# A First-Order Characterization and Topological Properties of Gel'fand $\mathcal{C}^\infty-$Rings

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**Definition 1.** *A $\mathcal{C}^\infty-$**structure** on a set $A$ is a pair $\mathfrak{A} = (A, \Phi)$, where:*

$$\begin{array}{rccc} \Phi: & \bigcup_{n\in\mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \to & \bigcup_{n\in\mathbb{N}} \mathrm{Func}\,(A^n; A) \\ & (f : \mathbb{R}^n \overset{\mathcal{C}^\infty}{\to} \mathbb{R}) & \mapsto & \Phi(f) := (f^A : A^n \to A) \end{array},$$

*that is, $\Phi$ interprets the **symbols** –here considered simply as syntactic symbols rather than functions– of all smooth real functions of $n$ variables as $n-$ary function symbols on $A$.*

*A $\mathcal{C}^\infty-$structure $\mathfrak{A} = (A, \Phi)$ is a $\mathcal{C}^\infty-$**ring** whenever it preserves projections and all equations between smooth functions.*

In this work we introduce the concept of a "Gel'fand $\mathcal{C}^\infty-$ring" in this first-order language as follows:

**Definition 2.** *A $\mathcal{C}^\infty$-ring $A$ is a **Gelfand $\mathcal{C}^\infty$-ring** whenever the following formula is true in it:*

$$(\forall x \in A)(\exists y \in A)(\exists y' \in A)((1 - x \cdot y) \cdot (1 - (1 - x) \cdot y') = 0) \tag{1}$$

Gel'fand $\mathcal{C}^\infty-$rings compose a full subcategory of $\mathcal{C}^\infty$**Ring**, which we denote by $\mathcal{C}^\infty$**GfRing**, and have the following remarkable properties:

- A $\mathcal{C}^\infty-$ring, $A$, is a Gel'fand $\mathcal{C}^\infty$ if, and only if, every prime and $\mathcal{C}^\infty-$radical prime ideal ( see [2]) is contained in a unique maximal ideal;
- $\mathcal{C}^\infty$**GfRing** is closed under products, quotients and directed colimits (for their definition, see [3]);
- Every $\mathcal{C}^\infty-$domain (for the definition, see [3]) is a Gel'fand $\mathcal{C}^\infty$ ring (the converse is not true);
- Every von Neumann regular $\mathcal{C}^\infty$-ring (see [1]) is a Gel'fand $\mathcal{C}^\infty$-Ring;
- A $\mathcal{C}^\infty$-domain is a Gel'fand $\mathcal{C}^\infty$-ring if, and only if, it is a local $\mathcal{C}^\infty$-ring;

Moreover, we show the following topological result concerning the topology of the smooth versions of the prime and maximal *spectra* (for the definitions, see [1]) of Gel'fand $\mathcal{C}^\infty$-rings:

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- Whenever $A$ is a Gel'fand $\mathcal{C}^\infty$-ring, the map:

$$\begin{array}{rccc} \mu: & \mathrm{Spec}^\infty(A) & \to & \mathrm{Specm}^\infty(A) \\ & \mathfrak{m} & \mapsto & \mathfrak{m}_\mathfrak{p} \end{array}$$

  which maps every prime ideal to the unique maximal ideal in which it is contained, is a continuous retraction; Conversely, if there is a continous retraction from $\mathrm{Spec}^\infty(A)$ to $\mathrm{Specm}^\infty(A)$, then $A$ is necessarily a Gel'fand $\mathcal{C}^\infty$-ring;

- A $\mathcal{C}^\infty$-ring is Gel'fand if, and only if, $\mathrm{Spec}^\infty(A)$ is a normal topological space.

The main contribution of this work is, thus, to show that the first order notion of a Gel'fand $\mathcal{C}^\infty$-ring proposed in **Definition 2** is a "fair" one, showing some of its interesting unfoldings.

## References


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