An Application of Algebraic Geometry Tools to Intuitionistic Logic

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Keywords: Logic, Category Theory, Algebraic Geometry

Makkai has established a few theorems extending Stone Duality to full First-Order logic (FOL) using categorical tools (see, for instance, [2] and [3]). More specifically, for any theory $\mathbb{T}$ in Classical FOL, we can construct its classifying Boolean pretopos $\mathcal{T}$. Models of this theory in any other Boolean pretopos $\mathcal{R}$ will be then given by Boolean pretopos functors $\mathcal{T} \to \mathcal{R}$. In this way, we can prove an adjunction

$$\mathbf{BP}^{*\mathrm{op}} \underset{F}{\overset{G}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{UG}$$

between the 2-category $\mathbf{BP}^*$ of Boolean pretopoi and natural isomorphisms between them and the 2-category $\mathbf{UG}$ of ultragroupoids. An ultracategory is a category with an additional "ultraproduct structure", the main example being the category Mod($\mathbb{T}$) of set valued models of a FOL theory $\mathbb{T}$ and the usual ultraproduct of models. An ultragroupoid is, then, an ultracategory in which all morphisms are invertible. One side of this adjunction is essentially Gödel's Completeness Theorem.

From the Algebraic Geometry side, the most intriguing highlights in Makkai's theorems are the connections established with the descent theorem for Grothendieck Toposes of Joyal and Tierney [1], as it was shown by Zawadowski [5]. This fact was first glimpsed by Pitts [4] when he proved in a sequence of papers that Heyting pretoposes satisfy the interpolation property. Zawadowski was able to show a more general theorem in respect to all pretoposes and make an explicit link between Makkai's Stone Duality, the interpolation property, and the proof of a (lax) descent theorem. It is worth noting that descent theory is at the heart of scheme theory, one of the first milestones of Algebraic Geometry.

However, something is missing in the literature. The Stone Duality was (explicitly) proven for Boolean pretoposes (Classical FOL), Zawadowski's main descent theorem was proven for general pretoposes (Coherent Logic) and Pitt's Interpolation Theorem was proven for Heyting pretoposes (Intuitionistic FOL). Therefore, our aim is to fill the gaps and prove Stone Duality and the Descent Theorem, but now for Heyting pretoposes. In the meantime, we wish to establish the explicit connections with several other pieces of work that somehow orbit this result.

Finally, the proposal of this talk is to present some the basic ingredients of the constructions we mentioned above, as well as give a conceptual map that could possibly illustrate how those ideas provide a link between the most of the category theoretical advances we have had in the last half of a century.

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