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A GENERAL METHOD FOR APPROXIMATING TO THE DISTRIBUTION OF SOME STATISTICS

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SUMMARY. The object of this paper is to show that for any statistic satisfying fairly general conditions, we can construct an adjusted statistic having the same distribution to order $O(n^{-1})$ of an arbitrary first-order approximating distribution. We prove that the multiplication of the statistic by a suitable stochastic correction improves the first order approximation to its distribution . This paper extends the results of the closely related paper by Cordeiro and Ferrari (1991) to cope with several other statistical tests. The resulting expression for the adjustment factor requires knowledge of the Edgeworth-type expansion to order $O(n^{-1})$ for the distribution of the unmodified statistic. In practice its functional form involves some derivatives of the first-order approximating distribution, certain differences between the cumulants of appropriate order in n of the unmodified statistic and those of its first order approximation, and the unmodified statistic itself. Some applications are discussed.

SOME KEY WORDS: Bartlett correction; Chi-squared distribution; Edgeworth-type expansion; Generalized Bartlett correction; Maximum likelihood estimate; Signed likelihood ratio statistic.

1. INTRODUCTION

In the past 15 years or so there has been a renewed interest on Bartlett corrections leading to better approximations of the null distribution of the likelihood ratio statistic by a chi-squared distribution. Computation of Bartlett corrections has been discussed by Lawley (1956), Barndorff-Nielsen and Cox (1984) and Cordeiro (1993a). General formulae for Bartlett corrections have been obtained explicitly in several regression models by Cordeiro (1983, 1985, 1987), Cordeiro and Paula (1989), Cordeiro, Paula and Botter (1994) and authors cited therein. The numerical benefits of Bartlett corrections have been demonstrated by Møller (1986) and Cordeiro (1993b, 1995) among others. The main goal of this paper is to show that Bartlett's technique can be carried out for general continuous statistics.

Several statistical tests rely in some way on first-order approximations derived from distributions other than chi-squared. We are often interested in computing significance levels or confidence intervals based on these first-order approximations. However, it also well known that these approximations may not work well for small or moderate-sized samples. The problem of developing a correction similar to the Bartlett correction to other test statistics was posed by Cox (1988) and solved three years later for statistics which converge to chi-squared by Cordeiro and Ferrari (1991). We now generalize this result to general continuous statistics.

Modified statistics have been widely used to obtain good approximations for classes of statistics associated with the normal and chi-squared distributions. The principal advantage of our main result is that it applies in full generality in a number of senses. First, for rather general parametric models, we can easily improve any continuous statistic by multipling it by an adequate adjustment factor. Second, our technique includes as special cases, some commonly used adjusted statistics, which enables us to study them within the same framewok, rather than as an unrelated collection of adjusted statistics. Third, it allows for nuisance parameters. Our arguments will be informal without explicit attention to regularity conditions, these being essentially those required for the expansions needed for maximum likelihood theory in regular estimation problems.

Let S be a continuous statistic whose distribution function has a known first-order approximating distribution. A natural question is then: Can we find a better approximation to the distribution of the statistic in use? The purpose of our paper is to answer this question to some extent. We propose a new statistic S^* whose distribution function agrees with the first-order approximating distribution to order $O(n^{-1})$, where n is the sample size. Thus, S^* is better approximated by this first-order distribution than S. The key idea for deriving the new statistic is to know the Edgeworth-type expansion for the distribution function of S in terms of the first-order approximating distribution to the required order. We assume the same conditions for the validity of the Edgeworth-type expansions (Feller, 1971; Skovgaard, 1981a, b, 1986). The infinite series expansion for the distribution of S can sometimes be divergent and we need to impose some further restrictions on the cumulants of S for using a truncated expansion to $O(n^{-1})$.

For most applications the ith cumulants κ_i's of S satisfy $\kappa_i = \kappa_{i0} + O(n^{-1/2})$ if $i \leq 2$ and $\kappa_i = \kappa_{i0} + O(n^{1-i/2})$ if $i \geq 3$, as $n \to \infty$, where the cumulants κ_{i0}'s refer to the first-order approximation and do not depend on n. Under these assumptions, we can obtain the Edgeworth-type expansion of the distribution of S corrected to order $O(n^{-1})$ using a truncated series with a few terms. The truncated series gives in general significant improvement over the first-order approximation.

In Section 2 we define a few important concepts which will be used in the following sections and discuss some theoretical aspects of using Edgeworth-type expansion for the density function of any continuous statistic S. In Section 3 we define a new statistic S^* whose distribution function is identical to the first order approximating distribution ignoring terms of order less than n^{-1}. We show that whenever a truncated Edgeworth-type expansion for the distribution function $F_S(x)$ of S to $O(n^{-1})$ is available, a new statistic S^* can be worked out in such a way that it will generally improve the original inference. The modified statistic is determined by a simple multiplicative adjustment to the statistic S which makes the terms of order

$n^{-1/2}$ and n^{-1} in the asymptotic expansion of the distribution function of the modified statistic S^* vanish. This scaling factor extends Bartlett's idea of correcting likelihood ratio statistics to several other types of statistics. It is called "generalized Bartlett correction" and can be given as a function of some derivatives of the first-order approximating distribution, the differences $(\kappa_i - \kappa_{i0})$'s between the cumulants of order greater than $O(n^{-3/2})$ and the unmodified statistic itself. Finally, in Section 4, we show through several applications that our method has the potential to be a very useful contribution to statistical literature since it comprises a very wide spectrum of improved statistics widely used to test hypotheses of interest including Bartlett-type corrected statistics which converge to chi-squared, the Cornish-Fisher polynomial transformation to normality and improved maximum likelihood estimates.

2. BACKGROUND

Suppose we are interested in the distribution of a continuous scalar statistic S, whose distribution depends on parameters θ and n. It is commom practice among applied statisticians to make inferences on θ by computing tail-area probabilities based on an approximating known distribution for S. In this section we discuss the nature of such approximations for arbitrary initial approximating distributions. In the next section we show how to improve the inferences.

We begin with the univariate Edgeworth-type expansion for the density of any continuous scalar statistic S, in which the error of the approximation approaches zero as some parameter n, typically the sample size, tends to infinity. In order to improve the approximation, even for small n, further terms in the asymptotic expansion may be required. One of the principal aims in this article is to introduce a very simple new procedure for modifying S to match the distribution function of the modified statistic S^* with some arbitrary known distribution function of reference, $F_Z(x)$ say, such that the error of the approximation is of order $O(n^{-3/2})$.

We now examine the density of S in terms of an arbitrary first-order approximating known density of a scalar random variable Z. Suppose that $F_Z(x)$ is absolutely continuous with known density function $f_Z(x)$. Further, we assume that the derivatives of $f_Z(x)$ of the requisite order are continuous in the support of S. Let the cumulants of S and Z be $\{\kappa_i\}$ and $\{\kappa_{i0}\}$, respectively. These cumulants are assumed to be known at least up to some order. In regular problems, the statistic S has a density function $f_S(x)$ which admits an expansion in terms of the initial known approximating density $f_Z(x)$ of Z, of the form (McCullagh, 1987; equation (5.3))

$$f_S(x) = f_Z(x) + \sum_{i=1}^{m} (-1)^i \eta_i \frac{D^i f_Z(x)}{i!}, \tag{1}$$

as $m \to \infty$, where $D^i f_Z(x) = d^i f_Z(x)/dx^i$.

In equation (1) the η_i's are "formal moments" obtained by treating the differences $(\kappa_i - \kappa_{i0})$'s as "formal cumulants". These "formal moments" are generally used to obtain a formal Edgeworth expansion (1) for the density function of S. It would be valid provided suitable regularity conditions hold. The truncated series approximation (1) for $f_S(x)$ is continuous and it does not hold in general if S has a discrete distribution. When $f_Z(x)$ is the standard normal density, equation (1) is known in the literature as Gram-Charlier expansion. It is important to emphasize that these differences between the cumulants of S and Z are not, in general, the cumulants of any real random variable (McCullagh, 1987; Section 5.2). The coefficients η_i's are obtained from the formulae that give moments in terms of cumulants (see, for example, Kendall and Stuart, 1977, equations (3.37)).

Equation (1) shows that a valid asymptotic expansion for the density function of any continuous statistic S, assuming that some conditions hold, can be obtained in general, up to any order of accuracy, from the knowledge of both cumulants of S and Z, and the derivatives of the reference density $f_Z(x)$. Such conditions imposed directly on the statistic are difficult to state in a broad sense, at least in the context of the present article. For details on regularity conditions, see McCullagh (1987, Chapter 16) and Skovgaard (1981a, b, 1986). In several applications we have in mind, Z has a normal distribution with zero mean and unit variance or a chi-squared distribution with known degrees of freedom. However, the choice of the first-order approximating density is usually made in order to minimize the number of correction terms in (1), and it may be more useful to consider a density $f_Z(x)$ other than the normal or chi-squared. A variety of distributions for Z can be exploited to give better finite sample approximations to the true density of S. Although the important conceptual advantage of (1) is that we can use any first-order approximating density, the limiting distribution of S (when it is known) is often the most convenient distribution to consider for Z in terms of theoretical calculations (see McCullagh, 1987, Chapter 5, for further details).

We are not really interested in exploiting the convergence of the series in (1) but our main objective is to take a small number of terms to guarantee a good approximation to $f_S(x)$. No attempt will be made here to state precisely the conditions in which the expansion in (1) is supposed to approximate the density of S. However, in most situations, excluding lattice problems, and under suitable smoothness conditions, we can approximate $f_S(x)$ by a trucated series as in (1) such that the error is of a specified order (see, for example, Skovgaard, 1981a, b, 1986). If S is a standardized sum of n independent random variables and Z has a standard normal density, we have $\kappa_1 - \kappa_{10} = \kappa_2 - \kappa_{20} = 0$ and $\kappa_i - \kappa_{i0} = O(n^{-(i-2)/2})$, for $i \geq 3$. Thus, the error of the expansion (1) with $m \geq 3$ terms is typically of order $O(n^{-(m-3)/2})$. For that reason, we content ourselves with 6 terms in the Gram-Charlier expansion, leaving an error that is of order $O(n^{-3/2})$.

Series expansion (1) can be expressed as a multiplicative correction to $f_Z(x)$

$$f_S(x) = f_Z(x)\Big\{1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!}\eta_i h_i(x)\Big\}, \tag{2}$$

where $h_i(x) = (-1)^i D^i f_Z(x)/f_Z(x)$. In many instances, the functions $h_i(x)$'s are simple with nice mathematical properties. In the case of the Gram-Charlier expansion, the leading term $f_Z(x)$ is the unit normal density and the $h_i(x)$'s reduce to the standard Hermite polynomials. Here, $\eta_1 = \kappa_1$, $\eta_2 = \kappa_2 - 1$ $\eta_i = \kappa_i$, for $i \geq 3$. If the basic approximating density is gamma, the $h_i(x)$'s are the Laguerre polynomials. Another special case of (2) is obtained when Z has a chi-squared distribution with q degrees of freedom, denoted here by χ^2_q. If $f_q(x)$ is the density function of χ^2_q, its derivatives are obtained recursively by

$$\frac{d^r f_q(x)}{dx^r} = \frac{1}{2^r}\sum_{j=0}^{r}(-1)^{r+j} f_{q-2j}(x)$$

and

$$f_{q+2l}(x) = \frac{\Gamma(q/2)x^l f_q(x)}{2^l\Gamma(q/2+l)},$$

where $\Gamma(.)$ is the gamma function.

From these equations we can prove that $h_i(x) = 2^{-1}\sum_{j=0}^{i}(-1)^j v_j/x^j$, where $v_j = \Pi_{r=1}^{j}(q-2r)$ if $j \geq 1$ and $v_0 = 1$. Thus,

$$h_i(x) = 2^{-1}\Big\{1 - \frac{q-2}{x} + \frac{(q-2)(q-4)}{x^2} + \ldots + (-1)^i\frac{(q-2)(q-4)\ldots(q-2i)}{x^i}\Big\}, \tag{3}$$

is an ith degree polynomial in x^{-1}. An expansion for the density of any statistic S to order $O(n^{-1})$, which has an asymptotic chi-squared distribution, can then be obtained by substituting the $h_i(x)$'s given in (3) into equation (2), and observing that the η_i's are the n^{-1} terms of the cumulants κ_i's of S, i.e., $\eta_i = \kappa_i - 2^i(i-1)!q$ for $i \geq 1$.

3. AN ADJUSTED STATISTIC

When testing a statistical hypothesis or estimating unknown parameters, it is often convenient to use an asymptotic approximation to the distribution of a statistic. Large sample assumptions are then commonly used in statistics since exact results are not always available. In such cases, inferences rely on what is called first-order asymptotics, i.e., they employ the quantiles of a known limiting distribution, but they may be inaccurate for small or moderate sample sizes. This section addresses the issue of obtaining a new statistic S^* which is better approximated by the first-order limiting distribution.

Let S be a given one-dimensional statistic which is assumed to be derived from a sample Y, of n independent observations having densities that depend on the vector parameter θ. We consider continuous scalar statistics whose density functions can be validly represented by expansion (1), as n, the dimension of Y increases. Let $F_S(x)$ and $F_Z(x)$ be the cumulative distribution functions of S and Z corresponding to $f_S(x)$ and $f_Z(x)$ respectively. Now suppose that $F_Z(x)$ is free from n and then $Z = O_p(1)$. Further, the distribution function $F_Z(x)$ will be assumed to be arbitrarily differentiable and it is also assumed that $f_Z(x) > 0$ for all x in the support of S. The formal Edgeworth-type expansion for the distribution function of S, obtained by integrating (1), is

$$F_S(x) = F_Z(x) + \sum_{i=1}^{m}(-1)^i \eta_i \frac{D^i F_Z(x)}{i!}, \tag{4}$$

as $m \to \infty$. Thus, equation (4) comes directly from (1) by changing $f_Z(x)$ for $F_Z(x)$. This equation is given in different notation by Hill and Davis (1968) for arbitrary analytic $F_Z(x)$. The truncated series approximation (4) is continuous. If S is a discrete random variable, $F_S(x)$ is discontinuous with jumps of order $n^{-1/2}$ at the support points of S. For this reason, no continuous series could approximate $F_S(x)$ with uniform accuracy in any non-trivial interval of $\boldsymbol{R}$.

In many statistical applications, we can group the terms of (4) according to their order in n. Then, sucessive terms in the re-grouped series can decrease (monotonically) in half-powers of n. Fortunately, many statistics can be given by sums of independent identically distributed random variables, and this approach can be achieved after suitable standardization of the statistic S and by choosing $F_Z(x)$ adequately, for example, as the limiting distribution function of S. We can therefore write $F_S(x)$ in the form

$$F_S(x) = F_Z(x) + A_1(x) + A_2(x) + O(n^{-3/2}), \tag{5}$$

where $A_1(x)$ and $A_2(x)$ are terms of orders $O(n^{-1/2})$ and $O(n^{-1})$, respectively, which depend on some differences $(\kappa_i - \kappa_{i0})$'s of the cumulants of S and Z. The terms $A_1(x)$ and $A_2(x)$ may be polynomials in x but this is not always the case.

Essentially, the idea behind our procedure of modifying S is based on the fact that the distribution function $F_S(x)$ may be formally expanded as in equation (5). We shall restrict ourselves to series expansions up to order n^{-1} leaving in (5) an error that is of order $O(n^{-3/2})$. We now prove that quite generally the statistic S can be modified by suitable functions $b_1(S)$ and $b_2(S)$ of the statistic S itself of orders $n^{-1/2}$ and n^{-1} to produce an adjusted statistic S^* which has the same distribution of Z to $O(n^{-1})$. The form (5) of the distribution function of S suggests the use of a modified statistic defined by

$$S^* = S - b_1(S) - b_2(S), \tag{6}$$

where $b_i(S) = O_p(n^{-i/2})$ for $i = 1, 2$ are additive stochastic correction terms as functions of the statistic S.

The functions $b_1(S)$ and $b_2(S)$ are now determined to make the distribution of S^* to order n^{-1}, $F_{S^*}(x)$ say, identical to $F_Z(x)$. The formula (1) of Cox and Reid (1987) is used in conjunction with (6) to derive an expansion for the distribution function of interest $F_{S^*}(x)$. Applying Cox and Reid's (1987) formula to equation (6) (see also equation (3.67) in Barndorff-Nielsen and Cox, 1989), under appropriate conditions, we find to $O(n^{-1})$

$$F_{S^*}(x) = F_S(x) - \mathrm{E}\{-b_1(S) \mid S = x\} f_S(x) - \mathrm{E}\{-b_2(S) \mid S = x\} f_S(x) + \frac{1}{2}\frac{d}{dx}[\mathrm{E}\{b_1(S)^2 \mid S = x\} f_S(x)]. \quad (7)$$

Some conditions are necessary to bound the remainder term in equation (7) (see Cox and Reid, 1987). It is now straightforward to conclude from equations (5) and (7) that the equality $F_{S^*}(x) = F_Z(x)$ holds to order n^{-1} if and only if

$$A_1(x) + A_2(x) + b_1(x)f_S(x) + b_2(x)f_S(x) + \frac{1}{2}\frac{d}{dx}\{b_1(x)^2 f_S(x)\} = 0.$$

Using (1) and collecting terms of orders $n^{-1/2}$ and n^{-1} in the last equation yields

$$A_1(x) + b_1(x)f_Z(x) = 0$$

and

$$A_2(x) + b_1(x)A_1'(x) + b_2(x)f_Z(x) + b_1(x)b_1'(x)f_Z(x) + \frac{1}{2}b_1(x)^2 f_Z'(x) = 0,$$

where primes denote derivatives with respect to x. It is easy to verify that these equations have at least one solution given by $b_1(x) = -A_1(x)/f_Z(x)$ and $b_2(x) = -A_2(x)/f_Z(x) + A_1(x)^2 f_Z'(x)/\{2f_Z(x)^3\}$, provided that $f_Z(x)$ is non-zero in the support of S.

Consequentely, the modified statistic S^* whose distribution function is $F_Z(x)$ to order n^{-1} is given by (assuming $S \neq 0$ a.s.)

$$S^* = S\Big[1 + \frac{A_1(S)}{f_Z(S)S} + \frac{1}{S}\Big\{\frac{A_2(S)}{f_Z(S)} - \frac{A_1(S)^2 f_Z'(S)}{2f_Z(S)^3}\Big\}\Big]. \quad (8)$$

The method that leads to (8) is formally correct provided only that the distribution of S has a valid Edgeworth expansion (4) up to and including the $O(n^{-1})$ term. The bracketed multiplying factor in equation (8) is a kind of stochastic adjustment involving the $n^{-1/2}$ and n^{-1} functions $A_1(x)$ and $A_2(x)$ of expansion (5), the density $f_Z(x)$ with its first derivative $f_Z'(x)$ and the statistic S itself. Clearly, the terms $A_1(x)$ and $A_2(x)$ are functions themselves of certain differences between the cumulants of S and Z and of some derivatives of the distribution function, a fact that may be seen from equations (4) and (8). In general, the stochastic multiplying factor in (8) may be written as $1 + b(S, \eta_i, D^i F_Z)$, where the notation emphasizes the dependence of the derivatives of the distribution function $F_Z(x)$ and "formal moments" η_i's and the unmodified statistic S. Given its similarity with the Bartlett-type correction for a class of chi-squared statistics (Cordeiro and Ferrari, 1991), the adjustment factor $1 + b(S, \eta_i, D^i F_Z)$ will be called "generalized Bartlett correction". This is a very general result which can be used to improve many important tests in statistics and econometrics.

Instead of modifying S, an alternative approach is to modify the quantiles of the reference distribution in order to make better inferences based on S. From formula (2) of Cox and Reid (1987) (see also expression (3.68) in Barndorff-Nielsen and Cox, 1989) and using the fact that S^* in (6) has distribution function $F_Z(x)$ to order n^{-1}, it follows that, to this order, $F_S(x^*) = F_Z(x)$, where $x^* = x + b_1(x) + B_2(x)$ with $B_2(x) = b_2(x) + b_1(x)b_1'(x)$. Then,

$$x^* = x\Big[1 - \frac{A_1(x)}{xf_Z(x)} - \frac{1}{x}\Big\{\frac{A_1(x)A_1'(x)}{f_Z(x)^2} + \frac{A_2(x)}{f_Z(x)} + \frac{A_1(x)^2 f_Z'(x)}{2f_Z(x)^3}\Big\}\Big]. \tag{9}$$

Therefore, improved inferences can be achieved from two distinct viewpoints, which are equivalent to $O(n^{-1})$. First, we can construct a new statistic in (8) which is better approximated by the first-order approximating distribution $F_Z(x)$. Second, we can obtain a new distribution based on the modified upper percentile point (9) of our statistic S which is closer to the true distribution of S than this first-order approximating distribution. It is clear that the functional forms of the multiplicative corrections to improve the upper tail of S and to improve the statistic S itself are not in general the same, unless the $n^{-1/2}$ term $A_1(x)$ is a constant not depending on x.

In the next section we show that some improved statistics widely used follow directly from equation (8). In general the η_i's are functions of the unknown parameters and we can use the statistic $\hat{S}^* = \{1+b(S, \hat{\eta}_i, D^i F_Z\})$ with the parameters η_i's replaced by consistent estimates $\hat{\eta}_i$'s because the error committed would typically be $O_p(n^{-3/2})$. This result follows from the equivalence of formulae (3c) and (4c) of Cox and Reid (1987).

4. SPECIAL CASES

In this section we shall consider some special cases of equation (8) in order to show its importance and usefulness to produce more accurate approximations to the distributions of statistics. Examples include Cornish-Fisher's formula for the polynomial transformation to normality, accurate formula for correcting statistics which converge to a chi-squared distribution (Cordeiro and Ferrari, 1991), improvements in goodness-of-fit tests based on V-statistics (Cordeiro and Pérez-Abreu, 1995), development of corrections to signed likelihood ratio statistics and corrected maximum likelihood estimates. Several other special cases could also be easily obtained because of the generality of this technique for correcting statistics to order $O(n^{-1})$.

4.1 STATISTICS HAVING A LIMITING NORMAL DISTRIBUTION

Let T be a statistic whose distribution depends on parameters n and θ. Assume that there exists $\mu = \mu(\theta)$ and $\sigma = \sigma(\theta)$ such that the standardized statistic $S = n^{1/2}(T - \mu)/\sigma$ has mean zero and unit variance and higher-order cumulants of the form $\kappa_r(S) = \rho_r n^{1-r/2}$, for $r \geq 3$, where the coefficients ρ_r's depend on the cumulants of the population distribution. Further, we assume that S converges in distribution to a standard normal random variable.

It is appropriate here to consider the basic limiting distribution of S in order to garantee an asymptotic expansion for the distribution of S in decreasing powers of $n^{-1/2}$. The Edgeworth expansion for the distribution function of S to $O(n^{-1})$ is derived in a straightforward way from (4) (see McCullagh, 1987, equation (5.12))

$$F_S(x;\rho) = \Phi(x) - \phi(x)\Big[\frac{\rho_3 h_2(x)}{6\sqrt{n}} + \frac{1}{n}\Big\{\frac{\rho_4 h_3(x)}{24} + \frac{\rho_3^2 h_5(x)}{72}\Big\}\Big], \tag{10}$$

where $\phi(x)$ and $\Phi(x)$ are the standard normal density and distribution functions, respectively. The polynomials appearing in (10) were defined after (2), and become the Hermite polynomials. They are given by $h_2(x) = x^2 - 1$, $h_3(x) = x^3 - 3x$ and $h_5(x) = x^5 - 10x^3 + 15x$. The remaining terms of order $O(n^{-r/2})$, for $r \geq 3$, can be found in Niki and Konishi (1986). Combining equations (5) and (10), we can see immediately that $A_1(x) = -\rho_3\phi(x)h_2(x)/(6\sqrt{n})$ and $A_2(x) = -\phi(x)\{\rho_4 h_3(x)/24 + \rho_3^2 h_5(x)/72\}/n$.

By substituting these results in equations (8) and (9), we find

$$S^* = S - \frac{\rho_3}{6\sqrt{n}}(S^2 - 1) + \frac{1}{12n}\Big\{\frac{\rho_3^2(4S^3 - 7S)}{3} - \frac{\rho_4(S^3 - 3S)}{2}\Big\}. \tag{11}$$

$$x^* = x + \frac{\rho_3}{6\sqrt{n}}(x^2 - 1) - \frac{1}{12n}\Big\{\frac{\rho_3^2(2x^3 - 5x)}{3} - \frac{\rho_4(x^3 - 3x)}{2}\Big\}. \tag{12}$$

Equation (11) is just the classical Cornish-Fisher polynomial transformation to normality when stochastic quantities of order $O_p(n^{-3/2})$ and smaller are neglected, i.e., $S^* \sim N(0,1) + O_p(n^{-3/2})$ (see McCullag, 1987, p. 166). Also, equation (12) gives the approximate percentage points of S expressed in terms of the standard normal percentage points ignoring quantities of order $O(n^{-3/2})$ and smaller (see McCullagh, p. 171). This special case can be regarded as a partial check of the validity of equations (9) and (10).

Many extensively used statistics can be expressed as sums of independent and identically distributed random variables $Y_1, \ldots, Y_n$ having finite cumulants δ_r to some order. Other statistics can be accurately approximated this way. In these cases, classical results due to the central limit theory show that, under fairly general conditions, the standardized sum $S = n^{1/2}(\sum Y_i - n\delta_1)/(n\delta_2)$ converges in distribution to a standard normal random variable. This means that equations (11) and (12) hold with the two constants ρ_3 and ρ_4 being the standardized cumulants corresponding to δ_3 and δ_4, namely $\rho_3 = \delta_3/\delta_2^{3/2}$ and $\rho_4 = \delta_4/\delta_2^2$. Stronger forms of the central limit theorem that are validy under substantially weaker conditions than those assumed here are available to apply equations (11) and (12). The assumption that the Y_i's are independent and identically distributed random variables is not essential.

Another simple example of (11) involves the standardized random variable $S = (\chi_n^2 - n)/\sqrt{2n}$ which is asymptotically normally distributed with zero mean and unit variance. The third and fourth cumulants of S yield $\rho_3 = 2\sqrt{2}$ and $\rho_4 = 12$. Thus the adjusted random variable S^* follows from (11) as $S^* = S - \sqrt{2}(S^2-1)/(3\sqrt{n}) + (7S^3 - S)/(18n)$, which is asymptotically $N(0,1)$ with error $O(n^{-3/2})$. We emphasize that the Wilson-Hilferty transformation $S_1 = (9n/2)^{1/2}\{(\chi_n^2)^{1/3} - 1\}$ is not asymptotically standard normal even to order $O(n^{-1/2})$, although Cox and Reid's (1987) modification $S_2 = S_1 n^{-1/3} + (\sqrt{2}/3)n^{-5/6}$ is asymptotically $N(0,1)$ with error $O(n^{-1})$. Clearly, the form S^* is superior to S_1 and S_2 in terms of normal approximations.

4.2 CORRECTED TEST STATISTICS WHOSE ASYMPTOTIC DISTRIBUTIONS ARE χ^2

We now apply the results of Section 3 by considering a class of statistics for testing simple or composite null hypotheses whose null asymptotic distributions are central chi-squareds. This is an important class of statistics since it includes some of the most used tests in econometrics, such as the likelihood ratio, Lagrange multiplier and Wald tests.

For any statistic S whose null asymptotic distribution is central chi-squared with q degrees of freedom, under mild regularity conditions, we can write its distribution function to $O(n^{-1})$ as (Chandra, 1985)

$$F_S(x) = F_q(x) + \sum_{i=0}^{k} a_i F_{q+2i}(x), \tag{13}$$

where the a_i's of order n^{-1} are functions of the unknown parameters and $F_q(x)$ is the distribution function of χ_q^2. In addition to (13), the condition $\sum a_i = 0$ is necessary to produce a distribution function to $O(n^{-1})$. Combining formulae (5) and (13) gives $A_1(x) = 0$ and $A_2(x) = \sum_{i=1}^{k} a_i F_{q+2i}(x)$. From (8) and using the recurrence relation $F_{r+2}(x) = F_r(x) - (2x/r)dF_r(x)/dx$, one can verify that the multiplying factor $1 + b(S, \eta_i, D^i F_Z)$ reduces to a polynomial in S of degree at most $k-1$. Hence,

$$S^* = S\{1 - 2\sum_{i=1}^{k}(\sum_{l=i}^{k} a_l)\mu_i'^{-1} S^{i-1}\}, \tag{14}$$

where $\mu_i' = \mathrm{E}\{(\chi_q^2)^i\}$. This result was first given by Cordeiro and Ferrari (1991, equation (16)). Formula (14) can be used to improve many important tests in statistics and econometrics (Cordeiro, Ferrari and Paula, 1993; Cribari-Neto and Ferrari, 1995a, b, c; Ferrari and Cordeiro, 1995). An alternative way of obtaining an improved test is to consider the unmodified statistic S together with the modified percentage points x^* given in (9). It can be easily seen that

$$x^* = x\{1 + 2\sum_{i=1}^{k}(\sum_{l=i}^{k} a_l)\mu_i'^{-1} x^{i-1}\}.$$

The usual Bartlett correction to improve the likelihood ratio statistic ω comes from (14) with $k = 1$ by noting that $a_0 = -a_1 = -b/2$, where b is the n^{-1} term in $E(\omega)$. Improved score and Wald statistics are special cases of (14) for $k = 3$.

4.3 SIGNED ROOTS OF LIKELIHOOD RATIO STATISTICS

Consider continuous random variables having density function that depends on an unknown scalar parameter θ. Let ω be the usual likelihood ratio statistic $\omega = 2\{l(\hat{\theta}) - l(\theta)\}$, where $l(\theta)$ is the total log-likelihood function. In recent years there has been considerable interest in the signed root of the likelihood ratio statistic $S = \text{sgn}(\hat{\theta} - \theta)\omega^{1/2}$. The standard normal approximation to the distribution of S can be used to construct approximate confidence limits for θ having coverage error of order $n^{-1/2}$. DiCiccio (1984), Jensen (1986) and Barndorff-Nielsen (1986, 1990, 1991) among others have worked with adjustments to S that improve the accuracy of the standard normal approximation.

The most commonly adjusted statistic of S is the signed likelihood ratio standardized with respect to its mean and variance given by

$$S_1 = \frac{S - a_1/\sqrt{n}}{(1 + (a_2 - a_1^2)/n)^{1/2}}, \tag{15}$$

where the quantities a_1 and a_2 are obtained from $E(S) = a_1/\sqrt{n} + O(n^{-1})$ and $E(\omega) = 1 + a_2/n + O(n^{-2})$. Thus, a_1 is the coefficient of the n^{-1} bias of S and a_2 is the n^{-1} term in the Bartlett correction for improving the distribution of ω. The standardized statistic (15) has limiting normal distribution correct to order $n^{-3/2}$. It is also possible to construct a kind of score statistic U (Barndorff-Nielsen, 1986, 1990) of the form $U = S + O_p(n^{-1/2})$, such that the distribution of $S_2 = S + S^{-1}\log(U/S)$ also follows a N(0,1) distribution with relative error $O(n^{-3/2})$. However, the calculation of U could be very difficult in practice. An alternative statistic S_3 to overcome such difficulties was proposed by DiCiccio and Martin (1991), although it is not as accurate as S_1 and S_2. They constructed an auxiliary statistic $T = l(\theta)J(\hat{\theta})^{-1/2}\{K(\hat{\theta})/K(\theta)\}^{1/2}$, where $J(\theta)$ and $K(\theta)$ are the observed and expected informations for θ, and showed that the distribution of $S_3 = S + S^{-1}\log(T/S)$ is asymptotically N(0,1) but with higher error of order n^{-1}. In general, T and U are parameterization invariants and $T = U + O_p(n^{-1})$.

We now give a fourth corrected signed likelihood ratio statistic as an alternative to S_1 and S_2 which is easily calculated from the asymptotic expansion for the distribution of S (Jensen, 1986)

$$F_S(x) = \Phi(x) - \phi(x)\left\{\frac{a_1}{\sqrt{n}} + \frac{a_2 x}{2n}\right\} + O(n^{-3/2}).$$

This result and equation (8) yield $S^* = S\left\{1 + (a_1^2 - a_2)/(2n)\right\} - a_1/\sqrt{n}$. The statistics S_1, S_2 and S^* are equivalent to order n^{-1}. Our next project is to compare them through Monte Carlo simulations.

4.3 IMPROVING V-STATISTICS

We shall address issues related to corrections for V-statistics, i.e., statistics which converge in distribution, under appropriate conditions, to a linear combination of independent chi-squareds with degrees of freedom equal to one. Let $Y_1, \ldots, Y_n$ be a random sample from a distribution F and $h(x, y)$ be a symmetric kernel. Consider the V-statistic $V = n^{-2} \sum_{i=1}^n \sum_{j=1}^n h(Y_i, Y_j)$ and assume that the limiting distribution of $S = nV$ is that of the random variable $Z = \sum_{i=1}^m \lambda_i \chi_{1i}^2$, where the λ_i's are positive numbers related to the kernel h and the χ_{1i}^2's are mutually independent random variables, each with a chi-squared distribution with one degree of freedom. The linear combination of chi-squareds may be finite or infinite. Many well known goodness-of-fit test statistics are V-statistics having the distribution of Z as their limiting distributions. Examples of these are, amongst others, the Cramer-von Mises statistic w^2 (Durbin, 1973), the test of a circle of Watson (1961), the Poissonness tests considered by Baringhauss and Henze (1992), the exponential test of Baringhauss and Henze (1991) and the general chi-squared tests considered by Quiroz and Dudley (1991) and references therein. The case of statistics whose asymptotic distributions are χ^2 covered in Section 4.2 corresponds to finite m with the first q λ_i's equal to one and the remaining $(k-q)$ λ_i's equal to zero.

The main difficulty here in obtaining a generalized Bartlett correction $1 + b(S, \eta_i, D^i F_Z)$ for S using equation (8) is the fact that there is no closed-form expression for the distribution function $F_Z(x)$ nor for the density $f_Z(x)$, even when m is finite and the λ_i's are different from one. However, instead of working with distribution functions and their corresponding expansions, we could work in the Fourier domain through the characteristic functions of S and Z. Let $C_S(t)$ and $C_Z(t)$ be the characteristic functions of S and Z, respectively. The characteristic function of Z $C_Z(t) = \Pi_{j=1}^m (1 - 2it\lambda_j)^{-1/2}$ is valid for m finite or infinite. Inverting (2) yields the following asymptotic expansion for $C_S(t)$

$$C_S(t) = C_Z(t) \Big\{ 1 + \sum_{j=1}^{\infty} \eta_j \frac{(ti)^j}{j!} \Big\}.$$

Let $P_2(t)$ be the term of order n^{-1} in the above expansion of the characteristic function of $S = nV$, i.e., $C_S(t) = C_Z(t)\{1 + P_2(t)\} + O(n^{-3/2})$. For $m = \infty$, the function $P_2(t)$ has been obtained by Götze (1979). When m is finite, however, $P_2(t)$ is a very complicated function given by a Fourier transform of signed measures which are finite linear combinations of convolutions of chi-squared distributions, unless $\lambda_1 = \ldots = \lambda_m = 1$.

Under general conditions, Cordeiro and Pérez-Abreu (1995) obtained the generalized Bartlett correction $b(S)$ to adjust S in $S^* = S\{1 - b(S)\}$ such that S^* and Z have identical distributions to $O(n^{-1})$. In any situation (m finite or infinite), they showed after complicated algebra that

$$b(x) = \{2\pi y f_Z(x)\}^{-1} \int_0^x \int_0^\infty e^{-ity} C_Z(t) P_2(t) dt\, dy, \tag{16}$$

where $f_Z(x)$ is the density function of $Z = \sum_{i=1}^{m} \lambda_i \chi_{1i}^2$. One can not expect to obtain a closed-form formula for the generalized Bartlett correction in (16) even in cases where $C_Z(t)$ and $P_2(t)$ have closed-form expressions, because the n^{-1} terms in the asymptotic expansions for the distribution and density functions of $S = nV$ do not have in general closed-form. Equation (16) must be solved by numerical methods like the algorithm proposed by Imhof (1961). Cordeiro and Pérez-Abreu (1995) give examples of corrections for well known statistics arising from goodness-of-fit tests. For the special case $\lambda_1 = \ldots = \lambda_q = 1$ and $\lambda_{q+1} = \ldots = \lambda_m = 0$, equation (16) produces the polynomial of order $k-1$ in (14).

4.5. MODIFICATION OF STANDARDIZED MAXIMUM LIKELIHOOD ESTIMATES

We seek a statistic that is a function of the maximum likelihood estimate and whose distribution is normal excluding terms of order $O(n^{-3/2})$ and smaller. Let Y be the data vector of length n with total likelihood function $L(\theta) = L(\theta; Y)$ depending on a scalar parameter θ. We assume that the region of the sample space for which $L(\theta; Y) > 0$ does not depend on θ and that some conditions concerning smoothness of $L(\theta; Y)$ and its derivatives with respect to θ hold. The derivatives of the log-likelihood function $l(\theta) = \log L(\theta)$ are denoted by $U_\theta = dl(\theta)/d\theta$, $U_{\theta\theta} = d^2 l(\theta)/d\theta^2$, etc. The standard notation will be adopted for the cumulants of log-likelihood derivatives (Lawley, 1956): $\kappa_{\theta\theta} = \mathrm{E}(U_{\theta\theta})$, $\kappa_{\theta\theta\theta} = \mathrm{E}(U_{\theta\theta\theta})$, $\kappa_{\theta,\theta} = \mathrm{E}(U_\theta^2)$, $\kappa_{\theta,\theta\theta} = \mathrm{E}(U_\theta U_{\theta\theta})$, etc. We define the derivatives of the cumulants by $\kappa_{\theta\theta}^{(\theta)} = d\kappa_{\theta\theta}/d\theta$, etc. All κ's refer to a total over the components of Y and are, in general, of order $O(n)$. Let $\hat{\theta}$ be the maximum likelihood estimate of θ assumed unique for large n.

Under regularity conditions on $L(\theta; Y)$ (Cox and Hinkley 1974, Section 9.1), it follows quite generally that the score function U_θ is asymptotically $\mathrm{N}(0, \kappa_{\theta,\theta})$, so $\hat{\theta}$ satisfies $U_{\hat{\theta}} = 0$ at least for large n. Also, the asymptotic distribution of $\hat{\theta}$ is $\mathrm{N}(\theta, \kappa_{\theta,\theta}^{-1})$, with error apparently $O(n^{-1/2})$. These limiting results apply directly to situations in which the components of Y, while independent, need not be identically distributed. However, they still hold to dependent data under various conditions on the type of dependence.

We shall work with the standardized statistic $S = (\hat{\theta} - \theta)\kappa_{\theta,\theta}^{1/2}$ as a pivot function for θ which is frequently used to test the null hypothesis $\mathrm{H}_0 : \theta = \theta_0$, or to construct confidence limits for θ. The normal approximation for S is unsatisfactory in one important respect: the exact and approximate distributions of S differ by an $O(n^{-1/2})$ term. It is then desirable to improve on this result by adjusting S to have more nearly a standard normal distribution. On this basis, $Z \sim \mathrm{N}(0,1)$ with cumulants $\kappa_{rZ} = 0$ for $r \geq 3$ and we can obtain the formal moments η's after some algebra:

$$\begin{aligned} \eta_1 &= \kappa_{\theta,\theta}^{1/2} b_1(\theta) + O(n^{-3/2}), \\ \eta_2 &= \kappa_{\theta,\theta}\{v_2(\theta) + b_1(\theta)^2\} + O(n^{-2}), \end{aligned}$$

$$\eta_3 = \rho_{3\hat\theta} + O(n^{-3/2}),$$
$$\eta_4 = \rho_{4\hat\theta} + 4\rho_{3\hat\theta}\kappa_{\theta,\theta}^{1/2} b_1(\theta) + O(n^{-2}).$$

Here, $b_1(\theta)$ and $v_2(\theta)$ are the n^{-1} and the n^{-2} terms in the bias and variance of $\hat\theta$, respectively. Also, $\rho_{3\hat\theta}$ and $\rho_{4\hat\theta}$ are the third and fourth cumulants of $\hat\theta$ of orders $n^{-1/2}$ and n^{-1}, respectively. Formulae for $b_1(\theta)$, $v_2(\theta)$, $\rho_{3\hat\theta}$ and $\rho_{4\hat\theta}$ are given by Shenton and Bowman (1977, Sections 2.7.6 e 2.7.). We can also show that η_r is of order smaller than n^{-1} for $r \geq 5$.

The distribution function of S to $O(n^{-1})$ follows from equation (4) as

$$F_S(x;\kappa) = \Phi(x) - \phi(x)\Big\{\frac{6\eta_1 + \eta_3 h_2(x)}{6} + \frac{12\eta_2 h_1(x) + \eta_4 h_3(x)}{24}\Big\},$$

from which we obtain $A_1(x)$ and $A_2(x)$. Substituting these functions into (8) and simplifying, we find

$$S^* = S + \frac{\eta_3 - 6\eta_1}{6} + \frac{1}{72}\{(\eta_3 - \eta_1)^2 + 9(\eta_4 - 4\eta_1)\} - \frac{\eta_3}{6}S^2 + \frac{1}{72}\{2\eta_3(\eta_1 - \eta_3) - 3\eta_4\}S^3 + \frac{\eta_3^2}{72}S^5, \quad (17)$$

where $S = (\hat\theta - \theta)\kappa_{\theta,\theta}^{1/2}$. The adjusted pivotal quantity (17) is therefore a polynomial of 5th degree in the maximum likelihood estimate itself. Let $S^* = S + \sum_{i=0}^{5} \alpha_i S^i$ be this polynomial, where

$$\begin{aligned}
\alpha_0 &= (\eta_3 - 6\eta_1)/6,\\
\alpha_1 &= \{(\eta_3 - \eta_1)^2 + 9(\eta_3 - 4\eta_2)\}/72,\\
\alpha_2 &= -\eta_3/6, \quad \alpha_3 = \{2\eta_3(\eta_1 - \eta_3) - 3\eta_4\}/72,\\
\alpha_4 &= 0, \quad \alpha_5 = \frac{\eta_3^2}{72}.
\end{aligned}$$

Using general formulae for $b_1(\theta)$, $v_2(\theta)$, $\rho_{3\hat\theta}$ and $\rho_{4\hat\theta}$ given by Shenton and Bowman (1977, equations (2.30a, b), (2.31a, b)) and some Bartlett identities, which usually facilitate the computation of the κ's, we can obtain after some algebra

$$\begin{aligned}
\alpha_0 &= (4\kappa_{\theta\theta}^{(\theta)} - \kappa_{\theta\theta\theta})/(12\kappa_{\theta,\theta}^{3/2}),\\
\alpha_1 &= (\kappa_{\theta\theta\theta\theta} - \kappa_{\theta\theta,\theta\theta} - 2\kappa_{\theta\theta}^{(\theta\theta)})/(8\kappa_{\theta,\theta}^2) + (109\kappa_{\theta\theta\theta}^2 - 368\kappa_{\theta\theta}^{(\theta)}\kappa_{\theta\theta\theta} + 448\kappa_{\theta\theta}^{(\theta)2})/(288\kappa_{\theta,\theta}^3),\\
\alpha_2 &= (\kappa_{\theta\theta\theta} - 3\kappa_{\theta\theta}^{(\theta)})/(6\kappa_{\theta,\theta}^{3/2}), \qquad (18)\\
\alpha_3 &= -(\kappa_{\theta\theta\theta\theta} - 4\kappa_{\theta\theta\theta}^{(\theta)} + 6\kappa_{\theta\theta}^{(\theta\theta)} + 3\kappa_{\theta\theta,\theta\theta})/(24\kappa_{\theta,\theta}^2) + (73\kappa_{\theta\theta}^{(\theta)}\kappa_{\theta\theta\theta} - 7\kappa_{\theta\theta\theta}^2 - 120\kappa_{\theta\theta}^{(\theta)2})/(72\kappa_{\theta,\theta}^3),\\
\alpha_5 &= (\kappa_{\theta\theta\theta} - 3\kappa_{\theta\theta}^{(\theta)})^2/(72\kappa_{\theta,\theta}^3).
\end{aligned}$$

Notice that α_0 and α_2 are $O(n^{-1/2})$ while α_1, α_3 and α_5 are $O(n^{-1})$. Computing the α_i's from equations (18) for the model under consideration, the improved statistic S^* for the pivot $S = (\hat\theta - \theta)\kappa_{\theta,\theta}^{1/2}$ follows imediately. Then, in wide generality, the new pivotal quantity S^* is asymptotically standard normal distributed to a high degree of approximation, the relative error being typically $O(n^{-3/2})$. The statistical behavior of S^* and S can be quite different in finite samples.

Formula for S^* provides the basis for obtaining the corrected version of the maximum likelihood estimate $\hat{\theta}$. It is easy to check that $\hat{\theta}^* = \hat{\theta} + \sum_{i=0}^{5} \alpha_i (\hat{\theta} - \theta)^i \kappa_{\theta,\theta}^{(i-1)/2}$ follows a $N(\theta, \kappa_{\theta,\theta}^{-1})$ distribution and typically the error of approximation is $O(n^{-2})$. The above polynomial transformation for $\hat{\theta}^*$ looks very much like a truncated power series in the pivot $\hat{\theta} - \theta$. The statement that $S^* = (\hat{\theta}^* - \theta)\kappa_{\theta,\theta}^{1/2} \sim N(0,1) + O_p(n^{-3/2})$ implies that $\theta = \hat{\theta}^* \pm z\kappa_{\theta,\theta}^{-1/2}$ is an improved set of approximate confidence intervals for θ, where z is a normal upper point, i.e., values of θ outside this set are incompatible with the data.

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