}\delta_y. \tag{3.29}$$

The solutions become near the boundary at $z = 0$

$$A_x^0 = \delta_x + \alpha_x(\gamma_x, \gamma_y),$$

$$G_x^0 = -\frac{i\bar{\omega}}{h}\delta_x + \gamma_x,$$

$$A_y^0 = \delta_y + \alpha_y(\gamma_x, \gamma_y),$$

$$G_y^0 = \frac{i\bar{\omega}}{h}\delta_y + \gamma_y. \tag{3.30}$$

3.3 Anisotropy

We can introduce anisotropy as in the previous section, via a metric ansatz with $k_x \neq k_y$,

$$\frac{1}{L^2} ds^2 = -\frac{\alpha^2}{z^2} f(z) dt^2 + \frac{\alpha^2}{z^2} (k_x dx^2 + k_y dy^2) + \frac{1}{z^2} \frac{dz^2}{f(z)}, \tag{3.31}$$

and now it is not trivial anymore. The Einstein equations give now different values for K_{xx} and K_{yy} , and we find

Repeating the procedure from the previous subsection, we obtain the γ_x and γ_y given in Appendix 2.

3.4 Conductivity from Entropy Function

Considering the same $AdS_2 \times \mathbb{R}^2$ ansatz for the near-horizon metric of the planar extremal black hole in four dimensions,

$$ds^2 = -\frac{v}{z^2} (dt^2 - dz^2) + w(dx^2 + dy^2), \tag{3.33}$$

we compute as before the boundary spatial integral of the Lagrangian density,

$$f = \int dx dy \int d^4 x \sqrt{-g} \mathcal{L}, \tag{3.34}$$

and finally Sen's entropy function, which becomes

$$\mathcal{E} = \frac{\pi}{\alpha^2 v w z^2} \Gamma + \frac{\pi}{\alpha^2 e_A v w^2 z^2} \Theta, \tag{3.35}$$

where

$$\begin{aligned} \Gamma = & 4\alpha^4 h^2 v w z^2 (w^2 - L^2 v Z_0) - 4e^2 w^2 z^4 (v - L^2 Z_0) + \\ & + \alpha^2 (L^2 Z_0 (-B^2 v^2 + e^2 w^2 z^2 (z^2 - 4w) + 4h^2 v^2 z^4) + \\ & + v w (2e z^2 (2e w^2 + Z_0) - 4h^2 w z^4 - 13w)) \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} \Theta = & \alpha L^2 v W(z) (e^2 w^2 z^2 (-2\alpha B + \alpha^2 h (z^2 - 4w) + 4h z^2) + \\ & - \alpha^2 h v^2 (B^2 - 4h^2 z^2 (z^2 - \alpha^2 w))). \end{aligned} \tag{3.37}$$

The attractor equations are, as before

$$\frac{\partial \mathcal{E}}{\partial v} = 0, \quad \frac{\partial \mathcal{E}}{\partial w} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_A} = 0. \tag{3.38}$$

The full solutions for v , w , and e are too big to be shown here, though it should be possible to find them numerically.

3.5 S-duality

One important reason to consider the more general action (3.1) is S-duality. As explained in the previous section, such an action is manifestly invariant under S-duality acting on $Z(\phi)$ and $W(\phi)$.

However, we considered the ‘‘toy model’’ with only z dependence for $W(z)$, $Z(z)$, $V(z)$, and $\Phi(z)$, so we need to check S-duality on the solutions. Of course, we see that the Maxwell Eq. (3.11) is indeed S-duality invariant, but in order to have the solutions be as well, we need that $Z_0 = 0$ in (3.12).

Since $W(z)$ and $Z(z)$ are functions of the radial coordinate z , now we have to ask: at what position z is the action of S-duality relevant to transport coefficients to be considered? On the one hand, by virtue of calculating the holographic Green’s functions and using the Kubo formulas at the boundary, that is where it seems we should consider them. But on the other hand, the conductivity calculated from Sen’s entropy function in the attractor mechanism is obtained at the horizon, so that is where it seems to be needed in this case.

The two calculations are related by the application to AdS/CFT of the membrane paradigm, as done by Iqbal and

Liu, and as shown for instance in [15], but in the case of our toy model, that is guaranteed by the fact that $Z(z)$, as well as $V(z)$ and $\Phi(z)$, is related to $W(z)$, and $Z(z)$ is related to it via a duality-invariant proportionality relation,

$$Z(z) = \frac{h}{q} W(z), \tag{3.39}$$

which is one reason why the toy model set-up is a sensible one.

4 Conclusions

In this paper, we have considered the introduction of a topological term $W F_{\mu\nu} \tilde{F}^{\mu\nu}$ in the 3 + 1 dimensional gravitational action for the Einstein-Maxwell model used to holographically calculate transport coefficients in strongly coupled 2 + 1 dimensional materials, using a dyonic black hole background that asymptotes to AdS space.

Considering first a constant W , we have found that the results match the general results in [15] from the calculation in an a priori unknown metric, at the horizon of the black hole. Using also the attractor mechanism and Sen’s entropy function, the transport coefficients were written in terms of $\rho = \tilde{Q}$, B , and W , and we have found the action of S-duality on them.

A toy model for the complicated case of a general solution with nontrivial $W(\phi)$, $Z(\phi)$, $V(\phi)$, and $\Phi(\phi)$ was the case with functions of the radial coordinate z , $W(z)$, $Z(z)$, $V(z)$, and $\Phi(z)$. We obtained the solutions for fluctuations, which were very complicated, so we did not proceed to find the transport coefficients, though those could be found numerically from our results. We have also shown how to introduce anisotropy in this case, and found the solutions for fluctuations in this case. We have set up the case of the attractor mechanism and Sen’s entropy function, again leaving the complete (and very ugly) formulas for later. S-duality arguments sharpened the idea of the toy model, as well as its relevance.

There are many things left for further work, for instance the numerical evaluation of the transport coefficients in the toy model with z dependence. This is also viewed as a first step towards a case based on a more complete solution, involving not only a dyonic black hole, but nontrivial $\phi(z)$ and $W(\phi)$, $Z(\phi)$, $V(\phi)$, and $\Phi(\phi)$.

Solutions for Fluctuations in the Case of $W(z)$

$$\begin{aligned} \mathcal{W}(u) = & -\frac{i}{\gamma_x} \int_0^u -i \left(\frac{4(h^2 + q^2)W(z)(\gamma_y qz - \alpha_y)}{q} \right) dz \\ & - u \frac{\gamma_x h^3 (u^2 (2q^2(u-1)(4u-3) - 7u + 6) + 4)}{\gamma_x h (h^2 + q^2 - 3)} \\ & - u \frac{2h^2 (\gamma_y q^3 (3 - 2u) + (2q^2 - 3)Z_0 (\gamma_y qu - 2\alpha_y) + 3\gamma_y q(u-1))}{\gamma_x h (h^2 + q^2 - 3)} \\ & - u \frac{h^4 (\gamma_y q (2u(Z_0 - 1) + 3) - 4\alpha_y Z_0) + \gamma_x h^5 (u-1)u^2 (4u-3)}{\gamma_x h (h^2 + q^2 - 3)} \\ & - u \frac{\gamma_x h (q^2 (u^2 (q^2(u-1)(4u-3) - 7u + 6) + 4) + 3u^2)}{\gamma_x h (h^2 + q^2 - 3)} \\ & - u \frac{q(q^2 - 3) (3\gamma_y + \gamma_y q^2 (2u(Z_0 - 1) + 3) - 4\alpha_y q Z_0)}{\gamma_x h (h^2 + q^2 - 3)}. \end{aligned} \tag{A.1}$$

$$\begin{aligned} \mathcal{A}_x^0(u) = & i \frac{(h^3 z^2 (4z - 3)(\gamma_x qz - \alpha_x) + h^2 q (\gamma_y qz - \alpha_y))}{h(z-1)(h^2 + q^2 - 3)(z(z(z(h^2 + q^2) - 1) - 1) - 1)} \\ & + i \frac{h z^2 (q^2 (4z - 3) - 3)(\gamma_x qz - \alpha_x) + q(q^2 - 3)(\gamma_y qz - \alpha_y)}{h(z-1)(h^2 + q^2 - 3)(z(z(z(h^2 + q^2) - 1) - 1) - 1)} \\ & + \frac{i \gamma_y q (h^2 (3 - 4z) + q^2 (3 - 4z) + 3)}{4h(z-1)(z(z(z(h^2 + q^2) - 1) - 1) - 1)(hW(z) + qZ_0)} \end{aligned} \tag{A.2}$$

and similarly for $\mathcal{A}_y^0(u)$.

Solution for Fluctuations for the Anisotropic Model

$$\begin{aligned} \gamma_y = & \frac{(h^2 + q^2 - 3)}{3hk_x (h^2 + q^2 + 1)} \left(-ihk_x \int_0^1 \frac{4ik_y W(z)(h^2 + k_x k_y q^2)(\gamma_x qz - \alpha_x)}{q(k_x k_y)^{3/2}} dz \right. \\ & \left. - 4\alpha_x Z_0 (h^2 + k_x k_y q^2) \right. \\ & \left. + \gamma_x q (k_x k_y (h^2 + 2q^2 Z_0 + q^2 + 3) + 2h^2 Z_0) \right) \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} \gamma_x = & \frac{(h^2 + q^2 - 3)}{3hk_y (h^2 + q^2 + 1)} \left(ihk_y \int_0^1 \frac{4ik_x W(z)(h^2 + k_x k_y q^2)(\gamma_y qz - \alpha_y)}{q(k_x k_y)^{3/2}} dz \right. \\ & \left. - 4\alpha_y Z_0 (h^2 + k_x k_y q^2) \right. \\ & \left. + \gamma_y q (k_x k_y (h^2 + 2q^2 Z_0 + q^2 + 3) + 2h^2 Z_0) \right) \end{aligned} \tag{B.2}$$

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Declarations

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