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# COPIES OF $c_0(\tau)$ SPACES IN PROJECTIVE TENSOR PRODUCTS

VINÍCIUS CORTES, ELÓI MEDINA GALEGO, AND CHRISTIAN SAMUEL

(Communicated by Stephen Dilworth)

*Dedicated to the memory of our friend Eve Oja.*

ABSTRACT. Let $X$ and $Y$ be real Banach spaces and consider the projective tensor product $X\widehat{\otimes}_\pi Y$. Suppose that $\tau$ is an infinite cardinal, $X$ has the bounded approximation property, and the density character of $X$ is strictly smaller than the cofinality of $\tau$. We prove the following $c_0(\tau)$ generalizations of classical $c_0$ results due to Oja (1991) and Kwapień (1974), respectively:

(i) If $c_0(\tau)$ is isomorphic to a complemented subspace of $X\widehat{\otimes}_\pi Y$, then $c_0(\tau)$ is isomorphic to a complemented subspace of $Y$.
(ii) If $c_0(\tau)$ is isomorphic to a subspace of $X\widehat{\otimes}_\pi Y$, then $c_0(\tau)$ is isomorphic to a subspace of $Y$.

We also show that the result (i) is optimal for regular cardinals $\tau$ and Banach spaces $X$ without copies of $c_0(\tau)$. In order to do so, we provide a $c_0(\tau)$ extension of a classical $c_0$ result due to Emmanuele (1988) concerning the $c_0(\tau)$ complemented subspaces of $L_p(D^\tau, Y)$ spaces, $1 \le p \le \infty$, where $D^\tau$ is the Cantor cube. Finally, as a consequence of (i) we conclude that under the continuum hypothesis, the space $c_0(\aleph_\alpha)$, with $\alpha > 1$, is not isomorphic to a complemented subspace of $l_\infty\widehat{\otimes}_\pi l_\infty(\aleph_\alpha)$.

## 1. Introduction

Let $X$ and $Y$ be infinite dimensional real Banach spaces. We say that $Y$ contains a copy (resp., a complemented copy) of $X$, and write $X \hookrightarrow Y$ (resp., $X \overset{c}{\hookrightarrow} Y$), if $X$ is isomorphic to a subspace (resp., complemented subspace) of $Y$. We denote by $X\widehat{\otimes}_\pi Y$ the projective tensor product of $X$ and $Y$ [21, p. 17]. For a non-empty set $\Gamma$, $c_0(\Gamma, X)$ denotes the Banach space of all $X$-valued maps $f$ on $\Gamma$ with the property that for each $\varepsilon > 0$, the set $\{\gamma \in \Gamma : \|f(\gamma)\| \ge \varepsilon\}$ is finite, equipped with the supremum norm. This space will be denoted by $c_0(\Gamma)$ when $X = \mathbb{R}$. We will refer to $c_0(\Gamma)$ as $c_0(\tau)$ when the cardinality of $\Gamma$ (denoted by $|\Gamma|$) is equal to $\tau$. As usual, the dual and bidual of $c_0(\tau)$ will be denoted by $l_1(\tau)$ and $l_\infty(\tau)$, respectively, and when $\tau = \aleph_0$ these spaces will be denoted by $c_0$, $l_1$, and $l_\infty$, respectively.

In this paper we study subspaces and complemented subspaces of projective tensor products which are isomorphic to some $c_0(\tau)$ space. In the particular case where $\tau = \aleph_0$, this research topic is really very old and offers many difficulties due to the intriguing structure of the projective tensor products; see for instance the

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work “Unexpected subspaces of tensor products” [2]. We just point out that Pisier in [17, Theorem 4.1] showed the existence of Banach spaces $X$ and $Y$ such that

$$c_0 \overset{c}{\hookrightarrow} X\widehat{\otimes}_\pi Y \text{ without } c_0 \hookrightarrow X \text{ or } c_0 \hookrightarrow Y. \tag{1.1}$$

In contrast to Pisier's result, Oja [15, Théorème 3] proved that if a Banach space $X$ has a boundedly complete finite dimensional decomposition, then for any Banach space $Y$ we have

$$c_0 \overset{c}{\hookrightarrow} X\widehat{\otimes}_\pi Y \implies c_0 \overset{c}{\hookrightarrow} Y. \tag{1.2}$$

Moreover, Kwapień [13], [3, Theorem 2.1.6] had already shown that for any Banach space $Y$,

$$c_0 \hookrightarrow L_1[0,1]\widehat{\otimes}_\pi Y \implies c_0 \hookrightarrow Y. \tag{1.3}$$

The purpose of this work is to obtain $c_0(\tau)$ generalizations of the results of Oja (1.2) and Kwapień (1.3) to the case where $\tau$ is an uncountable cardinal. In this direction we have already studied the $c_0(\tau)$ complemented subspaces of $X\widehat{\otimes}_\pi Y$ in the particular cases when $X$ is one of the classical spaces $\ell_p(I)$ [6] or $L_p[0,1]$ [5]. We have shown that if $X = \ell_p(I)$, $1 \leq p < \infty$, or $X = L_q[0,1]$, $1 < q < \infty$, then

$$c_0(\tau) \overset{c}{\hookrightarrow} X\widehat{\otimes}_\pi Y \implies c_0 \overset{c}{\hookrightarrow} Y$$

for any Banach space $Y$. But we left the problem below unsolved [6, Problem 8.5].

**Problem 1.1.** Let $Y$ be a Banach space and let $\tau$ be an uncountable cardinal. Is it true that

$$c_0(\tau) \overset{c}{\hookrightarrow} l_\infty\widehat{\otimes}_\pi Y \implies c_0(\tau) \overset{c}{\hookrightarrow} Y?$$

Our generalization of (1.2) will provide a positive solution to Problem 1.1 whenever the cofinality of $\tau$ is strictly greater than the continuum (Corollary 1.8 with $\mathfrak{m} = \aleph_0$). Recall that the *cofinality* of an infinite cardinal $\tau$, denoted by $\mathrm{cf}(\tau)$, is the least cardinal $\alpha$ such that there exists a family of ordinals $\{\beta_j : j \in \alpha\}$ satisfying $\beta_j < \tau$ for all $j \in \alpha$, and $\sup\{\beta_j : j \in \alpha\} = \tau$ [12]. A Banach space $X$ has the $\lambda$-*bounded approximation property* (in short, $X$ has the $\lambda$-bap), $\lambda \geq 1$, if given $K$ a compact subset of $X$ and $\varepsilon > 0$, there exists a finite rank operator $T : X \to X$ satisfying $\|T\| \leq \lambda$ and $\|x - T(x)\| < \varepsilon$ for every $x \in K$. $X$ has the *bounded approximation property* if $X$ has the $\lambda$-bap for some $\lambda \geq 1$ [1]. The *density character* of $X$, denoted by $\mathrm{dens}(X)$, is the smallest cardinality of a dense subset of $X$.

**Theorem 1.2.** *Let $X$ be a Banach space with the bounded approximation property and let $\tau$ be an infinite cardinal satisfying $\mathrm{dens}(X) < \mathrm{cf}(\tau)$. For any Banach space $Y$,*

$$c_0(\tau) \overset{c}{\hookrightarrow} X\widehat{\otimes}_\pi Y \implies c_0(\tau) \overset{c}{\hookrightarrow} Y.$$

*Remark* 1.3. Even if $\tau = \aleph_0$ and $X$ contains no copies of $c_0$, we cannot replace the hypothesis $\mathrm{dens}(X) < \aleph_0$ with $\mathrm{dens}(X) \leq \aleph_0$ in Theorem 1.2. Indeed, the space $X$ mentioned above in the example of Pisier can be chosen having a Schauder basis ([16, Remark, p. 84] and [18, Corollary 10.1, p. 68]) and, therefore, having the bounded approximation property.

Recall that a cardinal $\tau$ is said to be *regular* when $\mathrm{cf}(\tau) = \tau$. As a consequence of Theorem 1.4, we will show in Remark 1.5 that Theorem 1.2 is optimal for any regular cardinal $\tau$ and any Banach space $X$ containing no copies of $c_0(\tau)$. In order to state Theorem 1.4, we need some notation. If $\Gamma$ is a non-empty set, we endow

$\{-1,1\}$ with the probability measure which assigns to each point the value 1/2 and denote by $D^\Gamma = \{-1,1\}^\Gamma$ the Cantor cube with the corresponding product measure $\mu$. Given $Y$ a Banach space and $p \in [1,\infty)$, we denote by $L_p(D^\Gamma, Y)$ the Lebesgue-Bochner space of all (classes of equivalence of) measurable functions $f : D^\Gamma \to Y$ such that the scalar function $\|f\|^p$ is integrable, equipped with the complete norm

$$\|f\|_p = \left[\int_0^1 \|f(t)\|^p d\mu(t)\right]^{\frac{1}{p}}.$$

These spaces will be denoted by $L_p(D^\Gamma)$ when $Y = \mathbb{R}$. A measurable function $f : D^\Gamma \to Y$ is *essentially bounded* if there exists $\varepsilon > 0$ such that the set $\{t \in D^\Gamma : \|f(t)\| \geq \varepsilon\}$ has measure zero, and we denote by $\|f\|_\infty$ the infimum of all such numbers $\varepsilon > 0$. By $L_\infty(D^\Gamma, Y)$ we will denote the space of all (classes of equivalence of) essentially bounded functions $f : D^\Gamma \to X$, equipped with the complete norm $\|\cdot\|_\infty$.

**Theorem 1.4.** *Given $Y$ a Banach space, $\tau$ an infinite cardinal, and $1 \leq p \leq \infty$,*

$$c_0(\tau) \hookrightarrow Y \implies c_0(\tau) \overset{c}{\hookrightarrow} L_p(D^\tau, Y).$$

*Remark* 1.5. Let us show that the hypothesis $\text{dens}(X) < \text{cf}(\tau)$ in Theorem 1.2 cannot be improved in the case of regular cardinals $\tau$ and Banach spaces $X$ with no copies of $c_0(\tau)$. According to a well-known result, see, e.g., [7, Corollary 11, p. 156],

$$c_0(\tau) \overset{c}{\not\hookrightarrow} l_\infty(\tau). \tag{1.4}$$

Moreover, $L_1(D^\Gamma)$ has the bounded approximation property [7, Example 11] and the density character of $L_1(D^\Gamma)$ is equal to $|\Gamma|$ [14, Theorem 2.12]. Recall also that the spaces $L_1(D^\Gamma, Y)$ and $L_1(D^\Gamma)\widehat{\otimes}_\pi Y$ are isometrically isomorphic [21, Example 2.19]. So, applying Theorem 1.4 with $X = L_1(D^\tau)$ and $Y = l_\infty(\tau)$ yields

$$c_0(\tau) \overset{c}{\hookrightarrow} L_1(D^\tau)\widehat{\otimes}_\pi l_\infty(\tau) \text{ although } c_0(\tau) \not\hookrightarrow L_1(D^\tau) \text{ and } c_0(\tau) \overset{c}{\not\hookrightarrow} l_\infty(\tau).$$

*Remark* 1.6. Using [10, Lemma 2.1] it is easy to see that $L_p([0,1], Y)$ is isomorphic to $L_p(D^{\aleph_0}, Y)$, $1 \leq p \leq \infty$, for every Banach space $Y$. Thus, the case $\tau = \aleph_0$ of Theorem 1.4 is due to Emmanuele [8, Main Theorem], [3, Theorem 4.3.2]. In fact, the proof of Theorem 1.4 is almost the same as that of Emmanuele, but for the sake of completeness we include it in the fourth section.

We will also prove the following theorem which can be seen as a version of Theorem 1.2 for Lebesque-Bochner spaces $L_p(D^\Gamma, Y)$ with $1 \leq p < \infty$.

**Theorem 1.7.** *Let $Y$ be a Banach space, let $\Gamma$ be an infinite set, let $\tau$ be an infinite cardinal, and $1 \leq p < \infty$. If $|\Gamma| < \text{cf}(\tau)$, then*

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p(D^\Gamma, Y) \implies c_0(\tau) \overset{c}{\hookrightarrow} Y.$$

Note that in the particular case $\mathfrak{m} = \aleph_0$, the following direct consequence of Theorem 1.2 provides a positive solution to Problem 1.1 for many large enough cardinals $\tau$. Recall that the density character of $\ell_\infty(\tau)$ is $2^\tau$ [9, Theorem 3.6.11] and this space has the bounded approximation property [7, p. 245].

**Corollary 1.8.** *Let $\tau$ and $\mathfrak{m}$ be infinite cardinals satisfying $2^\mathfrak{m} < \text{cf}(\tau)$. Then, for any Banach space $Y$,*

$$c_0(\tau) \overset{c}{\hookrightarrow} \ell_\infty(\mathfrak{m})\widehat{\otimes}_\pi Y \implies c_0(\tau) \overset{c}{\hookrightarrow} Y.$$

Under the continuum hypothesis $2^{\aleph_0} = \aleph_1$ we also get the following consequence of Theorem 1.2 which can be seen as a generalization of (1.4) whenever $\tau > \aleph_1$.

**Corollary 1.9.** *Assuming the continuum hypothesis, for any $\alpha > 1$ we have*

$$c_0(\aleph_\alpha) \overset{c}{\not\hookrightarrow} l_\infty \widehat{\otimes}_\pi l_\infty(\aleph_\alpha).$$

*Proof.* Assume by contradiction that there exists $\alpha > 1$ such that

$$c_0(\aleph_2) \overset{c}{\hookrightarrow} c_0(\aleph_\alpha) \overset{c}{\hookrightarrow} l_\infty \widehat{\otimes}_\pi l_\infty(\aleph_\alpha).$$

By Theorem 1.2 it follows that $l_\infty(\aleph_\alpha)$ contains a complemented subspace isomorphic to $c_0(\aleph_2)$, and this contradicts [7, Corollary 11, p. 156]. □

*Remark* 1.10. It is an open problem to know if the statement of Corollary 1.9 remains true for $\alpha = 0$ [2, Remark 3]. We also left open whether that statement holds when $\alpha = 1$.

Now we turn our attention to generalizations of the result (1.3). We will prove the following.

**Theorem 1.11.** *Let $X$ be a Banach space with the bounded approximation property and let $\tau$ be an infinite cardinal satisfying $\mathrm{dens}(X) < \mathrm{cf}(\tau)$. For any Banach space $Y$,*

$$c_0(\tau) \hookrightarrow X \widehat{\otimes}_\pi Y \implies c_0(\tau) \hookrightarrow Y.$$

Next, denote by $X \widehat{\otimes}_\varepsilon Y$ the injective tensor product of the Banach spaces $X$ and $Y$ [21, p. 46]. As a direct consequence of Theorem 1.11 we obtain the following.

**Corollary 1.12.** *Let $X$ and $Y$ be Banach spaces such that $X^*$ has both the bounded approximation property and the Radon-Nikodým property. Suppose that $\tau$ is a cardinal satisfying $\mathrm{dens}(X) < \mathrm{cf}(\tau)$. Then*

$$l_1(\tau) \overset{c}{\hookrightarrow} X \widehat{\otimes}_\varepsilon Y \implies l_1(\tau) \overset{c}{\hookrightarrow} Y.$$

*Proof.* Since $X^*$ has the bounded approximation property and the Radon-Nikodým property, by [21, Theorem 5.33] we know that the dual of $X \widehat{\otimes}_\varepsilon Y$ is isomorphic to $X^* \widehat{\otimes}_\pi Y^*$. Thus we get

$$c_0(\tau) \hookrightarrow X^* \widehat{\otimes}_\pi Y^*.$$

Now, since $X^*$ has the Radon-Nikodým property it follows by [22, Theorem 6] that $\mathrm{dens}(X) = \mathrm{dens}(X^*)$. Thus, according to Theorem 1.11 we conclude that $c_0(\tau) \hookrightarrow Y^*$, and an appeal to [20, Corollary 1.2] yields $l_1(\tau) \overset{c}{\hookrightarrow} Y$. □

Moreover, notice that the result (1.3) was proved in [13, Main Theorem] to a more general setting; more precisely, for any Banach space $Y$ and $1 \le p < \infty$,

$$c_0 \hookrightarrow L_p([0,1], Y) \implies c_0 \hookrightarrow Y. \tag{1.5}$$

Thus, we will also state a $c_0(\tau)$ generalization of (1.5) by proving the following non-complemented version of Theorem 1.7.

**Theorem 1.13.** *Let $Y$ be a Banach space, let $\Gamma$ be an infinite set, let $\tau$ be an infinite cardinal, and $1 \le p < \infty$. If $|\Gamma| < \mathrm{cf}(\tau)$, then*

$$c_0(\tau) \hookrightarrow L_p(D^\Gamma, Y) \implies c_0(\tau) \hookrightarrow Y.$$

*Remark* 1.14. Of course, if $\tau = \aleph_0$ and $X$ contains no copies of $c_0$, the optimality of Theorem 1.11 follows from the example of Pisier cited in (1.1). However, we do not know if Theorem 1.11 is optimal for any regular cardinal $\tau$ and any Banach space $X$ containing no copies of $c_0(\tau)$. Finally, as far as Corollary 1.12 is concerned, we do not even know if it is optimal in the case $\tau = \aleph_0$.

## 2. Preliminary results and notation

Before proving our results we will recall some auxiliary results that will be used throughout this paper. We will denote by $(e_i)_{i\in\tau}$ the canonical basis of $c_0(\tau)$, that is, $e_i(j) = \delta_{ij}$ for each $i, j \in \tau$. If $\Gamma$ is a subset of $\tau$, we identify $c_0(\Gamma)$ with the closed subspace of $c_0(\tau)$ consisting of the maps $g$ on $\tau$ such that $g(i) = 0$ for each $i \in \tau \setminus \Gamma$. We begin by recalling the following classical result due to Rosenthal [19, Remark following Theorem 3.4].

**Lemma 2.1.** *Let $X$ be a Banach space and let $\tau$ be an infinite cardinal. Suppose that $T : c_0(\tau) \to X$ is a bounded linear operator satisfying*

$$\inf\{\|T(e_i)\| : i \in \tau\} > 0.$$

*Then there exists a subset $\Gamma \subset \tau$ such that $|\Gamma| = \tau$ and $T_{|c_0(\Gamma)}$ is an isomorphism onto its image.*

We recall that a family $(x_i^*)_{i\in\tau}$ in the dual space $X^*$ is said to be *weak*$^*$*-null* if for each $x \in X$ we have

$$(x_i^*(x))_{i\in\tau} \in c_0(\tau).$$

Recall also that a family $(x_i)_{i\in\tau}$ in a Banach space $X$ is said to be *equivalent* to the canonical basis of $c_0(\tau)$ if there exists $T : c_0(\tau) \to X$ an isomorphism onto its image satisfying $T(e_i) = x_i$ for each $i \in \tau$.

The main characterization of complemented copies of $c_0(\tau)$ we will use is the following result obtained in [4, Theorem 2.4].

**Lemma 2.2.** *Let $X$ be a Banach space and let $\tau$ be an infinite cardinal. The following are equivalent:*

(1) *$X$ contains a complemented copy of $c_0(\tau)$.*
(2) *There exist a family $(x_i)_{i\in\tau}$ in $X$ equivalent to the unit-vector basis of $c_0(\tau)$ and a weak*$^*$*-null family $(x_i^*)_{i\in\tau}$ in $X^*$ such that, for each $i, j \in \tau$,*

$$x_i^*(x_j) = \delta_{ij}.$$

(3) *There exist a family $(x_i)_{i\in\tau}$ in $X$ equivalent to the unit-vector basis of $c_0(\tau)$ and a weak*$^*$*-null family $(x_i^*)_{i\in\tau}$ in $X^*$ such that*

$$\inf_{i\in\tau} |x_i^*(x_i)| > 0.$$

## 3. On complemented copies of $c_0(\tau)$ spaces in $X\widehat{\otimes}_\pi Y$ spaces

The main aim of this section is to prove the generalization of Oja's result (1.2), i.e., Theorem 1.2. In order to do this we need to state two lemmas. The proof of Lemma 3.1 is straightforward.

**Lemma 3.1.** *Let $X$ be a Banach space with the $\lambda$-bap, $\lambda \geq 1$, and let $\mathfrak{n}$ be its density character. Then there exist a directed set $\mathcal{A}$ of cardinality $\mathfrak{n}$ and a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ of finite rank operators on $X$ which converges to $\mathrm{Id}_X$ in the strong operator topology and satisfies $\|T_\alpha\| \leq \lambda$ for every $\alpha \in \mathcal{A}$.*

**Lemma 3.2.** *Let $X$ and $Y$ be Banach spaces, let $T$ be a finite rank operator on $X$, and fix a normalized basis $(v_1, \dots, v_k)$ of $T(X)$. Then there exist bounded linear operators $S_1, \dots, S_k$ from $X \widehat{\otimes}_\pi Y$ to $Y$ satisfying*

$$(T \otimes \mathrm{Id}_Y)(u) = \sum_{j=1}^{k} v_j \otimes S_j(u)$$

*for every $u \in X \widehat{\otimes}_\pi Y$.*

*Proof.* For each $1 \le j \le k$ there exists $\varphi_j \in X^*$ such that $\varphi_j(v_i) = \delta_{ij}$, $1 \le i \le k$. The continuous bilinear operator $(x, y) \in X \times Y \to \varphi_j(x)y \in Y$ induces a bounded operator $R_j : X \widehat{\otimes}_\pi Y \to Y$ such that

$$R_j(u) = \sum_{n=1}^{m} \varphi_j(x_n) y_n$$

for each $u = \sum_{n=1}^{m} x_n \otimes y_n$. Put $S_j = R_j \circ (T \otimes \mathrm{Id}_Y)$. In order to finish the proof it suffices to notice that

$$(T \otimes \mathrm{Id}_Y)(u) = \sum_{n=1}^{m} T(x_n) \otimes y_n = \sum_{n=1}^{m} \sum_{j=1}^{k} \varphi_j(T(x_n)) v_j \otimes y_n = \sum_{j=1}^{k} v_j \otimes S_j(u)$$

for each $u = \sum_{n=1}^{m} x_n \otimes y_n \in X \widehat{\otimes}_\pi Y$, and apply standard density arguments. $\square$

*Proof of Theorem* 1.2. Let $\lambda > 1$ be such that $X$ has the $\lambda$-bap. By Lemma 3.1, there exist a directed set $\mathcal{A}$ of cardinality $\mathfrak{n} = \mathrm{dens}(X)$ and a net $(T_\alpha)_{\alpha \in \mathcal{A}}$ of finite rank operators on $X$ which converges to $\mathrm{Id}_X$ in the strong operator topology and satisfies $\sup_{\alpha \in \mathcal{A}} \|T_\alpha\| \le \lambda$. For every $\alpha \in \mathcal{A}$ we write $\pi_\alpha = T_\alpha \otimes \mathrm{Id}_Y$. It is easy to check that the net $(\pi_\alpha)_{\alpha \in \mathcal{A}}$ converges to $\mathrm{Id}_{X \widehat{\otimes}_\pi Y}$ in the strong operator topology.

Suppose that $c_0(\tau)$ is isomorphic to a complemented subspace of $X \widehat{\otimes}_\pi Y$. By Lemma 2.2, there exist families $(u_i)_{i \in \tau}$ in $X \widehat{\otimes}_\pi Y$ and $(u_i^*)_{i \in \tau}$ in $[X \widehat{\otimes}_\pi Y]^*$ such that $(u_i)_{i \in \tau}$ is equivalent to the canonical basis of $c_0(\tau)$, $(u_i^*)_{i \in \tau}$ is weak*-null, and $|u_i^*(u_i)| = 1$ for every $i \in \tau$. Let $C = \sup_{i \in \tau} \|u_i^*\|$.

For every $i \in \tau$ we have

$$1 = |u_i^*(u_i)| = \lim_{\alpha} |u_i^*(\pi_\alpha(u_i))|$$

and so there exists an index $\alpha_i \in \mathcal{A}$ such that $|u_i^*(\pi_{\alpha_i}(u_i))| \ge 1/2$.

For every $\alpha \in \mathcal{A}$ consider the set

$$I_\alpha = \{\, i \in \tau \,:\, |u_i^*(\pi_\alpha(u_i))| \ge 1/2 \,\}.$$

Since $\bigcup_{\alpha \in \mathcal{A}} I_\alpha = \tau$ and $\mathfrak{n} < \mathrm{cf}(\tau)$, there exist an index $\beta_0 \in \mathcal{A}$ and a subset $\tau_1 \subset \tau$ such that $|\tau_1| = \tau$ and, for every $i \in \tau_1$,

$$|u_i^*(\pi_{\beta_0}(u_i))| \ge \frac{1}{2},$$

Let $k = \dim T_{\beta_0}(X)$ and let $(v_j)_{1 \le j \le k}$ be a normalized basis of $T_{\beta_0}(X)$. By Lemma 3.2, there exist bounded linear operators $S_1, \dots, S_k : X \widehat{\otimes}_\pi Y \to Y$ satisfying

$$\pi_{\beta_0}(u) = \sum_{j=1}^{k} v_j \otimes S_j(u)$$

for each $u \in X\widehat{\otimes}_\pi Y$. Thus, for every $i \in \tau_1$,

$$\frac{1}{2} \leq |u_i^*(\pi_{\beta_0}(u_i))| = \left|\sum_{j=1}^{k} u_i^*(v_j \otimes S_j(u_i))\right| \leq \sum_{j=1}^{k} |u_i^*(v_j \otimes S_j(u_i))|.$$

Now, $\tau_1$ is infinite so there exist an integer $1 \leq k_0 \leq k$ and a subset $\tau_2 \subset \tau_1$ such that $|\tau_2| = \tau$ and, for every $i \in \tau_2$,

$$|u_i^*(v_{k_0} \otimes S_{k_0}(u_i))| \geq \frac{1}{2k}. \tag{3.1}$$

Hence,

$$\|S_{k_0}(u_i))\| \geq \frac{1}{2Ck}$$

for every $i \in \tau_2$.

Since $(u_i)_{i\in\tau}$ is equivalent to the canonical basis of $c_0(\tau)$, by Lemma 2.1 there exists a subset $\tau_3 \subset \tau_2$ such that $|\tau_3| = \tau$ and $(S_{k_0}(u_i))_{i\in\tau_3}$ is equivalent to the unit basis of $c_0(\tau_3)$. If we denote by $y_i^*$ the bounded linear functional on $Y$ defined by

$$y_i^*(y) = u_i^*(v_{k_0} \otimes y),$$

for every $i \in \tau_3$ and $y \in Y$, then it is clear that $(y_i^*)_{i\in\tau_3}$ is weak*-null in $Y^*$. Therefore, combining inequality (3.1) with Lemma 2.2, we obtain a complemented subspace of $Y$ isomorphic to $c_0(\tau)$.

## 4. Complemented copies of $c_0(\tau)$ spaces in $L_p(D^\tau, Y)$ spaces

This section is devoted to the proofs of Theorems 1.4 and 1.7. For each $i \in \Gamma$, denote by $\pi_i : D^\Gamma \to \{-1, 1\}$ the usual Rademacher function. We recall that the family $(\pi_i)_{i\in\Gamma}$ is an orthonormal system in $L_2(D^\Gamma)$.

Next, given a Banach space $Y$, $f \in L_1(D^\Gamma, Y)$ and $i \in \Gamma$, we write

$$c_i(f) = \int_{D^\Gamma} \pi_i(t) f(t) d\mu(t).$$

Thus, by using approximations of vector-valued functions by simple functions it is easy to obtain the following Riemann-Lebesgue lemma for $f \in L_1(D^\Gamma, Y)$.

**Lemma 4.1.** *Let $Y$ be a Banach space and let $\Gamma$ be an infinite cardinal. If $f \in L_1(D^\Gamma, Y)$, then $(c_i(f))_{i\in\Gamma} \in c_0(\Gamma, Y)$.*

*Proof of Theorem* 1.4. Let $(y_i)_{i\in\tau}$ be a family in $Y$ equivalent to the canonical basis of $c_0(\tau)$. There exist constants $0 < \delta \leq M$ such that

$$\delta \sup_{i\in F} |\lambda_i| \leq \left\| \sum_{i\in F} \lambda_i y_i \right\| \leq M \sup_{i\in F} |\lambda_i| \tag{4.1}$$

for all finite and non-empty subsets $F$ of $\tau$ and all families of scalars $(\lambda_i)_{i\in F}$. Let $(y_i^*)_{i\in\tau}$ be a bounded family in $Y^*$ such that $y_i^*(y_j) = \delta_{ij}$. For each $i \in \tau$, write $f_i = \pi_i y_i \in L_p(D^\tau, Y)$ and define $\psi_i \in (L_p(D^\tau, Y))^*$ by $\psi_i(f) = y_i^*(c_i(f))$. Notice that each $\psi_i$ is bounded, since, by Hölder's inequality,

$$\forall f \in L_p(D^\tau, Y), \quad |\psi_i(f)| \leq \|y_i^*\| \left\| \int_{D^\tau} \pi_i(t) f(t) d\mu(t) \right\| \leq \|y_i^*\| \|f\|_p.$$

Since $|\pi_i(t)| = 1$ for all $t \in D^\tau$ and $i \in \tau$, by (4.1) we have

$$\delta \sup_{i\in F} |\lambda_i| \leq \left\| \sum_{i\in F} \lambda_i f_i \right\|_p \leq M \sup_{i\in F} |\lambda_i|$$

for all finite and non-empty subsets $F$ of $\tau$ and all families of scalars $(\lambda_i)_{i\in F}$. This proves that $(f_i)_{i\in\tau}$ is equivalent to the canonical basis of $c_0(\tau)$.

Next, if $i \neq j$, then

$$\psi_i(f_j) = \int_{D^\tau} \pi_i(t)\pi_j(t) y_i^*(y_j) d\mu(t) = 0$$

and furthermore

$$\psi_i(f_i) = \int_{D^\tau} \pi_i^2(t) y_i^*(y_i) d\mu(t) = \mu(D^\tau) = 1.$$

Thus, $\psi_i(f_j) = \delta_{ij}$.

In order to finish the proof, it suffices to show that $(\psi_i)_{i\in\tau}$ is a weak$^*$-null family in $(L_p(D^\tau, Y))^*$. Given $f \in L_p(D^\tau, Y) \subset L_1(D^\tau, Y)$, Lemma 4.1 guarantees that $(\|c_i(f)\|)_{i\in\tau} \in c_0(\tau)$. Since

$$|\psi_i(f)| \leq \|y_i^*\| \|c_i(f)\|$$

and the family $(y_i^*)_{i\in\tau}$ is bounded, $(\psi_i(f))_{i\in\tau} \in c_0(\tau)$. This completes the proof.

Next, we turn our attention to Theorem 1.7. Given $\Gamma$ an infinite set and $\Omega$ a non-empty, proper subset of $\Gamma$, put $\overline{\Omega} = \Gamma \setminus \Omega$ and write each element of $D^\Gamma$ as $(t, \overline{t}) \in D^\Omega \times D^{\overline{\Omega}}$. We will also denote by $\mu_\Omega$ the product measure on $D^\Omega$.

Our next two results are vector-valued versions of [11, Lemma 22.11, p. 437] and [11, Theorem 22.14, p. 439], respectively. If $A$ is a subset of $\Gamma$, the characteristic function of $A$ will be denoted by $\chi_A$.

**Lemma 4.2.** *Let $Y$ be a Banach space, let $\Gamma$ be an infinite set, and $1 \leq p < \infty$. Given $f \in L_p(D^\Gamma, Y)$ and $F$ a finite, non-empty subset of $\Gamma$, consider the function $S_F(f) : D^\Gamma \to Y$ defined by*

$$S_F(f)(t, \overline{t}) = \int_{D^{\overline{F}}} f(t, s) d\mu_{\overline{F}}(s)$$

*for every $(t, \overline{t}) \in D^\Gamma$. Then $S_F$ is a bounded linear operator on $L_p(D^\Gamma, Y)$ with $\|S_F\| \leq 1$.*

*Proof.* Given $f \in L_p(D^\Gamma, Y)$, notice that $S_F(f)$ is a simple function. Indeed, we may write

$$S_F(f) = \sum_{t\in D^F} \chi_t(\cdot) y(f, t) \tag{4.2}$$

where, for each $t \in D^F$,

$$y(f, t) = \int_{D^{\overline{F}}} f(t, s) d\mu_{\overline{F}}(s) \in Y$$

and $\chi_t$ denotes the characteristic function of the measurable set

$$A_t = \left\{ (t, s) : s \in D^{\overline{F}} \right\}.$$

So, in particular, $S_F(f)$ is measurable.

Now, by the scalar Fubini theorem and Hölder's inequality, we obtain

$$\begin{aligned}
\|S_F(f)\|_p^p &= \int_{D^\Gamma} \|S_F(f)(t,\bar{t})\|^p d\mu(t,\bar{t}) \\
&= \int_{D^\Gamma} \left\| \int_{D^{\overline{F}}} f(t,s) d\mu_{\overline{F}}(s) \right\|^p d\mu(t,\bar{t}) \\
&\leq \int_{D^\Gamma} \left[ \int_{D^{\overline{F}}} \|f(t,s)\| d\mu_{\overline{F}}(s) \right]^p d\mu(t,\bar{t}) \\
&\leq \int_{D^\Gamma} \left[ \int_{D^{\overline{F}}} \|f(t,s)\|^p d\mu_{\overline{F}}(s) \right] d\mu(t,\bar{t}) \\
&= \int_{D^{\overline{F}}} \left[ \int_{D^F} \left[ \int_{D^{\overline{F}}} \|f(t,s)\|^p d\mu_{\overline{F}}(s) \right] d\mu_F(t) \right] d\mu_{\overline{F}}(\bar{t}) \\
&= \int_{D^{\overline{F}}} \left[ \int_{D^\Gamma} \|f(t,s)\|^p d\mu(t,s) \right] d\mu_{\overline{F}}(\bar{t}) \\
&= \int_{D^{\overline{F}}} \|f\|_p^p d\mu_{\overline{F}}(\bar{t}) = \|f\|_p^p.
\end{aligned}$$

Hence, $S_F(f) \in L_p(D^\Gamma, Y)$ and $\|S_F(f)\|_p \leq \|f\|_p$. The linearity of $S_F$ is clear. □

**Lemma 4.3.** *Let $Y$ be a Banach space, let $\Gamma$ be an infinite set, and $1 \leq p < \infty$. Given $f \in L_p(D^\Gamma, Y)$ and $\varepsilon > 0$, there exists a finite, non-empty subset $F_0$ of $\Gamma$ such that $\|S_F(f) - f\|_p < \varepsilon$ for all finite subsets $F$ of $\Gamma$ with $F_0 \subset F$.*

*Proof.* We split the proof into three cases.

*Case* 1. $f = g(\cdot)y$, where $g \in L_p(D^\Gamma)$ and $y \in Y$. This case follows immediately from [11, Theorem 22.14, p. 439], noticing that $S_F(f) = S_F(g)(\cdot)y$ for every finite, non-empty subset $F$ of $\Gamma$.

*Case* 2. $f \in L_p(D^\Gamma, Y)$ is a simple function. This follows from the previous case, by linearity.

*Case* 3. Arbitrary $f \in L_p(D^\Gamma, Y)$. Fix $\varepsilon > 0$. Since the simple functions are dense in $L_p(D^\Gamma, Y)$, there exists a simple function $\varphi \in L_p(D^\Gamma, Y)$ such that $\|f - \varphi\|_p < \varepsilon/3$. By the previous case, there exists a finite, non-empty subset $F_0$ of $\Gamma$ such that $\|S_F(\varphi) - \varphi\|_p < \varepsilon/3$, for all finite subsets $F$ of $\Gamma$ with $F_0 \subset F$. Therefore

$$\|S_F(f) - f\|_p \leq \|S_F(f - \varphi)\|_p + \|S_F(\varphi) - \varphi\|_p + \|\varphi - f\|_p < \varepsilon$$

for all such $F \subset \Gamma$, as desired. □

We are now ready to prove Theorem 1.7.

*Proof of Theorem* 1.7. Suppose that $L_p(D^\Gamma, Y)$ contains a complemented copy of $c_0(\tau)$. According to Lemma 2.2, there exist families $(f_i)_{i\in\tau}$ in $L_p(D^\Gamma, Y)$ and $(\psi_i)_{i\in\tau}$ in $(L_p(D^\Gamma, Y))^*$ such that $(f_i)_{i\in\tau}$ is equivalent to the canonical basis of $c_0(\tau)$, $(\psi_i)_{i\in\tau}$ is weak*-null, and $|\psi_i(f_i)| = 1$ for every $i \in \tau$. Let $C = \sup_{i\in\tau} \|\psi_i\|$.

Let $\mathcal{F}$ be the family of all finite, non-empty subsets of $\Gamma$. By Lemma 4.3, for every $i \in \tau$ there exists a set $F_i \in \mathcal{F}$ such that $\|S_{F_i}(f) - f\|_p < 1/2$, and thus

$$|\psi_i(S_{F_i}(f_i))| > \frac{1}{2}.$$

For every $F \in \mathcal{F}$ consider the set

$$I_F = \{\, i \in \tau \,:\, |\psi_i(S_F(f_i))| > 1/2 \,\}.$$

Since $\bigcup_{F\in\mathcal{F}} I_F = \tau$ and $|\mathcal{F}| = |\Gamma| < \mathrm{cf}(\tau)$, there exist subsets $G \in \mathcal{F}$ and $\tau_1 \subset \tau$ such that $|\tau_1| = \tau$ and, for every $i \in \tau_1$,

$$|\psi_i(S_G(f_i))| > \frac{1}{2}.$$

Similarly to (4.2), for each $f \in L_p(D^\Gamma, Y)$ we write

$$S_G(f) = \sum_{t\in D^G} \chi_t(\cdot) y(f,t) \tag{4.3}$$

where, for each $t \in D^G$,

$$y(f,t) = \int_{D^{\overline{G}}} f(t,s) d\mu_{\overline{G}}(s) \in Y$$

and $\chi_t$ denotes the characteristic function of the measurable set

$$A_t = \left\{ (t,s) : s \in D^{\overline{G}} \right\}.$$

Therefore

$$\frac{1}{2} < |\psi_i(S_G(f_i))| \le \sum_{t\in D^G} |\psi_i(\chi_t(\cdot) y(f_i,t))|$$

for every $i \in \tau_1$. Now, $\tau_1$ is infinite and so there exist an element $t_0 \in D^G$ and a subset $\tau_2 \subset \tau_1$ such that $|\tau_2| = \tau$ and, for every $i \in \tau_2$,

$$|\psi_i(\chi_{t_0}(\cdot) y(f_i,t_0))| > \frac{1}{2^{|G|+1}}. \tag{4.4}$$

Since $\mu(A_{t_0}) = 1/2^{|G|} < 1$, we obtain

$$\|y(f_i,t_0)\| > \frac{1}{C 2^{|G|+1}} \tag{4.5}$$

for every $i \in \tau_2$.

Next, consider the linear operator $T : L_p(D^\Gamma, Y) \to Y$ defined by $T(f) = y(f,t_0)$ for every $f \in L_p(D^\Gamma, Y)$. In order to see that $T$ is bounded, notice that (4.3) yields

$$\|S_G(f)(s_1,s_2)\|^p = \sum_{t\in D^G} \chi_t(s_1,s_2) \|y(f,t)\|^p$$

for every $(s_1,s_2) \in D^G \times D^{\overline{G}}$. This implies that

$$\begin{aligned}
\|S_G(f)\|_p^p &= \int_{D^\Gamma} \|S_G(f)(s_1,s_2)\|^p d\mu(s_1,s_2) \\
&= \int_{D^\Gamma} \sum_{t\in D^G} \chi_t(s_1,s_2) \|y(f,t)\|^p d\mu(s_1,s_2) \\
&= \sum_{t\in D^G} \|y(f,t)\|^p \int_{D^\Gamma} \chi_t(s_1,s_2) d\mu(s_1,s_2) \\
&= \sum_{t\in D^G} \mu(A_t) \|y(f,t)\|^p = \sum_{t\in D^G} 2^{-|G|} \|y(f,t)\|^p \\
&\ge 2^{-|G|} \|y(f,t_0)\|^p = 2^{-|G|} \|T(f)\|^p
\end{aligned}$$

and so

$$\|T(f)\| \leq 2^{|G|/p} \|S_G(f)\|_p \leq 2^{|G|/p} \|f\|_p,$$

proving that $T$ is bounded.

Notice that, by (4.5), we have

$$\|T(f_i)\| > \frac{1}{C 2^{|G|+1}}$$

for every $i \in \tau_2$. Also, by hypothesis, $(f_i)_{i\in\tau}$ is equivalent to the canonical basis of $c_0(\tau)$. Thus, by Lemma 2.1 there exists a subset $\tau_3 \subset \tau_2$ such that $|\tau_3| = \tau$ and $(T(f_i))_{i\in\tau_3}$ is equivalent to the canonical basis of $c_0(\tau_3)$. Denoting by $y_i^*$ the bounded linear function on $Y$ defined by

$$y_i^*(y) = \psi_i(\chi_{t_0}(\cdot)y),$$

for every $i \in \tau_3$ and $y \in Y$, it is clear that $(y_i^*)_{i\in\tau_3}$ is weak*-null in $Y^*$. Combining these facts with (4.4), Lemma 2.2 yields a complemented copy of $c_0(\tau)$ in $Y$, as desired.

## 5. On copies of $c_0(\tau)$ spaces in $X\widehat{\otimes}_\pi Y$ spaces

With a simple modification of the techniques developed so far, we prove Theorem 1.11.

*Proof of Theorem* 1.11. Let $\lambda > 1$ be such that $X$ has the $\lambda$-bap. As in the proof of Theorem 1.2, by Lemma 3.1 there exist a directed set $\mathcal{A}$ of cardinality $\mathfrak{n} = \mathrm{dens}(X)$ and a net $(T_\alpha)_{\alpha\in\mathcal{A}}$ of finite rank operators on $X$ which converges to $\mathrm{Id}_X$ in the strong operator topology and satisfies $\sup_{\alpha\in\mathcal{A}} \|T_\alpha\| \leq \lambda$. For every $\alpha \in \mathcal{A}$ we write $\pi_\alpha = T_\alpha \otimes \mathrm{Id}_Y$. The net $(\pi_\alpha)_{\alpha\in\mathcal{A}}$ converges to $\mathrm{Id}_{X\widehat{\otimes}_\pi Y}$ in the strong operator topology.

Let $(u_i)_{i\in\tau}$ be a family in $X\widehat{\otimes}_\pi Y$ equivalent to the canonical basis of $c_0(\tau)$. There exist constants $0 < \delta \leq M$ such that

$$\delta \sup_{i\in F} |\lambda_i| \leq \left\| \sum_{i\in F} \lambda_i u_i \right\|_\pi \leq M \sup_{i\in F} |\lambda_i|$$

for all finite and non-empty subsets $F$ of $\tau$ and all families of scalars $(\lambda_i)_{i\in F}$.

For every $i \in \tau$ we have

$$\delta \leq \|u_i\|_\pi = \lim_\alpha \|\pi_\alpha(u_i)\|_\pi$$

and so there exists an index $\alpha_i \in \mathcal{A}$ such that $\|\pi_{\alpha_i}(u_i)\|_\pi \geq \delta/2$.

For every $\alpha \in \mathcal{A}$ consider the set

$$I_\alpha = \{\, i \in \tau \,:\, \|\pi_\alpha(u_i)\|_\pi \geq \delta/2 \,\}.$$

Since $\bigcup_{\alpha\in\mathcal{A}} I_\alpha = \tau$ and $\mathfrak{n} < \mathrm{cf}(\tau)$, there exist an index $\beta_0 \in \mathcal{A}$ and a subset $\tau_1 \subset \tau$ such that $|\tau_1| = \tau$ and, for every $i \in \tau_1$,

$$\|\pi_{\beta_0}(u_i)\|_\pi \geq \frac{\delta}{2}.$$

Let $k = \dim T_{\beta_0}(X)$ and let $(v_j)_{1\leq j\leq k}$ be a normalized basis of $T_{\beta_0}(X)$. By Lemma 3.2, there exist bounded linear operators $S_1, \ldots, S_k : X\widehat{\otimes}_\pi Y \to Y$ satisfying

$$\pi_{\beta_0}(u) = \sum_{j=1}^{k} v_j \otimes S_j(u)$$

for each $u \in X\widehat{\otimes}_\pi Y$. Thus, for every $i \in \tau_1$,

$$\frac{\delta}{2} \leq \|\pi_{\beta_0}(u_i)\|_\pi = \left\| \sum_{j=1}^{k} v_j \otimes S_j(u_i) \right\|_\pi \leq \sum_{j=1}^{k} \|v_j \otimes S_j(u_i)\|_\pi = \sum_{j=1}^{k} \|S_j(u_i)\|.$$

Now, $\tau_1$ is infinite so there exist an integer $1 \leq k_0 \leq k$ and a subset $\tau_2 \subset \tau_1$ such that $|\tau_2| = \tau$ and, for every $i \in \tau_2$,

$$\|S_{k_0}(u_i)\| > \frac{\delta}{2k}.$$

Since $(u_i)_{i\in\tau}$ is equivalent to the canonical basis of $c_0(\tau)$, by Lemma 2.1 there exists a subset $\tau_3 \subset \tau_2$ such that $|\tau_3| = \tau$ and $(S_{k_0}(u_i))_{i\in\tau_3}$ is equivalent to the unit basis of $c_0(\tau_3)$. This finishes the proof.

## 6. On copies of $c_0(\tau)$ spaces in $L_p(D^\tau, Y)$ spaces

In this last section, we provide the proof of Theorem 1.13.

*Proof of Theorem* 1.13. The proof is a slight modification of the proof of Theorem 1.7. Let $(f_i)_{i\in\tau}$ be a family equivalent to the canonical basis of $c_0(\tau)$ in $L_p(D^\Gamma, Y)$. There exist constants $0 < \delta \leq M$ such that

$$\delta \sup_{i\in F} |\lambda_i| \leq \left\| \sum_{i\in F} \lambda_i f_i \right\|_p \leq M \sup_{i\in F} |\lambda_i|$$

for all finite and non-empty subsets $F$ of $\tau$ and all families of scalars $(\lambda_i)_{i\in F}$.

Denote by $\mathcal{F}$ the family of all finite, non-empty subsets of $\Gamma$. By Lemma 4.3, for every index $i \in \tau$ there exists a set $\Omega_i \in \mathcal{F}$ such that $\|S_{\Omega_i}(f_i) - f_i\|_p < \delta/2$, and thus

$$\|S_{\Omega_i}(f_i)\|_p > \frac{\delta}{2}.$$

For every $\Omega \in \mathcal{F}$ consider the set

$$I_\Omega = \{\, i \in \tau \,:\, \|S_\Omega(f_i)\|_p > \delta/2 \,\}.$$

Since $\bigcup_{\Omega\in\mathcal{F}} I_\Omega = \tau$ and $|\mathcal{F}| = |\Gamma| < \mathrm{cf}(\tau)$, there exist subsets $G \in \mathcal{F}$ and $\tau_1 \subset \tau$ such that $|\tau_1| = \tau$ and, for every $i \in \tau_1$,

$$\|S_G(f_i)\|_p > \frac{\delta}{2}.$$

Using the exact same notation used in the proof of Theorem 1.7 (see (4.3)), recall that

$$\|S_G(f)\|_p^p = \sum_{t\in D^G} 2^{-|G|} \|y(f,t)\|^p$$

for every $f \in L_p(D^\Gamma, Y)$. In particular, we obtain

$$\frac{\delta^p}{2^p} < \|S_G(f_i)\|_p^p = \frac{1}{2^{|G|}} \sum_{t\in D^G} \|y(f_i,t)\|^p$$

for every $i \in \tau_1$. Now, $\tau_1$ is infinite and so there exist an element $t_0 \in D^G$ and a subset $\tau_2 \subset \tau_1$ such that $|\tau_2| = \tau$ and, for every $i \in \tau_2$,

$$\|y(f_i,t_0)\|^p > \frac{\delta^p}{2^p} \frac{2^{|G|}}{|G|} \geq \frac{\delta^p}{2^p} > 0,$$

that is,

$$\|y(f_i, t_0)\| > \frac{\delta}{2}.$$

Once again, consider the bounded linear operator $T : L_p(D^\Gamma, Y) \to Y$ defined by $T(f) = y(f, t_0)$ for every $f \in L_p(D^\Gamma, Y)$. The inequality above gives

$$\|T(f_i)\| > \frac{\delta}{2}$$

for every $i \in \tau_2$. Since $(f_i)_{i\in\tau}$ is equivalent to the canonical basis of $c_0(\tau)$, by Lemma 2.1 there exists a subset $\tau_3 \subset \tau_2$ such that $|\tau_3| = \tau$ and $(T(f_i))_{i\in\tau_3}$ is equivalent to the canonical of $c_0(\tau_3)$ in $Y$, and the proof is complete.

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Department of Mathematics, IME, University of São Paulo, Rua do Matão 1010, São Paulo, Brazil
*Email address*: vinicius.cortes@usp.br

Department of Mathematics, IME, University of São Paulo, Rua do Matão 1010, São Paulo, Brazil
*Email address*: eloi@ime.usp.br

Aix Marseille Université, CNRS, Centrale Marseille, I2M, Marseille, France
*Email address*: christian.samuel@univ-amu.fr