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Star countable spaces and ω-domination of discrete subspaces

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Abstract We establish that a first countable ω-monolithic space is star countable if and only if it has countable extent. A consistent example is given of a first countable normal star Lindelöf space of uncountable extent. Under the continuum hypothesis we prove that for any compact K, the space $C_p(K)$ is star countable if and only if it is Lindelöf. The above-mentioned results answer several published open questions.

Keywords ω-domination of discrete subsets · Star countable space · Compact space · Extent · Star Lindelöf space · Function space

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1 Introduction

A space X is called *star* $\mathcal{P}$ if for any open cover $\mathcal{U}$ of X, there exists a set $Y \subset X$ such that $\mathrm{St}(Y, \mathcal{U}) = X$ and the set Y (called the kernel of $\mathcal{U}$) has $\mathcal{P}$; here $\mathcal{P}$ is a topological property. The concept of star $\mathcal{P}$ space was introduced by Ikenaga in his paper [12] where he studied the cases of star countable, star Lindelöf and star σ-compact spaces. Star $\mathcal{P}$ properties were studied systematically in the survey of Matveev [16]. The paper [1] contains some results on star $\mathcal{P}$ spaces for compactness-like properties $\mathcal{P}$.

It is a well-known fact that every space is star discrete (and hence star metrizable) because for any cover $\mathcal{U}$ of a space X, there exists a closed discrete set $D \subset X$ such that $\mathrm{St}(D, \mathcal{U}) = X$. For some concrete classes $\mathcal{P}$, the star $\mathcal{P}$ properties were studied in the papers [4,7,10–12] and [13] with distinct terminology in each one. In particular, star countable spaces were called "star Lindelöf", "spaces of countable weak extent", "ω-star" and "$*$Lindelöf". However, after the paper [17] was published, all subsequent authors adopted its terminology so the term "star $\mathcal{P}$" can be now considered standard.

In the paper [1] star countable and star Lindelöf P-spaces were studied and it was shown that in the presence of normality, all these classes coincide with the class of spaces of countable extent. It was also established in [1] that $\mathbb{R}^\kappa$ is not star countable for any $\kappa \geq 2^{\mathfrak{c}^+}$. One of the sources of inspiration of the authors of [1] was Arhangel'skii's problem cited by Bonanzinga and Matveev in [4, Question 2.2.4]; Arhangel'skii asked whether for every compact space X, star countability of $C_p(X)$ is equivalent to its Lindelöf property. In [1] it was established that the answer is positive for ω_1-monolithic compact spaces. In this paper we show that under the Continuum Hypothesis, the answer is positive for all compact spaces.

2 Notation and terminology

If nothing is said about the axioms of separation of a space X, then X is assumed to be a T_1-space. Given a space X, the family $\tau(X)$ is its topology; if $x \in X$ then $\tau(x, X) = \{U \in \tau(X) : x \in U\}$. Suppose that $\mathcal{A}$ is a family of subsets of X; then $\mathrm{St}(Y, \mathcal{A}) = \bigcup\{A \in \mathcal{A} : Y \cap A \neq \emptyset\}$ for any $Y \subset X$. We denote by $\mathbb{R}$ the real line with its natural topology.

Our set-theoretic notation is standard; in particular, any ordinal is identified with the set of its predecessors. If κ is an infinite cardinal and A is a set, then let $[A]^{\leq\kappa} = \{B \subset A : |B| \leq \kappa\}$. Given a space X the cardinal $L(X) = \min\{\kappa$: every open cover of X has a subcover of cardinality $\leq \kappa\} + \omega$ is called *the Lindelöf number* of X. Besides, $hl(X) = sup\{L(Y) : Y \subset X\}$.

If X is a space and $\mathcal{U}$ is an open cover of X then a set $Y \subset X$ is a *kernel* of $\mathcal{U}$ if $\mathrm{St}(Y, \mathcal{U}) = X$. Suppose that $\mathcal{P}$ is a topological property; a space X is called *star* $\mathcal{P}$ if any open cover $\mathcal{U}$ of the space X has a kernel Y with the property $\mathcal{P}$. For an infinite cardinal κ, a space X is called κ*-monolithic* if $nw(\overline{A}) \leq \kappa$ for any set $A \subset X$ with $|A| \leq \kappa$. If X is a space then Δ usually denotes its diagonal $\{(x, x) : x \in X\}$; however, the cardinal $\Delta(X) = \min\{\kappa :$ the diagonal Δ is the intersection of κ-many open subsets of $X \times X\}$ is the diagonal number of X.

For any space X the *extent* of X (also denoted as $ext(X)$) is the sum of ω and the supremum of cardinalities of closed discrete subsets of X. Let $s(X) = \sup\{|D| : D$ is a discrete subset of $X\} + \omega$; the cardinal $s(X)$ is *the spread* of X. The sum of ω and the minimal cardinality of a local base at a point $x \in X$ is called the *character of* X *at* x; it is denoted by $\chi(x, X)$ and $\chi(X) = \sup\{\chi(x, X) : x \in X\}$. If X is a space and $x \in X$ then let

$\psi(x, X) = \min\{|\mathcal{U}| : \mathcal{U} \subset \tau(X) \text{ and } \bigcap \mathcal{U} = \{x\}\} + \omega$ and $\psi(X) = \sup\{\psi(x, X) : x \in X\}$; the cardinal $\psi(X)$ is called the *pseudocharacter* of the space X. Given an infinite cardinal κ we say that $t(X) \leq \kappa$ if, for any $A \subset X$ and $x \in \overline{A}$ there exists a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \overline{B}$.

We say that X is a *P-space* if every G_δ-subset of X is open in X. A family $\mathcal{N}$ is a network of X if every $U \in \tau(X)$ is the union of some subfamily of $\mathcal{N}$. Let $nw(X) = \min\{|\mathcal{N}| : \mathcal{N}$ is a network in $X\} + \omega$. The spaces with a countable network are called *cosmic*.

Given spaces X and Y, we denote by $C(X, Y)$ the set of all continuous functions from X to Y; we write $C(X)$ instead of $C(X, \mathbb{R})$. The space $C_p(X, Y)$ is the set $C(X, Y)$ endowed with the pointwise convergence topology.

The rest of our notation is standard and follows the book [8]; the definitions of cardinal invariants can be consulted in the survey of Hodel [9]. All necessary facts and notions of C_p-theory can be found in the books [22–24].

3 Countable domination of discrete subspaces

We will define and study a property called ω-domination of discrete sets. This concept, which is stronger than star countability, will enable us to solve an open question published in [1]. Besides, we will see that spaces in which discrete sets are ω-dominated, are interesting in themselves.

Definition 3.1 Given a space X, we will say that *a class $\mathcal{A}$ of subsets of X is dominated by a class $\mathcal{B}$* if for any set $A \subset X$ with $A \in \mathcal{A}$, there exists a set $B \subset X$ such that $B \in \mathcal{B}$ and $A \subset \overline{B}$. In particular, all (closed) discrete subsets of X are countably dominated (which we frequently abbreviate as ω-dominated) if, for any (closed) discrete set $D \subset X$, there exists a countable set $B \subset X$ such that $D \subset \overline{B}$.

The following proposition shows why domination of discrete subspaces is worth studying.

Proposition 3.2 *Suppose that X is a space in which Lindelöf* (*countable*) *subsets dominate closed discrete subsets. Then X is star Lindelöf* (*countable*).

Proof If $\mathcal{U}$ is an open cover of X, then there exists a closed discrete set $D \subset X$ such that $X = \mathrm{St}(D, \mathcal{U})$. Take a Lindelöf (countable) set $L \subset X$ such that $D \subset \overline{L}$. Then $\mathrm{St}(L, \mathcal{U}) = X$ and hence X is star Lindelöf (countable). □

It is easy to see that countable domination of closed discrete sets is strictly stronger than star countability. However, there is an important class of spaces in which they coincide.

Proposition 3.3 *A first countable space X is star Lindelöf* (*countable*) *if and only if Lindelöf* (*countable*) *subsets of X dominate closed discrete subsets of X.*

Proof We must only prove necessity, so assume that X is star Lindelöf (countable) and D is a closed discrete subset of X. Fix a countable decreasing local base $\{O_d^n : n \in \omega\}$ at the point $d \in D$ in such a way that $O_d^0 \cap D = \{d\}$ for every $d \in D$.

For the open cover $\mathcal{U}_n = \{O_d^n : d \in D\} \cup \{X \backslash D\}$ of the space X we can find a Lindelöf (countable) set $L_n \subset X$ such that $X = \mathrm{St}(L_n, \mathcal{U}_n)$ and hence $L_n \cap O_d^n \neq \emptyset$ for any $d \in D$. The set $L = \bigcup_{n \in \omega} L_n$ is Lindelöf (countable) and $L \cap O_d^n \neq \emptyset$ for any $d \in D$ and $n \in \omega$. Therefore $D \subset \overline{L}$. □

Theorem 3.4 *A first countable ω-monolithic space X is star countable if and only if X has countable extent.*

Proof Only necessity has to be established, so assume that X is a first countable ω-monolithic star countable space. If $D \subset X$ is a closed discrete subset of X, then we can apply Proposition 3.3 to find a countable set $A \subset X$ such that $D \subset \overline{A}$. By ω-monolithity of X, the set $\overline{A}$ is cosmic and hence $|D| \leq s(\overline{A}) \leq nw(\overline{A}) = \omega$. This proves that $ext(X) \leq \omega$. □

The following corollary answers Question 2.2 from the paper [1].

Corollary 3.5 *If X is strongly monotonically monolithic and star countable, then X is Lindelöf.*

Proof Just observe that X is first countable and ω-monolithic so $ext(X) = \omega$ by Theorem 3.4. Since in monotonically monolithic spaces the extent coincides with the Lindelöf number (see [21, Theorem 2.14]), the space X must be Lindelöf. □

It is Question 2.10 of the paper [25] where the authors ask whether a first countable star Lindelöf normal space must have countable extent. The following example gives a consistent negative answer.

Example 3.6 Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of ω. Then $M(\mathcal{A}) = \omega \cup \mathcal{A}$ is the Mrowka space in which all points of ω are isolated and the local base at $A \in \mathcal{A}$ is given by all sets $\{A\} \cup (A \backslash F)$ where F is a finite subset of ω. The space $M(A)$ is even star countable being separable and the set $\mathcal{A}$ is closed and discrete in $M(\mathcal{A})$. It was proved in [19] that there exists an uncountable family $\mathcal{A}$ such that $M(\mathcal{A})$ is normal if and only if there exists a Q-set in $\mathbb{R}$. Since the existence of a Q-set in $\mathbb{R}$ is consistent with ZFC, it is consistent that there exists an uncountable normal Mrowka space and hence there exists a first countable star Lindelöf normal space of uncountable extent.

The following theorem gives a consistent answer to Question 2.2.4 from the paper [4].

Theorem 3.7 *Suppose that the Continuum Hypothesis (CH) holds and K is a compact Hausdorff space such that $C_p(K)$ is star countable. Then the space K is ω_1-monolithic and hence $C_p(K)$ is Lindelöf.*

Proof We will show first that $t(K) \leq \omega$. Striving for a contradiction, assume that the tightness of X is uncountable. By Theorem 1.2 of [14] there exists a free sequence $S = \{x_\alpha : \alpha < \omega_1\} \subset X$ that converges to a point $p \in X$. Let $S_\alpha = \{x_\beta : \beta < \alpha\}$ and $Q_\alpha = \overline{S_\alpha}$ for every $\alpha < \omega_1$. It follows from CH that $w(Q_\alpha) \leq 2^\omega = \omega_1$ for each $\alpha < \omega_1$.

Take any point $x \in \overline{S} \backslash \{p\}$; since S converges to p, there exists $\alpha < \omega_1$ such that $x \notin \overline{S \backslash S_\alpha}$ and therefore $x \in Q_\alpha$. This proves that we have the equality $\overline{S} = \bigcup \{Q_\alpha : \alpha < \omega_1\} \cup \{p\}$ and hence the compact space $L = \overline{S}$ is represented as the union of ω_1-many subspaces of weight not exceeding ω_1. Therefore $w(L) \leq \omega_1$ by Theorem 2.1.11 of [2].

If $\pi_L : C_p(K) \to C_p(L)$ is the restriction map, then π_L is continuous and $C_p(L) = \pi_L(C_p(K))$. Since star countability is invariant under continuous images, the space $C_p(L)$ is star countable. It follows from $w(L) \leq \omega_1$ that L is ω_1-monolithic so $C_p(L)$ is Lindelöf by Theorem 1.35 of [1]. Therefore $t(L) \leq \omega$ (see [22, Problem 189]); this contradiction with the fact that S is an uncountable free sequence in L proves that $t(K) \leq \omega$.

Finally, take any set $D \subset K$ with $|D| \leq \omega_1$; let $\{d_\alpha : \alpha < \omega_1\}$ be an enumeration of D. The set $D_\alpha = \{d_\beta : \beta < \alpha\}$ is countable and hence $F_\alpha = \overline{D_\alpha}$ is a separable space;

applying CH again we convince ourselves that $w(F_\alpha) \le \omega_1$ for each $\alpha < \omega_1$. It follows from $t(K) \le \omega$ that $\overline{D} = \bigcup\{F_\alpha : \alpha < \omega_1\}$ and hence the compact space D is represented as the union of $\le \omega_1$-many subspaces of weight $\le \omega_1$. This shows that $w(\overline{D}) \le \omega_1$, i.e., the space K is ω_1-monolithic and hence we can apply [1, Theorem 1.35] once more to see that $C_p(K)$ is Lindelöf. □

It was asked in [1, Question 2.7] whether star countability of $C_p(K)$ for a compact space K implies that $t(K) \le \omega$. Theorem 3.7 easily implies that we have a positive answer under CH.

Corollary 3.8 *Assume that CH holds and K is a compact Hausdorff space such that $C_p(K)$ is star countable. Then K has countable tightness.*

Proof Apply Theorem 3.7 to see that $C_p(K)$ is Lindelöf and hence $t(K) \le \omega$ by [22, Problem 189]. □

We now turn to a systematic study of countable domination of discrete subspaces. Proposition 3.3 shows that it is an important concept; we hope to convince the reader that this notion has nice categorical properties and hence is interesting in itself. The following statement is evident

Proposition 3.9 *If either $d(X) \le \omega$ or $s(X) \le \omega$, then all discrete subsets of X are countably dominated.*

Therefore the class of spaces whose discrete subsets are ω-dominated is a generalization of separable spaces and spaces of countable spread.

Proposition 3.10 *If X is an ω-monolithic space in which all discrete subsets are ω-dominated, then $s(X) \le \omega$.*

Proof If D is a discrete subset of X, then $D \subset \overline{A}$ for some countable $A \subset X$. By ω-monolithity of X, we have $s(\overline{A}) \le nw(\overline{A}) \le \omega$ and therefore $|D| \le \omega$. □

Proposition 3.11 *Suppose that X is a space in which discrete subsets are ω-dominated. Then*

(a) *discrete subsets of U are ω-dominated for any $U \in \tau(X)$;*
(b) *discrete subsets are ω-dominated in any continuous image of X;*
(c) *the space X has the Souslin property;*
(d) *if $Y \subset X = \overline{Y}$, and $t(X) \le \omega$, then discrete subsets are ω-dominated in Y.*

Proof If U is an open subset of X and $D \subset U$ is discrete, then there exists a countable set $A \subset X$ such that $D \subset \overline{A}$. Then the set $B = A \cap U \subset U$ is countable and $D \subset \mathrm{cl}_U(B)$; this proves (a).

(b) Take a continuous onto map $f : X \to Y$; if $E \subset Y$ is a discrete set, then take a point $x_y \in f^{-1}(y)$ for every $y \in E$. The set $D = \{x_y : y \in E\}$ is easily seen to be discrete so there exists a countable set $A \subset X$ such that $D \subset \overline{A}$. Then $B = f(A)$ is a countable subset of Y and $E \subset \overline{B}$, i.e., all discrete subsets of Y are ω-dominated.
(c) If $\mathcal{U} \subset \tau^*(X)$ is an uncountable disjoint family, then pick a point $x_U \in U$ for every $U \in \mathcal{U}$. The set $D = \{x_U : U \in \mathcal{U}\}$ is clearly discrete. If $A \subset X$ is countable, then $A \cap U = \emptyset$ for some $U \in \mathcal{U}$ and hence $x_U \in D \backslash \overline{A}$. This shows that D is not contained in the closure of any countable subset of X which is a contradiction.

(d) If $D \subset Y$ is a discrete subspace of Y, then there is a countable set $E \subset X$ such that $D \subset \overline{E}$. It follows from $t(X) \leq \omega$, that there exists a countable set $H \subset Y$ such that $E \subset \overline{H}$ and hence $D \subset \mathrm{cl}_Y(H)$.

□

Proposition 3.12 *If X is a regular space in which all discrete subsets are ω-dominated, then $hl(X) \leq \mathfrak{c}$ and hence $|X| \leq 2^{\mathfrak{c}}$.*

Proof If D is a discrete subspace of X, then $D \subset \overline{A}$ for some countable set A. Therefore $\overline{D} \subset \overline{A}$ and hence $w(\overline{D}) \leq w(\overline{A}) \leq \mathfrak{c}$. In particular, $|D| \leq w(\overline{A}) \leq \mathfrak{c}$; this proves that $s(X) \leq \mathfrak{c}$ and $w(\overline{D}) \leq \mathfrak{c}$ for any discrete $D \subset X$.

If $hl(X) > \mathfrak{c}$, then there exists a scattered subspace Y in the space X such that $|Y| = \mathfrak{c}^+$; if D is the set of isolated points of Y, then D is discrete and $Y \subset \overline{D}$. Take a countable set $B \subset X$ such that $D \subset \overline{B}$. Then $Y \subset \overline{B}$ and $hl(\overline{B}) \leq w(\overline{B}) \leq \mathfrak{c}$; this contradiction with the fact that Y is a scattered subset of $\overline{B}$ of cardinality $\mathfrak{c}^+$ proves that $hl(X) \leq \mathfrak{c}$. □

Corollary 3.13 *If X is a space in which all discrete subsets are ω-dominated and $\chi(X) \leq \omega$, then $|X| \leq \mathfrak{c}$.*

Proof This follows from Proposition 3.11(c) and the Hajnal–Juhasz inequality $|X| \leq 2^{\chi(X)\cdot c(X)}$. □

Corollary 3.14 *If X is a compact Hausdorff space in which all discrete subsets are ω-dominated, then $w(X) \leq \mathfrak{c}$.*

Proof By Proposition 3.12, we have $\chi(X) = \psi(X) \leq hl(X) \leq \mathfrak{c}$. Since also $c(X) \leq \omega$ by Proposition 3.11, Shapirovsky's inequality $w(X) \leq \pi\chi(X)^{c(X)}$ (see Theorem 6.2 of [9]) shows that $w(X) \leq \mathfrak{c}^{\omega} = \mathfrak{c}$. □

It would be very interesting to find out whether $|X| \leq \mathfrak{c}$ for any regular space X of countable pseudocharacter in which discrete subsets are ω-dominated. The positive result in this direction would strengthen another Hajnal–Juhasz inequality $|X| \leq 2^{s(X)\cdot\psi(X)}$. So far, we could prove it only for topological groups.

Proposition 3.15 *Suppose that G is a topological group with $\psi(G) \leq \omega$ in which all discrete sets are countably dominated. Then G has a weaker second countable topology and hence $|G| \leq \mathfrak{c}$.*

Proof Apply Theorem 3.3.16 of the book [3] to see that there exists a weaker metrizable topology μ on X. Since the identity map $i : X \to (X, \mu)$ is continuous, it follows from Proposition 3.11(c) that (X, μ) has the Souslin property. The space (X, μ) being metrizable, it must be second countable and therefore $|X| \leq \mathfrak{c}$. □

Theorem 3.16 *The inequality $|X| \leq L(X)^{\Delta(X)}$ holds for any T_1-space X.*

Proof Let $\Delta(X) = \kappa$ and $L(X) = \lambda$. There exists a sequence $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ of open covers of X (called a G_κ-diagonal sequence) such that $\{x\} = \bigcap_{\alpha<\kappa} \mathrm{St}(x, \mathcal{U}_\alpha)$ for any $x \in X$. Observe that if we substitute every $\mathcal{U}_\alpha$ by an open refinement, then the respective family of covers will still be a G_κ-diagonal sequence. Therefore we can consider, without loss of generality, that $|\mathcal{U}_\alpha| \leq \lambda$ for any $\alpha < \kappa$.

If $\mathcal{U} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$, then for each $x \in X$, we can choose a family $\mathcal{V}_x \in [\mathcal{U}]^{\leq\kappa}$ such that $\bigcap \mathcal{V}_x = \{x\}$. Therefore the map $x \mapsto \mathcal{V}_x$ is an injection of X into $[\mathcal{U}]^{\leq\kappa}$ and hence $|X| \leq |[\mathcal{U}]^{\leq\kappa}| \leq (\kappa\cdot\lambda)^\kappa = \lambda^\kappa$. □

Corollary 3.17 *Let X be a regular space in which discrete subsets are countably dominated. If X has a G_δ-diagonal, then $|X| \leq \mathfrak{c}$.*

Proof Apply Proposition 3.12 to see that $L(X) \leq hl(X) \leq \mathfrak{c}$ and therefore $|X| \leq \mathfrak{c}^{\Delta(X)} = \mathfrak{c}^\omega = \mathfrak{c}$ by Theorem 3.16. □

Example 3.18 Let K be the Katetov extension of ω. Then $K = \omega \cup D$ where D is a closed discrete subset of K with $|D| = 2^\mathfrak{c}$ and ω is dense in K. Therefore K is a separable Hausdorff space and hence all discrete subsets of K are countably dominated. Observe that K is the union of countably many closed discrete subspaces of K so its diagonal is a G_δ-set. Since $|K| = hl(K) = 2^\mathfrak{c} > \mathfrak{c}$, this example shows that regularity cannot be omitted in Proposition 3.12 as well as in Corollary 3.17.

Recall that $X \in T_3$ is *a Lindelöf p-space* if there exists a perfect map of X onto a regular second countable space.

Theorem 3.19 *A Lindelöf p-space X is separable if and only if all discrete subsets of $X \times X$ are ω-dominated.*

Proof If X is separable, then so is $X \times X$ so necessity is trivial. Now, if all discrete subsets of $X \times X$ are ω-dominated, then apply Theorem 2.5 of [5] to find a discrete set $D \subset X \times X$ such that $p_1(D)$ is dense in X; here $p_1 : X \times X \to X$ is the projection of $X \times X$ onto its first factor. There exists a countable set $E \subset X \times X$ such that $D \subset \overline{E}$. Then $p_1(E)$ is a countable dense subset of X, i.e., X is separable as promised. □

Example 3.20 Under CH, there exists a hereditarily Lindelöf non-separable compact space K (see [15, Theorem 1.4]). Since $s(K) \leq hl(K) = \omega$, it follows from Proposition 3.9 that all discrete subsets of K are countably dominated and hence, under CH, the conclusion of Theorem 3.19 is false for compact spaces if we replace $X \times X$ with X.

Example 3.21 Under the Continuum Hypothesis there exists a non-separable space X such that X^ω is hereditarily Lindelöf (see Problem 99 of [23]) and hence all discrete subsets of X^ω are ω-dominated. Therefore, under CH, the p-property of the space X cannot be omitted in Theorem 3.19.

Example 3.22 If X is a space in which discrete subsets are countably dominated, then $X \times X$ does not necessarily have this property. Indeed, Moore constructed in [18] a ZFC example of a non-separable hereditarily Lindelöf space X such that $X \times X$ is d-separable, i.e., has a σ-discrete dense subspace. It follows from $s(X) \leq hl(X) = \omega$, that all discrete subsets of X are ω-dominated.

Fix a family $\{D_n : n \in \omega\}$ of discrete subsets of $X \times X$ such that the set $D = \bigcup_{n\in\omega} D_n$ is dense in $X \times X$. If every D_n is contained in the closure of a countable set E_n, then $E = \bigcup_{n\in\omega} E_n$ is a countable dense subset of $X \times X$. This implies that X is separable which is a contradiction. Therefore not all discrete subsets of $X \times X$ are countably dominated.

A function $f : X \to \mathbb{R}$ is called *ω-continuous* if $f|A$ is continuous for every countable $A \subset X$. Recall that a space X has *countable functional tightness* (which is denoted by $t_0(X) \leq \omega$) if every ω-continuous function $f : X \to \mathbb{R}$ is continuous. It is known that $t_0(X) \leq \omega$ if X is separable or has countable tightness (see Problems 418 and 419 of the book [22]). If X is a compact space, then $t(X) \leq s(X)$ and hence countable spread of X implies that X has countable functional tightness. The space X is *weakly discretely generated* if for every non-closed set $A \subset X$, there is a discrete set $D \subset A$ such that $\overline{D} \setminus A \neq \emptyset$.

Proposition 3.23 *If X is a weakly discretely generated space in which all discrete subsets are countably dominated, then $t_0(X) \leq \omega$.*

Proof Fix any ω-continuous function $f : X \to \mathbb{R}$. If f is discontinuous, then there is a closed subset F of $\mathbb{R}$ such that $G = f^{-1}(F)$ is not closed in X. Take a discrete set $D \subset G$ such that $\overline{D} \backslash G \neq \emptyset$ and pick a point $x \in \overline{D} \backslash G$. We can find a countable set $E \subset X$ such that $D \subset \overline{E}$ and hence $\overline{D} \subset \overline{E}$. The function f is continuous on the set $\overline{E}$ by Problem 418 of [22] so $f|\overline{D}$ is continuous as well. This implies $f(x) \in \overline{f(D)} \subset \overline{f(G)} \subset \overline{F} = F$ and therefore $f(x) \in F$ which is a contradiction with $x \notin G = f^{-1}(F)$. □

All compact spaces are weakly discretely generated by [6, Proposition 4.1]), so the following statement is an immediate consequence of Proposition 3.23.

Corollary 3.24 *If X is a compact Hausdorff space in which discrete sets are ω-dominated, then X has countable functional tightness.*

Observation 3.25 If a space X is the union of a separable subspace and a subspace of countable spread, then it is immediate that all discrete subsets of X are ω-dominated. Unfortunately, we could not find an example of a space X whose discrete sets are ω-dominated but X cannot be represented as the union of a separable subspace and a subspace of countable spread. We will see later that if only closed discrete subsets of X are ω-dominated, then X need not be the union of a separable subspace and a subspace of countable extent.

It seems to be much more difficult to prove analogous theorems for the spaces in which only closed discrete sets are countably dominated. We will first formulate some evident properties of this class.

Proposition 3.26 *If either $d(X) \leq \omega$ or $ext(X) \leq \omega$, then all closed discrete subsets of X are countably dominated.*

Proposition 3.27 *If X is an ω-monolithic space in which all closed discrete subsets are ω-dominated, then $ext(X) \leq \omega$.*

Proof If D is a closed discrete subset of X, then $D \subset \overline{A}$ for some countable $A \subset X$. By ω-monolithity of X, we have $ext(\overline{A}) \leq nw(\overline{A}) \leq \omega$ and therefore $|D| \leq \omega$. □

Example 3.28 Let Y be a separable Tychonoff space of uncountable extent, e.g., the square of the Sorgenfrey line. Consider a discrete space D of cardinality ω_1 and let $X = (Y \times D) \cup \{p\}$ where $p \notin Y \times D$. We consider that $Y \times D$ is open in X and has the product topology while the local base at p is the family $\{\{p\} \cup (Y \times (D \backslash F)) :$ the set $F \subset D$ is finite$\}$. We omit an easy proof that X is not representable as the union of a separable subspace and a subspace of countable extent. However, all closed discrete subsets of X are ω-dominated.

The following two statements can be established by straightforward modifications in the proof of Propositions 3.11 and 3.12.

Proposition 3.29 *Suppose that X is a space in which closed discrete subsets are ω-dominated. Then*

(a) *closed discrete subsets of U are ω-dominated for any clopen set $U \in \tau(X)$;*
(b) *closed discrete subsets are ω-dominated in any continuous image of X;*
(c) *any discrete family $\mathcal{U} \subset \tau^*(X)$ is countable.*

Proposition 3.30 *If X is a regular space in which all closed discrete subsets are ω-dominated, then $ext(X) \le \mathfrak{c}$.*

Corollary 3.31 *Suppose that X is a regular space with a G_δ-diagonal in which all closed discrete subsets are ω-dominated. Then $|X| \le 2^{\mathfrak{c}}$.*

Proof It suffices to apply Proposition 3.30 and the Ginsburg–Woods inequality $|X| \le 2^{\Delta(X)\cdot ext(X)}$ to see that $|X| \le 2^{\mathfrak{c}}$. □

Example 3.32 If κ is any uncountable regular cardinal, then it is easy to see that the set $F_\kappa = \{\alpha < \kappa : \mathrm{cof}(\alpha) \le \omega\}$ endowed with the order topology is countably compact and first countable. In particular, $ext(F_\kappa) = \omega$ and hence closed discrete subsets of F_κ are ω-dominated. Since $|F_\kappa| = \kappa$, this example shows that the conclusion of Corollary 3.13 does not hold for spaces in which closed discrete sets are countably dominated.

Theorem 3.33 *Assume that X is a normal space of uncountable extent whose closed discrete subsets are ω-dominated. Then*

(a) $2^{|D|} \le \mathfrak{c}$ *for any closed discrete set $D \subset X$;*
(b) *if there exists a closed discrete set $D \subset X$ such that $|D| = ext(X)$, then $2^{ext(X)} = \mathfrak{c}$;*
(c) *If $|D| < ext(X)$ for any closed discrete set $D \subset X$, then $2^{<ext(X)} = \mathfrak{c}$.*

Proof Let $D \subset X$ be a closed discrete subset of X; take a countable set $E \subset X$ such that $D \subset \overline{E}$. Observe first that $|C(E)| \le \mathfrak{c}$ and by normality of $\overline{E}$ the set $C(D) = \mathbb{R}^D$ is the image of $C(E)$ under the restriction map. Therefore $2^{|D|} = |C(D)| \le |C(E)| = \mathfrak{c}$ and hence $2^{|D|} \le \mathfrak{c}$; this settles (a).

If there exists a closed discrete set $D \subset X$ such that $|D| = ext(X)$, then $2^{ext(X)} = 2^{|D|} \le \mathfrak{c}$ by the property (1). Since D is infinite, we have the equality $2^{ext(X)} = 2^{|D|} = \mathfrak{c}$; this proves (b).

To see that (c) holds, observe that for any infinite $\kappa < ext(X)$, there exists a closed discrete set $D_\kappa \subset X$ with $|D_\kappa| = \kappa$. Apply (a) again to see that $2^\kappa = 2^{|D_\kappa|} = \mathfrak{c}$ and hence $2^{<ext(X)} = \sup\{2^\kappa : \omega \le \kappa < ext(X)\} = \mathfrak{c}$. □

Corollary 3.34 *Suppose that X is a normal space whose closed discrete sets are ω-dominated. If $\mathfrak{c} < 2^{\omega_1}$, then $ext(X) \le \omega$.*

Proof If the extent of X is uncountable, then there exists a closed discrete set $D \subset X$ such that $|D| = \omega_1$. Then $2^{|D|} = 2^{\omega_1} > \mathfrak{c}$ which is a contradiction with Theorem 3.33(a). □

Corollary 3.35 *Suppose that X is a hereditarily normal space in which discrete sets are ω-dominated. If $\mathfrak{c} < 2^{\omega_1}$, then $s(X) \le \omega$.*

Proof If D is a discrete subset of X, then $F = \overline{D}\backslash D$ is closed in X and hence in the space $U = X\backslash F$ all discrete subsets are ω-dominated by Proposition 3.11(a). Since U is a normal space by our hypothesis, we can apply Corollary 3.34 to conclude that $ext(U) \le \omega$. The set D being closed and discrete in U, we have the inequality $|D| \le ext(U) \le \omega$. □

Corollary 3.36 *Suppose that X is a normal space with a G_δ-diagonal whose closed discrete sets are ω-dominated. If $\mathfrak{c} < 2^{\omega_1}$, then $|X| \le \mathfrak{c}$.*

Proof Apply Corollary 3.34 to see that $ext(X) \le \omega$; now it follows from the Ginsburg–Woods inequality $|X| \le 2^{\Delta(X)\cdot ext(X)}$ that $|X| \le \mathfrak{c}$. □

Corollary 3.37 *Suppose that X is a hereditarily normal space with $\psi(X) \leq \omega$ in which discrete sets are ω-dominated. If $\mathfrak{c} < 2^{\omega_1}$, then $|X| \leq \mathfrak{c}$.*

Proof By Corollary 3.35, the space X must have countable spread. Therefore we can apply the Hajnal–Juhasz inequality $|X| \leq 2^{s(X)\cdot\psi(X)}$ to conclude that $|X| \leq \mathfrak{c}$. □

Proposition 3.38 *Suppose that X is a normal space with a G_δ-diagonal whose closed discrete sets are ω-dominated. If $\mathfrak{c}$ is a non-limit cardinal with $2^{<\mathfrak{c}} = \mathfrak{c}$, then $|X| \leq \mathfrak{c}$.*

Proof Assume that $D \subset X$ is a closed discrete set and observe that it follows from Theorem 3.33(a) that $|D| < \mathfrak{c}$. Since the cardinal $\mathfrak{c}$ is a non-limit, we have the inequality $ext(X) < \mathfrak{c}$ and therefore

$$|X| \leq 2^{\Delta(X)\cdot ext(X)} \leq 2^{\omega\cdot ext(X)} \leq 2^{<\mathfrak{c}} = \mathfrak{c}.$$

as promised. □

Given an infinite cardinal κ, a space X is called *κ-metalindelöf* if every open cover of X has a refinement of order $\leq \kappa$. The following statement is straightforward.

Proposition 3.39 *If κ is an infinite cardinal and X is a κ-metalindelöf star countable space, then $L(X) \leq \kappa$.*

Corollary 3.40 *Suppose that X is a T_1-space with a G_δ-diagonal whose closed discrete subsets are ω-dominated. Then $|X| \leq \mathfrak{c}$ if and only if X is $\mathfrak{c}$-metalindelöf.*

Proof The necessity being trivial, assume that the space X is $\mathfrak{c}$-metalindelöf. Since X is star countable by Proposition 3.2, it follows from Proposition 3.39 that $L(X) \leq \mathfrak{c}$. Finally, apply Theorem 3.16 to see that $|X| \leq \mathfrak{c}^\omega = \mathfrak{c}$. □

Proposition 3.41 *Assume that X is a countably paracompact regular space whose closed discrete subsets are ω-dominated. If CH holds, then $ext(X) \leq \omega$.*

Proof Given a closed discrete set $D \subset X$ take a countable set $A \subset X$ such that $D \subset \overline{A}$. The space $Y = \overline{A}$ being separable, the cardinality of the family $RO(Y)$ of regular open subsets of Y does not exceed $\mathfrak{c} = \omega_1$. By Theorem 3.9 of the paper [20], we have the inequality $ext(Y) < \omega_1$ and hence $ext(Y) = \omega$ which shows that $|D| \leq \omega$. □

Corollary 3.42 *Under CH, if X is a countably paracompact T_3-space with a G_δ-diagonal in which closed discrete sets are countably dominated, then $|X| \leq \mathfrak{c}$.*

Proof Observe first that $ext(X) \leq \omega$ by Proposition 3.41; the Ginsburg–Woods inequality $|X| \leq 2^{ext(X)\cdot\Delta(X)}$ does the rest. □

4 Open questions

The results of this paper clearly demonstrate that the notion of ω-domination turned out to be interesting in itself. The following list of open questions shows that the information we have about the respective classes is far from being complete.

Question 4.1 *Suppose that X is a regular space in which all discrete subsets are ω-dominated. Is it true that there exist subspaces Y and Z of the space X such that $X = Y \cup Z$ and $s(Y) = d(Z) = \omega$?*

Question 4.2 *Suppose that a regular space X has a G_δ-diagonal and a Lindelöf dense subspace. Then, trivially, $|X| \leq 2^{\mathfrak{c}}$ but is it true that $|X| \leq \mathfrak{c}$?*

Question 4.3 *Let X be a regular space in which discrete subsets are countably dominated. Is it true that $d(X) \leq \mathfrak{c}$?*

Question 4.4 *Let X be a regular space in which discrete subsets are countably dominated. Is it true that $nw(X) \leq \mathfrak{c}$?*

Question 4.5 *Let X be a regular space of countable pseudocharacter in which discrete subsets are ω-dominated. Is it true that $|X| \leq \mathfrak{c}$?*

Question 4.6 *Suppose that X is a regular space with $t(X) = \psi(X) = \omega$ in which discrete subsets are ω-dominated. Is it true that $|X| \leq \mathfrak{c}$?*

Question 4.7 *Suppose that X is a regular sequential space such that $\psi(X) = \omega$ and all discrete subsets of X are ω-dominated. Is it true that $|X| \leq \mathfrak{c}$?*

Question 4.8 *Suppose that X is a regular Fréchet–Urysohn space such that $\psi(X) = \omega$ and all discrete subsets of X are ω-dominated. Is it true that $|X| \leq \mathfrak{c}$?*

Question 4.9 *Suppose that X is a regular Lindelöf space such that $\psi(X) = \omega$ and all discrete subsets of X are ω-dominated. Is it true that $|X| \leq \mathfrak{c}$?*

Question 4.10 *Suppose that X is a regular Lindelöf space such that $\psi(X) = \omega$ and all discrete subsets of $X \times X$ are ω-dominated. Is it true that $|X| \leq \mathfrak{c}$?*

Question 4.11 *Suppose that X is a regular Lindelöf Σ-space and all discrete subsets of $X \times X$ are ω-dominated. Must X be separable?*

Question 4.12 *Suppose that a regular space X has a G_δ-diagonal and all closed discrete subsets of X are ω-dominated. Is it true that $|X| \leq \mathfrak{c}$?*

Question 4.13 *Suppose that a regular space X of countable tightness has a G_δ-diagonal and all closed discrete subsets of X are ω-dominated. Is it true that $|X| \leq \mathfrak{c}$?*

Question 4.14 *Suppose that X is a compact Hausdorff space and $C_p(X)$ is star countable. Is it true in ZFC that $C_p(X)$ is Lindelöf?*

Question 4.15 *Suppose that X is a regular monotonically monolithic star countable space and $t(X) \leq \omega$. Must X be Lindelöf?*

Question 4.16 *Suppose that X is a regular monotonically monolithic star countable Fréchet–Urysohn space. Must X be Lindelöf?*

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