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LAGRANGIAN AND HAMILTONIAN FORMALISM FOR CONSTRAINED VARIATIONAL PROBLEMS

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ABSTRACT. We consider solutions of Lagrangian variational problems with linear constraints on the derivative. These solutions are given by curves γ in a differentiable manifold M that are everywhere tangent to a smooth distribution $\mathcal{D}$ on M; such curves are called horizontal. We study the manifold structure of the set $\Omega_{P,Q}(M,\mathcal{D})$ of horizontal curves that join two submanifolds P and Q of M. We consider an action functional $\mathcal{L}$ defined on $\Omega_{P,Q}(M,\mathcal{D})$ associated to a time-dependent Lagrangian defined on $\mathcal{D}$. If the Lagrangian satisfies a suitable hyper-regularity assumption, it is shown how to construct an associated degenerate Hamiltonian H on TM^* using a general notion of *Legendre transform* for maps on vector bundles. We prove that the solutions of the Hamilton equations of H are precisely the critical points of $\mathcal{L}$.

1. INTRODUCTION

The aim of this paper is to generalize to constrained variational problem the classical results about the correspondence between Lagrangian and Hamiltonian formulations (see for instance [1]). Particular cases of this theory are the *sub-Riemannian geodesic problem*, and the so called *Vakonomic* approach to the non holonomic mechanics.

The constrained variational problem is modeled by the following setup: we consider an n-dimensional differentiable manifold M endowed with a smooth distribution $\mathcal{D} \subset TM$ of rank k; moreover, L is a (possibly time dependent) Lagrangian function on $\mathcal{D}$. In the non holonomic mechanics, M represents the configuration space, $\mathcal{D}$ the constraint, and L is typically the difference between the kinetic and a potential energy. In the sub-Riemannian geodesic problem, L is simply the quadratic form corresponding to a positive definite metric on $\mathcal{D}$.

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The solutions to the constrained variational problem are given by horizontal curves γ in M which are stationary points of the action functional

$$\mathcal{L}(\gamma) = \int_a^b L(t, \gamma(t), \dot{\gamma}(t))\, \mathrm{d}t,$$

and that satisfy suitable boundary conditions.

We consider the set $\Omega_{P,Q}([a,b], M, \mathcal{D})$ of horizontal curves in M of class C^1 joining two submanifolds P and Q of M. If either P or Q is transversal to $\mathcal{D}$, then we show that $\Omega_{P,Q}([a,b], M, \mathcal{D})$ has a natural structure of a Banach manifold. More in general, we give conditions that guarantee the existence of a differentiable structure on $\Omega_{P,Q}([a,b], M, \mathcal{D})$ in terms of the symplectic structure of the cotangent bundle TM^*. In this situation, $\mathcal{L}$ is a smooth map on $\Omega_{P,Q}([a,b], M, \mathcal{D})$ and we describe its critical points.

In order to be able to treat the case of a general Lagrangian function, in the paper we have considered as domain of the action functional the set of horizontal curves of class C^1. If one considers a Lagrangian of some specific form, like for instance $L(t, q, \dot{q})$ quadratic in $\dot{q}$, then one can extend the domain of the action functional to include curves that satisfy weaker regularity conditions, for instance of Sobolev type. Considering such extension may be more appropriate for developing an existence theory for the solutions of the variational problem by techniques of Global Analysis. We remark here that virtually all the results presented in this paper may be extended in this direction by minor modifications of the arguments.

When the Lagrangian function L satisfies a hyper-regularity condition, we introduce an associated Hamiltonian H_0 on $\mathcal{D}^*$ using a suitable version of the Legendre transform for general vector bundles. The Hamiltonian H_0 has a canonical extension to a Hamiltonian H in TM^*, which is degenerate, given by $H(t, q, p) = H_0(t, q, p|_{\mathcal{D}})$. The solutions of the Hamilton equations of H whose momenta annihilate TP and TQ at the endpoints are shown to be precisely the critical points of the action functional $\mathcal{L}$ in $\Omega_{P,Q}([a,b], M, \mathcal{D})$. In this way, we obtain a Hamiltonian formulation of our variational principle.

In the particular case where $\mathcal{D}$ is endowed with smoothly varying positive definite inner product g and L is given by $L(t, q, \dot{q}) = \frac{1}{2} g(\dot{q}, \dot{q})$, then the solutions of the corresponding Hamiltonian are known in the context of *sub-Riemannian geometry* as the *normal extremals* of $(M, \mathcal{D}, g)$. The critical points of the constraint defining $\Omega([a,b], M, \mathcal{D})$ are called *abnormal extremals*. In particular, we obtain a variational proof of [8, Theorem 1].

In [2, Theorem 1.17] it is proven that the normal sub-Riemannian extremals between two fixed points of a sub-Riemannian manifold are critical points of the sub-Riemannian action functional. The proof is presented in the context of the *Malliavin calculus*, employed to study some problems connected with the asymptotics of the semi-group associated with a hypoelliptic diffusion. For this purposes, the author's proof is restricted to the case that the image of the normal extremal be contained in an open subset of M on which the distribution $\mathcal{D}$ is globally generated by k smooth vector fields. In this paper we reprove the result of [2, Theorem 1.17] under the more general assumptions that:

- the Lagrangian function may be time-dependent, and it is not necessarily quadratic in the derivatives;
- the vector bundle $\mathcal{D}$ is not necessarily trivial around the image of the normal extremizer;
- the endpoints of the normal extremizers are free to move on two submanifolds of M.

As to the first generalization of the extremizing property of the normal extremizers, it is interesting to observe that in the proof it is employed the Lagrangian multipliers technique that uses *time-dependent referentials* of $\mathcal{D}$ defined in a neighborhood of the graph of any continuous curve in M. The existence of such referentials is obtained by techniques of calculus with affine connections, and it is likely that the method of time-dependent referentials may be applied to other situations where global geometrical results are to be proven.

Another observation that is worth making about the Lagrangian multipliers is that, in the functional setup of the method, the constraint is given by the kernel of a suitable submersion (see formula (3.4.1)) from the set of C^1-curves in an open subset of M taking values in the Banach space of $\mathbb{R}^{n-k}$-valued continuous functions. This submersion is defined using time-dependent referentials of the annihilator $\mathcal{D}^o$ of $\mathcal{D}$ in the cotangent bundle TM^*, and the surprising result is that such map fails to be a submersion precisely at the abnormal extremizers (Corollary 3.4.4). We therefore obtain a new variational description of the abnormal extremizers.

In Reference [6] it is studied the case of Vakonomic mechanics, by considering a Lagrangian L of the form $L(t,q,\dot{q}) = \frac{1}{2}g(\dot{q},\dot{q}) - V(q)$, where $V : M \mapsto \mathbb{R}$ represents the *potential energy* of the force acting on the system.

We conclude with a remark about a possible index theory for trajectories of Vakonomic mechanics. Every such solution comes with a well defined Morse index, possibly infinite, which is the dimension of a maximal negative space for the second variation of the Lagrangian action functional. In the case of non constrained hyper-regular Lagrangians, this number is related to the *Maslov index* of the corresponding solution of the Hamilton equations (see [13]). However, an index theorem for the action functional of a general constrained Lagrangian is not known yet, and we suggest that further investigation can be done in this direction. A sub-Riemannian version of the Morse index theorem can be found in [5].

We give a brief description of the material presented in each section of the paper.

In Subsection 2.1 we present an abstract version of the Legendre transform for maps defined on general vector bundles. In Subsection 2.2 we recall the classical theory concerning the relations between the critical points of the action functional associated to a time-dependent Lagrangian function and the solutions of the corresponding Hamiltonian obtained by the Legendre transform. We consider rather weak regularity assumptions on the Lagrangian L and we also study the case of variable endpoints. For the case of time independent non constrained smooth Lagrangians and curves with fixed endpoints, we refer to [1].

In Subsection 3.1 we show the existence of local time-dependent referentials for a general vector bundle defined in the neighborhood of the graph of a given continuous curve. Using these referentials, in Subsection 3.2 we describe a convenient atlas for the Banach manifold structure on the set of horizontal curves with free final endpoint. In Subsection 3.3 we study the differential structure of $\Omega_{P,Q}([a,b],M,\mathcal{D})$ in terms of critical points of the *endpoint map* on $\Omega_P([a,b],M,\mathcal{D})$. Such critical points are completely characterized in terms of *characteristic curves* of $\mathcal{D}$, which are the curves in TM^* everywhere tangent to the kernel of the restriction to $\mathcal{D}^o$ of the canonical symplectic form of TM^*. Some questions concerning the genericity of the property of existence of critical points of the endpoint map are answered in [3]. In Subsection 3.4 we study the differentiable structure of $\Omega_{P,Q}([a,b],M,\mathcal{D})$ in terms of local referentials of the annihilator $\mathcal{D}^o$ of $\mathcal{D}$.

In Section 4 we state the main result of the paper (Theorem 4.0.5), that establishes the correspondence between the critical points of the action functional of a hyper-regular constrained Lagrangian and the solutions of the corresponding degenerate Hamiltonian. In Subsection 4.1 it is presented a suitable version of Schwarz's distributional calculus, needed for technical reasons in the proof of Theorem 4.0.5. In Subsection 4.2 we give the proof of Theorem 4.0.5.

2. The Legendre Transform. Lagrangians and Hamiltonians on Manifolds

2.1. *The Legendre transform*

Let ξ_0 be a real finite dimensional vector space, let ξ_0^* denote its dual, and let $Z : U \mapsto \mathbb{R}$ be a function of class C^2 defined on the open subset $U \subset \xi_0$.

Definition 2.1.1. Assume that the differential $\mathrm{d}Z$ is a diffeomorphism onto an open subset $V \subset \xi_0^*$. The *Legendre transform* of Z is the C^1 map $Z^* : V \mapsto \mathbb{R}$ defined by:

$$Z^* = E_Z \circ (\mathrm{d}Z)^{-1}, \tag{2.1.1}$$

where $E_Z : U \mapsto \mathbb{R}$ is given by

$$E_Z(v) = \mathrm{d}Z(v)\,v - Z(v), \quad v \in U. \tag{2.1.2}$$

Lemma 2.1.2. *Using the canonical identification of ξ_0 and its bi-dual ξ_0^{**}, the map $\mathrm{d}Z^*$ is the inverse of $\mathrm{d}Z$. Therefore, Z^* is a map of class C^2.*

Proof. Differentiating the equality $Z^* \circ \mathrm{d}Z = E_Z$ and (2.1.2), we obtain:

$$\mathrm{d}Z^*(\mathrm{d}Z(v)) \circ \mathrm{d}^2 Z(v) = \mathrm{d}E_Z(v), \quad \mathrm{d}E_Z(v) = \hat{v} \circ \mathrm{d}^2 Z(v),$$

where $\hat{v} \in \xi_0^{**}$ is the evaluation at v. Since $\mathrm{d}^2 Z(v) : \xi_0 \mapsto \xi_0^*$ is an isomorphism, the conclusion follows. □

Corollary 2.1.3. $Z^{**} = Z$

Proof. By Lemma 2.1.2, we have:

$$Z^{**} = E_{Z^*} \circ (\mathrm{d}Z^*)^{-1} = E_{Z^*} \circ \mathrm{d}Z.$$

Hence, by definition of E_{Z^*}, we get

$$\begin{aligned} E_{Z^*}(\mathrm{d}Z(v)) &= \mathrm{d}Z^*(\mathrm{d}Z(v))\,\mathrm{d}Z(v) - Z^*(\mathrm{d}Z(v)) = \\ &= \mathrm{d}Z(v)\,v - E_Z(v) = Z(v). \end{aligned}$$

□

Let now M be a smooth manifold and $\pi : \xi \mapsto M$ be a smooth vector bundle over M; for $m \in M$, we denote by ξ_m the fiber $\pi^{-1}(m)$. The dual bundle of ξ will be denoted by ξ^*; the bi-dual ξ^{**} is canonically identified with ξ.

Let $Z : U \subset \xi \mapsto I\!R$ be a map such that, for every $m \in M$, $U \cap \xi_m$ is open in ξ_m and such that the restriction of Z to $U \cap \xi_m$ is of class C^2.

Definition 2.1.4. The *fiber derivative* $\mathbf{F}Z : U \mapsto \xi^*$ is the map defined by:

$$\mathbf{F}Z(v) = \mathrm{d}(Z|_{U \cap \xi_m})(v), \quad v \in U, \tag{2.1.3}$$

where $m = \pi(v)$. Let $V \subset \xi^*$ be the image of $\mathbf{F}Z$; we say that Z is *regular* if for each $m \in M$, the restriction of $\mathbf{F}Z$ to $U \cap \xi_m$ is a local diffeomorphism; Z is said to be *hyper-regular* if for each m such restriction is a diffeomorphism onto $V \cap \xi_m^*$. If Z is hyper-regular, we define the *Legendre transform* of Z as the map $Z^* : V \mapsto I\!R$ whose restriction to $V \cap \xi_m$ is the Legendre transform of the restriction of Z to $U \cap \xi_m$.

Applying Lemma 2.1.2 and Corollary 2.1.2 fiberwise, we obtain immediately the following:

Proposition 2.1.5. *Let $Z : U \subset \xi \mapsto I\!R$ be hyper-regular. Then, for each $m \in M$, the restriction of Z^* to $V \cap \xi_m^*$ is of class C^2. Moreover, $\mathbf{F}Z$ and $\mathbf{F}Z^*$ are mutually inverse bijections, and $Z^{**} = Z$.* □

2.2. *Time dependent Lagrangians and Hamiltonians on manifolds*

Let M be a smooth n-dimensional manifold, let $\pi : TM \mapsto M$ and $\pi : TM^* \mapsto M$ be respectively the tangent and the cotangent bundle of M; we consider the following vector bundles:

$$\xi = \mathbb{R} \times TM \xrightarrow{\mathrm{Id}\times\pi} \mathbb{R} \times M, \quad \xi^* = \mathbb{R} \times TM^* \xrightarrow{\mathrm{Id}\times\pi} \mathbb{R} \times M.$$

Observe that the fiber $\xi_{(t,m)}$ is $\{t\} \times T_mM$, and $\xi^*_{(t,m)} = \{t\} \times T_mM^*$.

Definition 2.2.1. A *(time-dependent) Lagrangian on M* is a function $L : U \subset \xi \mapsto \mathbb{R}$ defined on the open set U and satisfying the following continuity and differentiability conditions:

(1) L is continuous;
(2) for each $t \in \mathbb{R}$, the map $L(t, \cdot)$ is of class C^1 in $U \cap \big(\{t\} \times TM\big)$, and its differential is continuous in U;
(3) for each $t \in \mathbb{R}$, the map $\mathbb{F}L(t, \cdot) : U \cap \big(\{t\} \times TM\big) \mapsto \{t\} \times TM^*$ is of class C^1.

A *(time-dependent) Hamiltonian on M* is a function $H : V \subset \xi^* \mapsto \mathbb{R}$ defined on the open set V and satisfying the following properties:

(1) for all $t \in \mathbb{R}$, $H(t, \cdot)$ is of class C^1;
(2) for each $(t, m) \in \mathbb{R} \times M$, the restriction of H to $V \cap \xi^*_{(t,m)}$ is of class C^2.

We use the notions of regularity and hyper-regularity given in Definition 2.1.4 for Lagrangians and Hamiltonians on manifolds.

Using the Legendre transform defined in Subsection 2.1 (Definition 2.1.4), given a hyper-regular Lagrangian L on M, the map $H = L^*$ is a hyper-regular Hamiltonian on M. To see that $H(t, \cdot)$ is of class C^1, one applies the Inverse Function Theorem to the map $\mathbb{F}L(t, \cdot)$.[1]

If H is the hyper-regular Hamiltonian obtained by Legendre transform from the Lagrangian L, then by Proposition 2.1.5, we have that $H^* = L$, and that $\mathbb{F}H$ and $\mathbb{F}L$ are mutually inverse bijections.

Let $L : U \subset \mathbb{R} \times TM \mapsto \mathbb{R}$ be a Lagrangian on M and $\gamma : [a, b] \mapsto M$ be a curve of class C^1, with $(t, \dot\gamma(t)) \in U$. The *action* $\mathcal{L}(\gamma)$ of L on the curve γ is given by the integral:

$$\mathcal{L}(\gamma) = \int_a^b L(t, \dot\gamma(t))\, dt. \tag{2.2.1}$$

[1] As a matter of fact, the Hamiltonian $H = L^*$ is continuous. This can be seen by applying the Theorem of Invariance of Domain (see [11]) to conclude that $\mathbb{F}L$ is a homeomorphism onto an open subset of $\mathbb{R} \times TM^*$.

$\mathcal{L}$ defines a functional on the set:

(2.2.2)

$$\Omega_{P,Q}([a,b],M;U) = \\ = \left\{\gamma : [a,b] \xrightarrow{C^1} M : \gamma(a) \in P,\ \gamma(b) \in Q,\ (t,\dot\gamma(t)) \in U,\ \forall t\right\},$$

where P and Q are two smooth embedded submanifolds of M. It is well known that $\Omega_{P,Q}([a,b],M;U)$ has the structure of an infinite dimensional smooth Banach manifold (see for instance [12]), and $\mathcal{L}$ is a functional of class C^1 on $\Omega_{P,Q}([a,b],M;U)$. We will call $\mathcal{L}$ the *action functional* of the Lagrangian L.

We have the following characterization of the critical points of $\mathcal{L}$:

Proposition 2.2.2. *A curve $\gamma \in \Omega_{P,Q}([a,b],M;U)$ is a critical point of $\mathcal{L}$ if and only if the following three conditions are satisfied:*

1. $\mathbf{F}L(a,\dot\gamma(a))|_{T_{\gamma(a)}P} = 0$ *and* $\mathbf{F}L(b,\dot\gamma(b))|_{T_{\gamma(b)}Q} = 0$;
2. $t \mapsto \mathbf{F}L(t,\dot\gamma(t))$ *is of class* C^1;
3. *for all* $[t_0,t_1] \subset [a,b]$ *and for any chart* $q = (q_1,\ldots,q_n)$ *on* M *whose domain contains the image* $\gamma([t_0,t_1])$, *the following equation is satisfied in* $[t_0,t_1]$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q}(t,q(t),\dot q(t)) = \frac{\partial L}{\partial q}(t,q(t),\dot q(t)), \tag{2.2.3}$$

where $L(t,q,\dot q)$ is the coordinate representation of L.

Proof. Let $\gamma \in \Omega_{P,Q}([a,b],M;U)$ be a critical point of $\mathcal{L}$. Let $[t_0,t_1] \subset [a,b]$ be an interval and consider a chart $q = (q_1,\ldots,q_n)$ in M whose domain contains the image $\gamma([t_0,t_1])$. Let us consider an arbitrary C^1 variational vector field v along γ with support contained in $]t_0,t_1[$; by standard computations it follows:

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial q}(t,q(t),\dot q(t))\, v(t) + \frac{\partial L}{\partial \dot q}(t,q(t),\dot q(t))\, \dot v(t)\ \mathrm{d}t = 0. \tag{2.2.4}$$

Remark 2.2.3. The term $\frac{\partial L}{\partial \dot q}(t,q(t),\dot q(t))$ is of class C^1; this will follow immediately from Corollary 4.1.3 and the generalized functions calculus developed in Subsection 4.1.

Integration by parts in (2.2.4) and the Fundamental Lemma of Calculus of Variations imply then that equation (2.2.3) is satisfied.

Observe that the coordinate representation of the map $\mathbf{F}L(t,\dot\gamma(t))$ is given by the partial derivative $\frac{\partial L}{\partial \dot q}(t,q(t),\dot q(t))$.

The equalities $\mathbf{F}L(a,\dot\gamma(a))|_{T_{\gamma(a)}P} = 0$ and $\mathbf{F}L(b,\dot\gamma(b))|_{T_{\gamma(b)}Q} = 0$ follow easily from integrating by parts (2.2.4) in intervals of the form $[a,t_1]$ and $[t_0,b]$.

Conversely, if conditions 1, 2 and 3 are satisfied, equation (2.2.4) follows easily, which implies that γ is a critical point. □

We now pass to the study of the Hamiltonian formalism, and we consider the canonical symplectic form ω on TM^*, given by $\omega = -\mathrm{d}\vartheta$, where the canonical 1-form ϑ on TM^* is defined by $\vartheta_p(\zeta) = p(\mathrm{d}\pi_p(\zeta))$. If $q = (q_1, \ldots, q_n)$ is a chart in M and $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ is the corresponding chart on TM^*, ϑ and ω are given by:

$$\vartheta = \sum_{i=1}^{n} p_i \,\mathrm{d}q_i, \quad \omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

Given a Hamiltonian H on M, we define the *Hamiltonian vector field* $\vec{H}$ to be the time-dependent vector field on TM^* defined by:

$$\omega(\vec{H}, \cdot) = \mathrm{d}H_t,$$

where $H_t = H(t, \cdot)$.

We say that a curve $\gamma : [a, b] \mapsto M$ is a *solution of the Hamiltonian* H if there exists a C^1-curve $\Gamma : [a, b] \mapsto TM^*$ with $\pi \circ \Gamma = \gamma$ and such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t) = \vec{H}(t, \Gamma(t)) \tag{2.2.5}$$

for all t. In this case, we say that Γ is a *Hamiltonian lift* of γ. In coordinates (q, p), equation (2.2.5) is written as:

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{\partial H}{\partial p}(t, q(t), p(t)), \\[2mm] \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\dfrac{\partial H}{\partial q}(t, q(t), p(t)). \end{cases} \tag{2.2.6}$$

These are called the *Hamilton equations* of H; observe that the first equation in (2.2.6) can be written intrinsically as:

$$\dot{\gamma}(t) = \mathbf{F}H(t, \Gamma(t)). \tag{2.2.7}$$

Theorem 2.2.4. *Let L be a hyper-regular Lagrangian on M and let H be the corresponding hyper-regular Hamiltonian by the Legendre transform. Let P and Q be smooth submanifolds of M; a curve $\gamma \in \Omega_{P,Q}([a, b], M; U)$ is a critical point of $\mathcal{L}$ if and only if γ is a solution of the Hamiltonian H which admits a Hamiltonian lift Γ such that*

$$\Gamma(a)|_{T_{\gamma(a)}P} = 0, \quad \Gamma(b)|_{T_{\gamma(b)}Q} = 0. \tag{2.2.8}$$

Proof. Let $\gamma \in \Omega_{P,Q}([a, b], M; U)$ be a critical point of $\mathcal{L}$; set $\Gamma(t) = \mathbf{F}L(t, \dot{\gamma}(t))$. Since $\mathbf{F}H$ and $\mathbf{F}L$ are mutually inverse, equation (2.2.7) follows. Moreover, by Proposition 2.2.2, Γ is of class C^1 and (2.2.8) holds. We now prove that the second

Hamilton equation holds, in a chart (q,p) of TM^*. To this aim, we differentiate with respect to q the equality:

$$H\Big(t,q,\frac{\partial L}{\partial \dot q}(t,q,\dot q)\Big)=\frac{\partial L}{\partial \dot q}(t,q,\dot q)\,\dot q-L(t,q,\dot q),$$

obtaining:

$$\frac{\partial H}{\partial q}(t,q,p)+\frac{\partial H}{\partial p}(t,q,p)\,\frac{\partial^2 L}{\partial q\,\partial \dot q}(t,q,\dot q)=\frac{\partial^2 L}{\partial q\,\partial \dot q}(t,q,\dot q)\,\dot q-\frac{\partial L}{\partial q}(t,q,\dot q), \tag{2.2.9}$$

where $p=\frac{\partial L}{\partial \dot q}(t,q,\dot q)$. Using that $\mathbf{F}H$ and $\mathbf{F}L$ are mutually inverse, we get

$$\frac{\partial H}{\partial p}(t,q,p)=\dot q;$$

it follows from (2.2.9):

$$\frac{\partial H}{\partial q}(t,q,p)=-\frac{\partial L}{\partial q}(t,q,\dot q). \tag{2.2.10}$$

The second Hamilton equation follows from formula (2.2.10) and from Proposition 2.2.2.

Conversely, suppose that γ is a solution of the Hamiltonian H which admits a Hamiltonian lift Γ satisfying (2.2.8). Since $\mathbf{F}H$ and $\mathbf{F}L$ are mutually inverse, from (2.2.7) it follows that $\Gamma(t)=\mathbf{F}L(t,\dot\gamma(t))$. Equality (2.2.10) and the second Hamilton equation imply (2.2.3), and the conclusion follows from Proposition 2.2.2. □

3. The Space of Horizontal Curves and its Differentiable Structure

Let M be an n dimensional manifold; a smooth *distribution* $\mathcal{D}$ *of rank* k on M is a smooth subbundle $\mathcal{D}\subset TM$ whose fibers are k-dimensional spaces. This means that, for each $m\in M$, $\mathcal{D}_m=\mathcal{D}\cap T_mM$ is a k-dimensional subspace of T_mM which is smoothly varying with m, i.e., there exist k smooth vector fields around each point of M which form a pointwise basis for $\mathcal{D}$.

We will consider throughout a manifold M with a fixed distribution $\mathcal{D}$; a C^1-curve $\gamma:[a,b]\mapsto M$ will be called *horizontal* if $\dot\gamma(t)\in\mathcal{D}$ for all t.

Let $\Omega([a,b],M)$ denote the space of curves $\gamma:[a,b]\mapsto M$ of class C^1 and let P, Q be two submanifolds of M. We define the following subsets of $\Omega([a,b],M)$:

$$
\begin{aligned}
\Omega_P([a,b],M) &= \Big\{\gamma\in\Omega([a,b],M):\gamma(a)\in P\Big\},\\
\Omega_{P,Q}([a,b],M) &= \Big\{\gamma\in\Omega_P([a,b],M):\gamma(b)\in Q\Big\},\\
\Omega([a,b],M,\mathcal{D}) &= \Big\{\gamma\in\Omega([a,b],M):\gamma\text{ horizontal}\Big\},\\
\Omega_P([a,b],M,\mathcal{D}) &= \Big\{\gamma\in\Omega([a,b],M,\mathcal{D}):\gamma(a)\in P\Big\},\\
\Omega_{P,Q}([a,b],M,\mathcal{D}) &= \Big\{\gamma\in\Omega_P([a,b],M,\mathcal{D}):\gamma(b)\in Q\Big\}.
\end{aligned}
\tag{3.0.11}
$$

Given a subset $U\subset\mathbb{R}\times TM$, we will denote by $\Omega([a,b],M;U)$ the set of curves $\gamma\in\Omega([a,b],M)$ such that $(t,\dot\gamma(t))\in U$ for all t; similar notations will be used for all the spaces appearing in formula (3.0.11).

In this section we will prove that $\Omega_P([a,b],M,\mathcal{D})$ is a Banach submanifold of the manifold $\Omega_P([a,b],M)$, while $\Omega_{P,Q}([a,b],M,\mathcal{D})$ may have singularities.

In the Subsection 3.1 we give a couple of preliminary results needed to the study of the geometry of the set of horizontal paths in $(M,\mathcal{D})$.

The main reference for the geometry of infinite dimensional manifolds is [7]; for the basics of Riemannian geometry we refer to [4].

3.1. *Existence of time-dependent referentials.*

Definition 3.1.1. Let $(M,\overline{g})$ be a Riemannian manifold and $x\in M$. A positive number $r\in\mathbb{R}^+$ is said to be a *normal radius* for x if $\exp_x:\mathrm{B}_r(0)\mapsto B_r(x)$ is a diffeomorphism, where exp is the exponential map of $(M,\overline{g})$, $\mathrm{B}_r(0)$ is the open ball of radius r around $0\in T_xM$ and $B_r(x)$ is the open ball of radius r around $x\in M$. We say that r is *totally normal* for x if r is a normal radius for all $y\in B_r(x)$.

By a simple argument in Riemannian geometry, it is easy to see that if $K\subset M$ is a compact subset, then there exists $r>0$ which is totally normal for all $x\in K$.

Given an vector bundle $\pi:\xi\mapsto M$ of rank k over a manifold M, a *time-dependent local referential* of ξ is a family of smooth maps $X_i:A\mapsto\xi$, $i=1,\ldots,k$, defined on an open subset $A\subseteq\mathbb{R}\times M$ such that $\{X_i(t,x)\}_{i=1}^k$ is a basis of the fiber ξ_x for all $(t,x)\in A$.

Lemma 3.1.2. *Let M be a finite dimensional manifold, let $\pi:\xi\mapsto M$ be a vector bundle over M and let $\gamma:[a,b]\mapsto M$ be a continuous curve. Then, there exists an open subset $A\subseteq\mathbb{R}\times M$ containing the graph of γ and a smooth time-dependent local referential of ξ defined in A.*

Proof. We first consider the case that γ is a smooth curve.

Let us choose an arbitrary connection in ξ, an arbitrary Riemannian metric $\overline{g}$ on M and a smooth extension $\gamma : [a-\varepsilon, b+\varepsilon] \mapsto M$ of γ, with $\varepsilon > 0$. Since the image of γ is compact in M, there exists $r > 0$ which is a normal radius for all $\gamma(t)$, $t \in [a-\varepsilon, b+\varepsilon]$. We define A to be the open set:

$$A = \Big\{(t,x) \in I\!R \times M : t \in]a-\varepsilon, b+\varepsilon[,\ x \in B_r(\gamma(t))\Big\}.$$

Let now $\overline{X}_1, \ldots, \overline{X}_k$ be a referential of ξ along γ; for instance, this referential can be chosen by parallel transport along γ relative to the connection on ξ. Finally, we obtain a time-dependent local referential for ξ in A by setting, for $(t,x) \in A$ and for $i = 1, \ldots, k$, $X_i(t,x)$ equal to the parallel transport (relative to the connection of ξ) of $\overline{X}_i(t)$ along the radial geodesic joining $\gamma(t)$ and x.

The general case of a continuous curve is easily obtained by a density argument. For, let $\gamma : [a,b] \mapsto M$ be continuous and let $r > 0$ be a totally normal radius for $\gamma(t)$, for all $t \in [a,b]$. Let $\gamma_1 : [a,b] \mapsto M$ be any smooth curve such that $\mathrm{dist}(\gamma(t), \gamma_1(t)) < r$ for all t, where dist is the distance induced by the Riemannian metric $\overline{g}$ on M. Then, if we repeat the above proof for the curve γ_1, the open set A thus obtained will contain the graph of γ, and we are done. □

The abstract result of Lemma 3.1.2 will now be used in the situation that we are interested in. Namely, let us consider a manifold M endowed with a smooth distribution $\mathcal{D}$ of rank k.

Let $A \subset I\!R \times M$ be an open set and let $X_1, \ldots, X_n$ be a time-dependent referential of TM defined in A. We say that such referential is *adapted* to the distribution $\mathcal{D}$ if $X_1, \ldots, X_k$ form a referential for $\mathcal{D}$.

Corollary 3.1.3. *Given any continuous curve $\gamma : [a,b] \mapsto M$, there exists a time-dependent referential of TM adapted to $\mathcal{D}$ defined in an open set $A \subset I\!R \times M$ containing the graph of γ.*

Proof. Let $\mathcal{D}'$ be any fixed complementary bundle to $\mathcal{D}$ in TM, for instance, $\mathcal{D}'$ can be chosen to be the orthogonal complement of $\mathcal{D}$ with respect to an arbitrarily fixed Riemannian metric on M. Then apply Lemma 3.1.2 to the vector bundles $\mathcal{D}$ and $\mathcal{D}'$. □

3.2. *Charts in $\Omega([a,b],M)$ adapted to a time dependent referential.*

Given a time-dependent referential of TM defined in an open set $A \subset I\!R \times M$, we are going to associate to it a map

$$\mathcal{B}_0 : \Omega([a,b], M; \hat{A}) \longmapsto C^0([a,b], I\!R^n),$$

where $\hat{A}$ denotes the open subset of $\mathbb{R} \times TM$ given by:

$$\hat{A} = \Big\{(t, v) \in \mathbb{R} \times TM : (t, \pi(v)) \in A\Big\}, \tag{3.2.1}$$

and $C^0([a, b], \mathbb{R}^n)$ is the Banach space of continuous $\mathbb{R}^n$-valued functions on $[a, b]$. We define $\mathcal{B}_0$ by:

$$\mathcal{B}_0(\gamma) = h, \tag{3.2.2}$$

where $h = (h_1, \ldots, h_n)$ is given by

$$\dot{\gamma}(t) = \sum_{i=1}^{n} h_i(t) X_i(t, \gamma(t)), \tag{3.2.3}$$

for all $t \in [a, b]$. The map $\mathcal{B}_0$ is smooth; its differential is computed in the following:

Lemma 3.2.1. *Let $\gamma \in \Omega([a, b], M; \hat{A})$ and v be C^1 vector field along γ. Set $h = \mathcal{B}_0(\gamma)$, $z = (\mathrm{d}\mathcal{B}_0)_\gamma(v)$. We define a time-dependent vector field in A by*

$$X(t, x) = \sum_{i=1}^{n} h_i(t) X_i(t, x), \quad (t, x) \in A \tag{3.2.4}$$

and a vector field w along γ by

$$w(t) = \sum_{i=1}^{n} z_i(t) X_i(t, \gamma(t)). \tag{3.2.5}$$

Given a chart $(q_1, \ldots, q_n)$ defined in an open set $V \subset M$, denote by $\tilde{v}(t)$, $\tilde{X}(t, q)$ and $\tilde{w}(t)$ the representation in coordinates of v, X and w respectively. Then, the following relation holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{v}(t) = \frac{\partial \tilde{X}}{\partial q}(t, \gamma(t))\tilde{v}(t) + \tilde{w}(t), \tag{3.2.6}$$

for all $t \in [a, b]$ such that $\gamma(t) \in V$.

Proof. Simply consider a variation of γ with variational vector field v and differentiate relation (3.2.3) with respect to the variation parameter, using the local chart. □

Corollary 3.2.2. *Let $\phi : V \mapsto \tilde{V} \subset \mathbb{R}^n$ be a local chart in M, and let $X_1, \ldots, X_n$ be a time-dependent referential of TM defined on the open set $A \subset \mathbb{R} \times M$. Let $\mathcal{B}$ be the map:*

$$\mathcal{B}(\gamma) = (\phi(\gamma(a)), \mathcal{B}_0(\gamma)),$$

defined whenever $\gamma \in \Omega([a, b], M; \hat{A})$ and $\gamma(a) \in V$.

Then, $\mathcal{B}$ is a local chart in $\Omega([a, b], M)$ taking values in an open subset of $\mathbb{R}^n \times C^0([a, b], \mathbb{R}^n)$.

Proof. The differential of $\mathcal{B}$ is given by:

$$\mathrm{d}\mathcal{B}_\gamma(v) = \big(\mathrm{d}\phi_{\gamma(a)}(v(a)), (\mathrm{d}\mathcal{B}_0)_\gamma(v)\big)$$

where $(\mathrm{d}\mathcal{B}_0)_\gamma(v)$ is given in Lemma 3.2.1. By standard results on the existence and uniqueness of the solution of a linear differential equation with given initial conditions, it follows that $\mathrm{d}\mathcal{B}_\gamma$ is an isomorphism, hence $\mathcal{B}$ is a local diffeomorphism. The injectivity of $\mathcal{B}$ follows easily from the uniqueness of solutions of ordinary differential equations with given initial conditions. This concludes the proof. □

Now, using the chart $\mathcal{B}$, we can prove that $\Omega_P([a,b], M, \mathcal{D})$ has the structure of a smooth Banach manifold:

Proposition 3.2.3. *$\Omega_P([a,b], M, \mathcal{D})$ is a submanifold of $\Omega_P([a,b], M)$.*

Proof. If the referential $X_1, \ldots, X_n$ defining $\mathcal{B}$ is adapted to the distribution $\mathcal{D}$, then a curve γ in $\Omega([a,b], M; \hat{A})$ is horizontal if and only if $\mathcal{B}_0(\gamma) = h$ satisfies $h_{k+1} = \ldots = h_n = 0$. This means that, if ϕ is a *submanifold chart* for P then $\mathcal{B}$ is a submanifold chart for $\Omega_P([a,b], M, \mathcal{D})$. The conclusion follows from Corollary 3.1.3. □

We can now give a good description of the space $T_\gamma\Omega_P([a,b], M, \mathcal{D})$ using the map $\mathcal{B}$.

Let $\gamma \in \Omega([a,b], M; \hat{A})$ and set $h = \mathcal{B}_0(\gamma)$. Define a time-dependent vector field X in A as in (3.2.4). By Lemma 3.2.1, the kernel $\mathrm{Ker}\,(\mathrm{d}\mathcal{B}_0)_\gamma$ is the vector subspace of $T_\gamma\Omega([a,b], M)$ consisting of those v whose representation in coordinates $\tilde{v}$ satisfy the homogeneous part of the linear differential equation (3.2.6), namely:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{v}(t) = \frac{\partial \tilde{X}}{\partial q}(t, \gamma(t))\tilde{v}(t). \tag{3.2.7}$$

By the uniqueness of the solution of a Cauchy problem, it follows that, for all $t \in [a,b]$, the evaluation map

$$\mathrm{Ker}\,(\mathrm{d}\mathcal{B}_0)_\gamma \ni v \mapsto v(t) \in T_{\gamma(t)}M$$

is an isomorphism. Therefore, for every $t \in [a,b]$ we can define a linear isomorphism $\Phi_t : T_{\gamma(a)}M \mapsto T_{\gamma(t)}M$ by:

$$\Phi_t(v(a)) = v(t), \quad v \in \mathrm{Ker}\,(\mathrm{d}\mathcal{B}_0)_\gamma. \tag{3.2.8}$$

Using the maps Φ_t we can give a coordinate free description of the differential of $\mathcal{B}_0$, based on the "method of variation of constants" for solving non homogeneous linear differential equations.

Lemma 3.2.4. *Let $\gamma \in \Omega([a,b], M; \hat{A})$ and $v \in T_\gamma\Omega([a,b], M)$. Set $h = \mathcal{B}(\gamma)$ and $z = (\mathrm{d}\mathcal{B}_0)_\gamma(v)$. Define the objects X, w and Φ_t as in* (3.2.4), (3.2.5) *and* (3.2.8) *respectively. Then, the following equality holds:*

$$v(t) = \Phi_t\left(v_0 + \int_a^t \Phi_s^{-1} w(s)\,\mathrm{d}s\right), \tag{3.2.9}$$

where $v_0 = v(a)$.

Proof. Both sides of (3.2.9) coincide at $t = a$, therefore, to conclude the proof, one only has to show that its representation in local coordinates satisfies the differential equation (3.2.6). This follows by direct computation, observing that the representation in local coordinates of the maps Φ_t is a solution of the homogeneous linear differential equation (3.2.7). □

Corollary 3.2.5. *Suppose that the referential $X_1, \ldots, X_n$ that defines $\mathcal{B}_0$ is adapted to $\mathcal{D}$. Let γ be an horizontal curve in $\Omega_P([a,b], M; \hat{A})$.*

Then, the tangent space $T_\gamma\Omega_P([a,b], M, \mathcal{D})$ consists of all vector fields v of the form (3.2.9), *where w runs over all continuous horizontal vector fields along γ and $v_0 \in T_{\gamma(a)}P$.*

Proof. Follows directly from Lemma 3.2.4, observing that $\mathcal{B}$ is a submanifold chart for $\Omega_P([a,b], M, \mathcal{D})$, as it was remarked in the proof of Proposition 3.2.3, provided that the chart ϕ used to define $\mathcal{B}$ is chosen to be a submanifold chart for P. □

3.3. *Characteristic curves and the critical points of the endpoint map*

We have proven that $\Omega_P([a,b], M, \mathcal{D})$ is a submanifold of $\Omega([a,b], M)$. In order to study the differentiable structure of the set $\Omega_{P,Q}([a,b], M, \mathcal{D})$, we define the *endpoint map* end : $\Omega([a,b], M) \mapsto M$ by:

$$\mathrm{end}(\gamma) = \gamma(b).$$

Definition 3.3.1. A curve $\gamma \in \Omega_{P,Q}([a,b], M, \mathcal{D})$ is called *regular* if the differential at γ of the restriction of end to $\Omega_P([a,b], M, \mathcal{D})$ is *transversal* to Q, i.e., if

$$\mathrm{Im}\Big(\mathrm{d}(\mathrm{end}|_{\Omega_P([a,b],M,\mathcal{D})})(\gamma)\Big) + T_{\gamma(b)}Q = T_{\gamma(b)}M.$$

The curve γ will be called *singular* if it is not regular.

We recall the definition of transversality for maps between Banach manifolds.

Definition 3.3.2. Let $\mathcal{M}$ and $\mathcal{N}$ be Banach manifolds and $\mathcal{Q} \subset \mathcal{N}$ a submanifold. A smooth map $f : \mathcal{M} \mapsto \mathcal{N}$ is said to be *transversal to $\mathcal{Q}$ at $x \in f^{-1}(\mathcal{Q})$* if $\mathrm{Im}(\mathrm{d}f(x)) + T_{f(x)}\mathcal{Q} = T_{f(x)}\mathcal{N}$ and if $\mathrm{d}f(x)^{-1}(T_{f(x)}\mathcal{Q})$ is a complemented subspace of $T_x\mathcal{M}$. We say that f is a *submersion* at $x \in \mathcal{M}$ if f is transversal to $\{f(x)\}$

at x; f is transversal to $\mathcal{Q}$ (resp., a submersion) if f is transverse to $\mathcal{Q}$ (resp., a submersion) at every $x \in f^{-1}(\mathcal{Q})$ (resp., at every $x \in \mathcal{M}$).

It is well known (see for instance [7]) that if f is transverse to $\mathcal{Q}$ at x, then $f^{-1}(\mathcal{Q})$ is a smooth submanifold of $\mathcal{M}$ around x. We can now motivate the introduction of the endpoint map:

Proposition 3.3.3. *Suppose that $\gamma \in \Omega_{P,Q}([a,b], M, \mathcal{D})$ is a regular curve; then the set $\Omega_{P,Q}([a,b], M, \mathcal{D})$ is a submanifold of $\Omega_P([a,b], M, \mathcal{D})$ around γ.*

Proof. Clearly, $\Omega_{P,Q}([a,b], M, \mathcal{D}) = (\mathrm{end}|_{\Omega_P([a,b],M,\mathcal{D})})^{-1}(Q)$. If γ is regular, this restriction of the endpoint map is easily seen to be transverse to Q at γ, since it has finite dimensional range. □

We want to relate the differential of the endpoint map with the symplectic structure of TM^*. We denote by $\mathcal{D}^o \subset TM^*$ the annihilator of $\mathcal{D}$. The restriction $\omega|_{\mathcal{D}^o}$ of the canonical symplectic form of TM^* to $\mathcal{D}^o$ is in general no longer nondegenerate and its kernel $\mathrm{Ker}(\omega|_{\mathcal{D}^o})(p)$ at a point $p \in \mathcal{D}^o$ may be non zero. We say that an absolutely continuous curve $\eta : [a,b] \mapsto \mathcal{D}^o$ is a *characteristic curve* for $\mathcal{D}$ if

$$\dot{\eta}(t) \in \mathrm{Ker}(\omega|_{\mathcal{D}^o})(\eta(t)),$$

for almost all $t \in [a,b]$.

We take a closer look at the kernel of $\omega|_{\mathcal{D}^o}$. Let Y be a horizontal vector field in an open subset of M. We associate to it a Hamiltonian function H_Y defined by

$$H_Y(p) = p(Y(x)),$$

where $x = \pi(p)$. We can now compute the ω-orthogonal complement of $T_p\mathcal{D}^o$ in T_pTM^*. Recall that $\vec{H}_Y$ denotes the corresponding Hamiltonian vector field in TM^*.

Lemma 3.3.4. *Let $p \in TM^*$ and set $x = \pi(p)$. The ω-orthogonal complement of $T_p\mathcal{D}^o$ in T_pTM^* is mapped isomorphically by $\mathrm{d}\pi_p$ onto $\mathcal{D}_x$. Moreover, if Y is a horizontal vector field defined in an open neighborhood of x in M, then $\vec{H}_Y(p)$ is the only vector in the ω-orthogonal complement of $T_p\mathcal{D}^o$ which is mapped by $\mathrm{d}\pi_p$ into $Y(x)$.*

Proof. The function H_Y vanishes on $\mathcal{D}^o$ and therefore $\omega(\vec{H}_Y, \cdot) = \mathrm{d}H_Y$ vanishes on $T_p\mathcal{D}^o$. The conclusion follows by observing that, since ω is nondegenerate, the ω-orthogonal complement of $T_p\mathcal{D}^o$ in T_pTM^* has dimension $k = \dim(\mathcal{D}_x)$. □

Corollary 3.3.5. *The projection of a characteristic curve of $\mathcal{D}$ is automatically horizontal. Moreover, let $\gamma : [a,b] \mapsto M$ be a horizontal curve, let $X_1, \ldots, X_n$ be a time-dependent referential of TM adapted to $\mathcal{D}$, defined in an open subset $A \subset \mathbb{R} \times M$ containing the graph of γ. Define a time-dependent vector field X in A as in* (3.2.4).

Let $\eta : [a,b] \mapsto \mathcal{D}^o$ be a curve with $\pi \circ \eta = \gamma$. Then η is a characteristic curve of $\mathcal{D}$ if and only if η is an integral curve of $\vec{H}_X$.

Proof. For $p \in \mathcal{D}^o$, the kernel of the restriction of ω to $T_p\mathcal{D}^o$ is equal to the intersection of $T_p\mathcal{D}^o$ with the ω-orthogonal complement of $T_p\mathcal{D}^o$ in T_pTM^*. By Lemma 3.3.4, it follows that the kernel of $\omega|_{\mathcal{D}^o}$ projects by $\mathrm{d}\pi$ into $\mathcal{D}$, and therefore the projection of a characteristic is always horizontal.

For the second part of the statement, observe that for $t \in [a,b]$, $X(t,\cdot)$ is a horizontal vector field in an open neighborhood of $\gamma(t)$ whose value at $\gamma(t)$ is $\dot\gamma(t)$. Therefore $\dot\eta(t)$ is ω-orthogonal to $T_{\eta(t)}\mathcal{D}^o$ if and only if $\dot\eta(t) = \vec{H}_X(\eta(t))$. □

Corollary 3.3.6. *Let $\gamma : [a,b] \mapsto M$ be a horizontal curve and $X_1, \ldots, X_n$ be a time-dependent referential of TM adapted to $\mathcal{D}$, defined in an open subset $A \subset \mathbb{R} \times M$ containing the graph of γ. Let X be defined as in (3.2.4). A curve $\eta : [a,b] \mapsto \mathcal{D}^o$ with $\pi \circ \eta = \gamma$ is a characteristic of $\mathcal{D}$ if and only if its representation $\tilde\eta(t) \in \mathbb{R}^{n*}$ in any coordinate chart of M satisfies the following first order homogeneous linear differential equation:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde\eta(t) = -\frac{\partial \tilde X}{\partial q}(t,\gamma(t))^*\tilde\eta(t), \tag{3.3.1}$$

where $\tilde X$ is the representation in coordinates of X.

Proof. Simply use Corollary 3.3.5 and write the Hamilton equations of $\vec{H}_X$ in coordinates. □

Differential equation (3.3.1) is called the *adjoint system* of (3.2.7). It is easily seen that $\tilde\eta$ is a solution of (3.3.1) if and only if $\tilde\eta(t)\tilde v(t)$ is constant for every solution $\tilde v$ of (3.2.7). From this observation we get:

Lemma 3.3.7. *Let $\gamma : [a,b] \mapsto M$ be a horizontal curve and suppose that the referential $X_1, \ldots, X_n$ defining Φ_t in (3.2.8) is adapted to $\mathcal{D}$. Then a curve $\eta : [a,b] \mapsto \mathcal{D}^o$ with $\pi \circ \eta = \gamma$ is a characteristic for $\mathcal{D}$ if and only if $\eta(t) = (\Phi_t^*)^{-1}(\eta(a))$ for every $t \in [a,b]$.*

Proof. By Corollary 3.3.6 and the observation above we get that η is a characteristic if and only if $\eta(t)v(t)$ is constant for every $v \in \mathrm{Ker}\,(\mathrm{d}\mathcal{B}_0)_\gamma$. The conclusion follows. □

We can finally prove the main theorem of the subsection.

Theorem 3.3.8. *The annihilator of the image of the differential of the restriction of the endpoint mapping to $\Omega_P([a,b],M,\mathcal{D})$ is given by:*

$$\operatorname{Im}\Big(\mathrm{d}(\mathrm{end}|_{\Omega_P([a,b],M,\mathcal{D})})(\gamma)\Big)^o = \Big\{\eta(b) : \eta \text{ is a characteristic for } \mathcal{D},\ \eta(a) \in T_{\gamma(a)}P^o \text{ and } \pi\circ\eta = \gamma\Big\}. \tag{3.3.2}$$

Proof. By Lemma 3.2.5, we have:

$$\operatorname{Im}\Big(\mathrm{d}(\mathrm{end}|_{\Omega_P([a,b],M,\mathcal{D})})(\gamma)\Big) = \Big\{\Phi_b\Big(v_0 + \int_a^b \Phi_s^{-1}w(s)\,\mathrm{d}s\Big) : w \text{ is a continuous horizontal vector field along } \gamma \text{ and } v_0 \in T_{\gamma(a)}P\Big\}. \tag{3.3.3}$$

By Lemma 3.3.7, if η is a characteristic with $\pi\circ\eta = \gamma$ and with $\eta(a) \in T_{\gamma(a)}P^o$, then $\eta(b)$ annihilates the right hand side of (3.3.3). Namely:

(3.3.4)

$$\begin{aligned}\eta(b)\Big(\Phi_b\big(v_0 + \int_a^b \Phi_s^{-1}w(s)\,\mathrm{d}s\big)\Big) &= \\ &= (\Phi_b^*)^{-1}(\eta(a))\left(\Phi_b\big(v_0 + \int_a^b \Phi_s^{-1}w(s)\,\mathrm{d}s\big)\right) \\ &= \eta(a)\left(v_0 + \int_a^b \Phi_s^{-1}w(s)\,\mathrm{d}s\right) = \int_a^b \eta(a)\Phi_s^{-1}w(s)\,\mathrm{d}s \\ &= \int_a^b (\Phi_s^*)^{-1}\eta(a)w(s)\,\mathrm{d}s = \int_a^b \eta(s)w(s)\,\mathrm{d}s = 0.\end{aligned}$$

We have to prove that if $\eta_0 \in T_{\gamma(b)}M^*$ annihilates the righthand side of (3.3.3) then there exists a characteristic η with $\pi\circ\eta = \gamma$, $\eta(a) \in T_{\gamma(a)}P^o$ and $\eta(b) = \eta_0$.

Define η by $\eta(t) = (\Phi_t^*)^{-1}(\Phi_b^*(\eta_0))$ for all $t \in [a,b]$. By Lemma 3.3.7, we only have to prove that $\eta([a,b]) \subset \mathcal{D}^o$ and that $\eta(a) \in T_{\gamma(a)}P^o$. Computing as in (3.3.4) with $v_0 = 0$, we see that, since η_0 annihilates the righthand side of (3.3.3), then:

$$\int_a^b \eta(s)w(s)\mathrm{d}s = 0,$$

for any horizontal continuous vector field w along γ, which proves that $\eta([a,b]) \subset \mathcal{D}^o$.

Now, setting $w = 0$ in the righthand side of (3.3.3), we obtain that

$$\eta_0(\Phi_b(v_0)) = \eta(a)(v_0) = 0$$

for any $v_0 \in T_{\gamma(a)}P$, and this concludes the proof. □

Corollary 3.3.9. *The image of the differential of the restriction of the endpoint mapping to $\Omega_P([a,b],M,\mathcal{D})$ contains $\mathcal{D}_{\gamma(b)}$.*

Proof. By Theorem 3.3.8, the annihilator of the image of the differential of the restriction of the endpoint mapping to $\Omega_P([a,b],M,\mathcal{D})$ is contained in the annihilator of $\mathcal{D}_{\gamma(b)}$. The conclusion follows. □

Corollary 3.3.10. *If Q is transverse to $\mathcal{D}$, i.e., $T_xQ+\mathcal{D}_x = T_xM$ for all $x \in Q$, then every $\gamma \in \Omega_{P,Q}([a,b],M,\mathcal{D})$ is regular, and $\Omega_{P,Q}([a,b],M,\mathcal{D})$ is a submanifold of $\Omega_P([a,b],M,\mathcal{D})$.*

Proof. It follows easily from Proposition 3.3.3 and Corollary 3.3.9. □

The next corollary, which is obtained easily from (3.3.2), gives a characterization of singular curves in terms of characteristics:

Corollary 3.3.11. *A curve $\gamma \in \Omega_{P,Q}([a,b],M,\mathcal{D})$ is singular if and only if it is the projection of a non zero characteristic η of $\mathcal{D}$ with $\eta(a) \in T_{\gamma(a)}P^o$ and $\eta(b) \in T_{\gamma(b)}Q^o$.* □

Observe that by Lemma 3.3.7 a characteristic either never vanishes or is identically zero.

3.4. *Another description of the differentiable structure of $\Omega_{P,Q}([a,b],M,\mathcal{D})$*

The set $\Omega_{P,Q}([a,b],M,\mathcal{D})$ can be thought as the subset of $\Omega_P([a,b],M,\mathcal{D})$ consisting of curves with endpoint in Q, or as the subset of $\Omega_{P,Q}([a,b],M)$ consisting of curves that are horizontal. The first point of view was adopted in subsection 3.3; in this subsection we consider the second approach.

Lemma 3.4.1. *Let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be Banach manifolds, with $\mathcal{M}_1$ finite dimensional, and let $f : \mathcal{M} \mapsto \mathcal{M}_1$, $g : \mathcal{M} \mapsto \mathcal{M}_2$ be submersions. Let $p_1 \in \mathcal{M}_1$, $p_2 \in \mathcal{M}_2$ and choose $x \in f^{-1}(p_1) \cap g^{-1}(p_2)$. Then, $f|_{g^{-1}(p_2)}$ is a submersion at x if and only if $g|_{f^{-1}(p_1)}$ is a submersion at x.*

Proof. In first place, the closed subspace $\mathrm{Ker}(\mathrm{d}f(x)) \cap \mathrm{Ker}(\mathrm{d}g(x))$ is complemented in $\mathrm{Ker}(\mathrm{d}g(x))$, because it has finite codimension.

Since $\mathrm{Ker}(\mathrm{d}g(x))$ is complemented in $T_x\mathcal{M}$, it follows that the closed subspace $\mathrm{Ker}(\mathrm{d}f(x)) \cap \mathrm{Ker}(\mathrm{d}g(x))$ is complemented in $T_x\mathcal{M}$, thus $\mathrm{Ker}(\mathrm{d}f(x)) \cap \mathrm{Ker}(\mathrm{d}g(x))$ is complemented in $\mathrm{Ker}(\mathrm{d}f(x))$.

It remains to show that $\mathrm{d}f(x)|_{\mathrm{Ker}(\mathrm{d}g(x))}$ is surjective onto $T_{f(x)}\mathcal{M}_1$ if and only if $\mathrm{d}g(x)|_{\mathrm{Ker}(\mathrm{d}f(x))}$ is surjective onto $T_{g(x)}\mathcal{M}_2$. This follows from a general fact: if $T : V \mapsto V_1$ and $S : V \mapsto V_2$ are surjective linear maps between vector spaces, then

$T|_{\mathrm{Ker}(S)}$ is surjective if and only if $\mathrm{Ker}(T) + \mathrm{Ker}(S) = V$. Clearly, this relation is symmetric in S and T, and we obtain the thesis. □

We now consider an n-dimensional manifold M, endowed with a smooth distribution of rank k.

Using Lemma 3.1.2, we describe $\mathcal{D}$ locally as the kernel of a time-dependent $\mathbb{R}^{n-k}$-valued 1-form:

Proposition 3.4.2. *Let $\gamma : [a,b] \mapsto M$ be a continuous curve. Then, there exists an open subset $A \subseteq \mathbb{R} \times M$ containing the graph of γ and a smooth time-dependent $\mathbb{R}^{n-k}$-valued 1-form θ defined in A, with $\theta_{(t,x)} : T_xM \mapsto \mathbb{R}^{n-k}$ a surjective linear map and $\mathcal{D}_x = \mathrm{Ker}(\theta_{(t,x)})$ for all $(t,x) \in A$.*

Proof. Let ξ be the subbundle of the cotangent bundle TM^* given by the *annihilator* $\mathcal{D}^o$ of $\mathcal{D}$. Apply Lemma 3.1.2 to ξ and set $\theta = (\theta_1, \ldots, \theta_{n-k})$, where $\{\theta_i\}_{i=1}^{n-k}$ is a time-dependent local referential of ξ defined in an open neighborhood of the graph of γ. □

Let θ and A be as in Lemma 3.4.2, and define

$$\Theta : \Omega_{P,Q}([a,b], M; \hat{A}) \mapsto C^0([a,b], \mathbb{R}^{n-k})$$

to be the smooth map:

$$\Theta(\gamma)(t) = \theta_{(t,\gamma(t))}(\dot{\gamma}(t)), \tag{3.4.1}$$

where $\hat{A} \subset \mathbb{R} \times TM$ is defined in (3.2.1).

Clearly, $\Omega_{P,Q}([a,b], M, \mathcal{D}; \hat{A}) = \Theta^{-1}(0)$.

Proposition 3.4.3. *Θ is a submersion.*

Proof. Let $\gamma \in \Omega_{P,Q}([a,b], M; \hat{A})$ be a fixed curve; we choose a time-dependent referential $X_1, \ldots, X_n$ of TM defined in a neighborhood of the graph of γ such that $\theta_i(X_j) = \delta_{i,j-k}$, $i = 1, \ldots, k$ and $j = k+1, \ldots, n$. This is easily done by considering an extension of θ to a basis of TM^* (see Lemma 3.1.2), and then taking the X_i's obtained by suitably reindexing the corresponding dual basis. We now consider the chart $\mathcal{B}$ of $\Omega_P([a,b], M)$ associated to $X_1, \ldots, X_n$ constructed in subsection 3.2 (see Corollary 3.2.2). In such a chart, the map Θ is simply a projection, and therefore it is a submersion. □

It is easy to see that the endpoint map is a submersion on $\Omega_P([a,b], M)$, hence we have the following:

Corollary 3.4.4. *Let Θ be defined as in (3.4.1). Then, $\gamma \in \Omega_{P,Q}([a,b], M, \mathcal{D}; \hat{A})$ is regular if and only if the restriction of Θ to $\Omega_{P,Q}([a,b], M; \hat{A})$ is a submersion at γ.*

Proof. It follows easily from Lemma 3.4.1 and Proposition 3.4.3.

To apply Lemma 3.4.1, we take $\mathcal{M} = \Omega_P([a,b], M; \hat{A})$, $\mathcal{M}_1 = \mathbb{R}^{n-\dim(Q)}$, $\mathcal{M}_2 = C^0([a,b], \mathbb{R}^{n-k})$, $f = \psi \circ \mathrm{end}$, $g = \Theta$, where ψ is a submersion of an open subset $W \subset M$ around $\gamma(b)$ taking values in $\mathbb{R}^{n-\dim(Q)}$ such that $\psi^{-1}(0) = Q \cap W$. □

4. Lagrangians with linear constraints and degenerate Hamiltonians

Let M be an n-dimensional manifold and $\mathcal{D} \subset TM$ be a smooth distribution of rank k. We consider $\mathcal{D}$ as a vector bundle over M with projection $\pi : \mathcal{D} \mapsto M$. We apply the theory of Section 2 to the vector bundle $\xi = \mathbb{R} \times \mathcal{D}$ over the manifold $\mathbb{R} \times M$, with projection $\mathrm{Id} \times \pi$. The fiber $\xi_{(t,m)}$ is given by $\{t\} \times \mathcal{D}_m$.

Let $L : U \subset \xi \mapsto \mathbb{R}$ be a map of class C^2 defined in the open set U; we assume that L is hyper-regular in the sense of Definition 2.1.4. Let $H_0 = L^*$ be the Legendre transform of L, defined in an open subset $V \subset \mathbb{R} \times \mathcal{D}^*$. Define an *extension* H of H_0 by setting:

$$H(t,p) = H_0(t, p|_{\mathcal{D}}), \tag{4.0.2}$$

whenever $(t, p|_{\mathcal{D}}) \in V$. Observe that the maps H_0 and H are of class C^1.

In this context, we say that L is a *constrained Lagrangian* on M, and H is the corresponding *degenerate Hamiltonian*.

Given any two submanifolds P and Q of M, a constrained Lagrangian L on M defines an action functional $\mathcal{L}$ on $\Omega_{P,Q}([a,b], M, \mathcal{D}; U)$ by formula (2.2.1) whose stationary points are interpreted as the trajectories of the mechanical systems that we are interested in.

The following is the main result of the section and its proof is given in subsection 4.2:

Theorem 4.0.5. *Let M be an n-dimensional manifold, $\mathcal{D} \subset TM$ be a smooth distribution of rank k, $L : U \subset \mathbb{R} \times \mathcal{D} \mapsto \mathbb{R}$ be a hyper-regular constrained Lagrangian of class C^2, let $H_0 = L^*$ be its Legendre transform and let H be the corresponding degenerate Hamiltonian as in* (4.0.2).

Fix two submanifolds P and Q of M and let $\mathcal{L}$ be the action functional of L in the space $\Omega_{P,Q}([a,b], M, \mathcal{D}; U)$ defined by (2.2.1). *Let $\gamma \in \Omega_{P,Q}([a,b], M, \mathcal{D}; U)$ be a regular curve. Then, γ is a critical point of $\mathcal{L}$ if and only if it is a solution of H that admits a Hamiltonian lift Γ such that $\Gamma(a) \in T_{\gamma(a)}P^o$ and $\Gamma(b) \in T_{\gamma(b)}Q^o$.*

Theorem 4.0.5 is a generalization to the case of constrained Lagrangians of Theorem 2.2.4, where it was required a weaker regularity assumption on L. We emphasize that the rather awkward regularity assumption made in Theorem 2.2.4 is due to the fact that the result will be now applied to the case of a Lagrangian whose regularity in the variable t is not clear *a priori*.

The classical example of a constrained hyper-regular Lagrangian function L is given by:

$$L(t,v) = \frac{1}{2}\,g(v,v) - V(\pi(v)), \tag{4.0.3}$$

where g is a smoothly varying nondegenerate inner product on $\mathcal{D}$ and $V : M \mapsto \mathbb{R}$ is a map of class C^2. A version of Theorem 4.0.5 for Lagrangians of the form (4.0.3), with g positive definite, is proven in [6, Proposition 3.3].

To see that (4.0.3) defines a hyper-regular Lagrangian, simply observe that the fiber derivative $\mathbf{F}L$ is given by:

$$\mathbf{F}L(t,v) = g(v,\cdot).$$

For such Lagrangians, it is easily computed:

$$\begin{aligned} E_L(t,v) &= \frac{1}{2}\,g(v,v) + V(\pi(v)), \quad v \in \mathcal{D}, \\ H_0(t,p) &= \frac{1}{2}\,g^{-1}(p,p) + V(\pi(p)), \quad p \in \mathcal{D}^*. \end{aligned} \tag{4.0.4}$$

Theorem 4.0.5 implies that the critical points of the action functional $\mathcal{L}$ are the solutions of the Hamiltonian H given by:

$$H(t,p) = \frac{1}{2}\,g^{-1}(p|_{\mathcal{D}}, p|_{\mathcal{D}}) + V(\pi(p)), \quad p \in TM^*.$$

We emphasize that, in general, a minimum of the action functional $\mathcal{L}$ may not be a regular curve i $\Omega_{P,Q}([a,b],M,\mathcal{D})$, and in this situation it may not satisfy the Hamilton equations of H. Examples of this phenomenon are given in [8, 10] in the case $V = 0$. Hence, one can only conclude that a minimum of $\mathcal{L}$ is either a solution of the Hamilton equations or the projection of a non null characteristic of $\mathcal{D}$ (see Corollary 3.3.11).

4.1. *Generalized functions calculus*

For the proof of Theorem 4.0.5, and also to justify the computation of the Euler–Lagrange equations (see Remark 2.2.3), we will occasionally have to consider derivatives of functions that are only continuous. These derivatives must be understood in the sense of Schwarz-distributional calculus. However, the usual definition of distribution as the dual space of smooth compactly supported maps only allows products of distributions by smooth maps. To overcome this difficulty, we introduce a calculus for distributions of *stronger* regularity, that are the dual of a space of functions with *weaker* regularity.

Let V be a real finite dimensional vector space. For $k \geq 0$, we define $C_0^k([a,b],V)$ to be the Banach space of V-valued C^k maps on $[a,b]$ whose first k derivatives vanish at a and at b, endowed with the standard C^k-norm. We denote by $D^k([a,b],V)$

the dual Banach space of $C_0^k([a,b],V^*)$. Denoting by $L^p([a,b],V)$ the Banach space of V-valued measurable functions on $[a,b]$ whose p-th power is Lebesgue integrable, we have an inclusion:

$$L^1([a,b],V) \hookrightarrow D^k([a,b],V)$$

defined by

$$\langle f,\alpha\rangle = \int_a^b \alpha(t)\, f(t)\, \mathrm{d}t, \quad f \in L^1([a,b],V),\ \alpha \in C_0^k([a,b],V^*).$$

Moreover, we have inclusions $D^k \hookrightarrow D^{k+1}$ defined by restriction of the functionals. We summarize these observations by the following diagram:

$$\cdots \hookrightarrow C^1 \hookrightarrow C^0 \hookrightarrow L^1 \hookrightarrow D^0 \hookrightarrow D^1 \hookrightarrow \cdots$$

An element f of any space $D^k([a,b],V)$ is called a *generalized function*. We sometimes omit the parameters in C^k or D^k, whenever there is no risk of confusion.

In addition to the standard vector space operations in D^k, we define the following:

- *derivative operation:* for $f \in D^k([a,b],V)$, we denote by f' the element in $D^{k+1}([a,b],V)$ defined by

$$\langle f',\alpha\rangle = -\langle f,\alpha'\rangle$$

  for all $\alpha \in C_0^{k+1}([a,b],V^*)$;
- *product operation:* for $f \in D^k([a,b],V)$, $g \in C^k([a,b],W)$ and a fixed bilinear map $V \times W \mapsto U$, we define $fg \in D^k([a,b],U)$ as follows. The bilinear map $V \times W \mapsto U$ induces a bilinear map $W \times U^* \mapsto V^*$ by $(w \cdot u^*)(v) = u^*(v \cdot w)$; we set:

$$\langle fg,\alpha\rangle = \langle f, g\cdot\alpha\rangle,$$

  for all $\alpha \in C_0^k([a,b],U^*)$;
- *restriction operation:* for $f \in D^k([a,b],V)$ and $[c,d] \subset [a,b]$, we set:

$$\langle f|_{[c,d]},\alpha\rangle = \langle f,\overline{\alpha}\rangle,$$

  for all $\alpha \in C_0^k([c,d],V)$, where $\overline{\alpha} \in C_0^k([a,b],V)$ is the extension to zero of α outside $[c,d]$.

It is easily seen that when we apply the above operations to elements of D^k which correspond to functions then we obtain the standard operations on functions. Moreover, the standard Leibniz rule for derivatives of products holds for distributions:

$$(fg)' = f'g + fg',$$

for all $f \in D^k$ and $g \in C^{k+1}$.

In order to prove some regularity results we present the following elementary Lemmas.

Lemma 4.1.1. *Let $f \in D^k([a,b],V)$ be such that $f' = 0$. Then f is a constant function.*

Proof. We first consider the case $V = \mathbb{R}$. If $f' = 0$, then $\langle f, \alpha' \rangle = 0$ for all $\alpha \in C_0^{k+1}([a,b],\mathbb{R})$, hence $\langle f, \beta \rangle = 0$ for all $\beta \in C_0^k([a,b],\mathbb{R})$ with $\int_a^b \beta = 0$. Let $\beta_0 \in C_0^k([a,b],\mathbb{R})$ with $\int_a^b \beta_0 = 1$; set $c = \langle f, \beta_0 \rangle$. It is easily seen that $f = c$.

For the general case, observe that for all $\alpha \in V^*$, the product $\alpha f \in D^k([a,b],\mathbb{R})$ has vanishing derivative, hence it is constant. Since α is arbitrary, then f is constant. □

Lemma 4.1.2. *Let $f \in D^k([a,b],V)$ with $k \geq 1$; there exists an element $F \in D^{k-1}([a,b],V)$ with $F' = f$. If $f \in D^0([a,b],V)$, there exists $F \in L^2([a,b],V)$ with $F' = f$.*

Proof. Consider the map $\mathrm{d} : C_0^{k+1} \mapsto C_0^k$ given by derivative. It is easily seen that d is injective, with closed and complemented image. It follows that the *transpose map* $\mathrm{d}^* : D^k \mapsto D^{k+1}$ is surjective; clearly, the derivative of distributions is $-\mathrm{d}^*$, which proves the first part of the thesis.

For the case $k = 0$, let H_0^1 denote the Sobolev space of absolutely continuous functions $\alpha : [a,b] \mapsto V^*$ having square integrable derivative, and such that $\alpha(a) = \alpha(b) = 0$. Again, the derivation map $\mathrm{d} : H_0^1 \mapsto L^2$ is injective and has closed and complemented image. Therefore, given $f \in D^0$, we can find $F \in L^{2^*} \simeq L^2$ with $\mathrm{d}^* F = -f|_{H_0^1}$. It follows that $F' = f$. □

Corollary 4.1.3 (Bootstrap lemma). *Let f be a generalized function.*

(1) *If $f' \in D^0$ then $f \in L^2$;*
(2) *If $f' \in L^2$ then $f \in C^0$;*
(3) *If $f' \in C^0$ then $f \in C^1$.*

Proof. We prove, for example, the first item. By Lemma 4.1.2, we can find $F \in L^2$ with $F' = f'$. By Lemma 4.1.1, it follows that $F - f$ is constant, hence $f \in L^2$.

The other items are proven similarly. □

We now give a result that shows that *regularity* of a generalized function is a local property:

Lemma 4.1.4. *Let λ be a generalized function on $[a,b]$. Suppose that for all $t \in [a,b]$ there exists $\varepsilon > 0$ such that the restriction $\lambda|_{[t-\varepsilon,t+\varepsilon]\cap[a,b]}$ is of class C^k, $k \geq 0$. Then λ is of class C^k.*

Proof. Consider a partition $a = t_0 < t_1 < \ldots < t_r = b$ such that $f_i = \lambda|_{[t_i,t_{i+2}]}$ is of class C^k for all $i = 0,\ldots,r-2$. By applying λ to functions with support contained in $]t_{i+1},t_{i+2}[$, it is easily seen that, for all i, $f_i \equiv f_{i+1}$ in $[t_{i+1},t_{i+2}]$.

Hence there exists a C^k map f on $[a,b]$ such that $f|_{[t_i,t_{i+2}]} \equiv f_i$ for all i. It follows that f agrees with λ on maps α with support contained in some interval $[t_i, t_{i+2}]$, and such functions span the entire domain of λ. This concludes the proof. □

Finally, we need the following result that relates the dual spaces of C^0 and C^0_0. For $t \in [a,b]$ and $\sigma \in V$, we denote by $\delta_t^\sigma \in C^0([a,b],V^*)^*$ the *Dirac's delta*, defined by:

$$\delta_t^\sigma(\alpha) = \alpha(t)\,\sigma, \quad \alpha \in C^0([a,b],V^*).$$

Lemma 4.1.5. *Let λ be an element in $C^0([a,b],V^*)^*$ be such that λ vanishes identically on $C^0_0([a,b],V^*)$. Then, there exist σ_a and σ_b in V such that:*

$$\lambda = \delta_a^{\sigma_a} + \delta_b^{\sigma_b}. \tag{4.1.1}$$

Proof. The codimension of $C^0_0([a,b],V^*)$ in $C^0([a,b],V^*)$ is $2\dim(V)$, and so the annihilator of $C^0_0([a,b],V^*)$ in $C^0([a,b],V^*)^*$ has dimension equal to $2\dim(V)$. The conclusion follows immediately from the observation that the elements $\delta_a^{\sigma_a}+\delta_b^{\sigma_b}$ form a $2\dim(V)$-dimensional subspace of such annihilator. □

4.2. *Proof of Theorem 4.0.5*

The proof of Theorem 4.0.5 is based on the method of Lagrange multipliers, and we start with the precise statement of the result needed for our purposes.

Proposition 4.2.1. *Let $\mathcal{M}$ be a Banach manifold, E a Banach space, let $F : \mathcal{M} \mapsto I\!R$ and $g : \mathcal{M} \mapsto E$ be maps of class C^1. Let $p \in g^{-1}(0)$ be such that g is a submersion at p. Then, p is a critical point for $f|_{g^{-1}(0)}$ if and only if there exists $\lambda \in E^*$ such that p is a critical point for the functional $f_\lambda = f - \lambda \circ g$ in $\mathcal{M}$.*

Proof. The point p is critical for $f|_{g^{-1}(0)}$ if and only if $\mathrm{d}f(p)$ vanishes on the tangent space $T_p g^{-1}(0) = \mathrm{Ker}(\mathrm{d}g(p))$. The proof follows from elementary functional analysis arguments. □

The linear functional $\lambda \in E^*$ of Proposition 4.2.1 is called the *Lagrange multiplier* of the constrained critical point p; it is easily seen that such λ is unique. We can now prove of the main result of the section. In the argument we will need a regularity result for a Lagrangian multiplier, whose proof is postponed to Lemma 4.2.2.

Proof of Theorem 4.0.5. We start by choosing an arbitrary complementary distribution $\mathcal{D}'$ to $\mathcal{D}$, i.e., a smooth distribution of rank $n-k$ in M such that $T_mM = \mathcal{D}_m \oplus \mathcal{D}'_m$ for all $m \in M$; moreover, we fix an arbitrary smoothly varying positive definite inner product g on $\mathcal{D}'$ (for the existence of $\mathcal{D}'$ and g, see for instance the proof of Corollary 3.1.3). Let $\pi_{\mathcal{D}} : TM \mapsto \mathcal{D}$ and $\pi_{\mathcal{D}'} : TM \mapsto \mathcal{D}'$ be the projections and let $\overline{L} : \overline{U} \subset I\!R \times M \mapsto I\!R$ be the *extended* Lagrangian defined by:

$$\overline{L}(t,v) = L(t,\pi_{\mathcal{D}}(v)) + \frac{1}{2}\, g(\pi_{\mathcal{D}'}(v),\pi_{\mathcal{D}'}(v)), \tag{4.2.1}$$

where

$$\overline{U} = \Big\{(t,v) \in I\!R \times TM : (t,\pi_{\mathcal{D}}(v)) \in U\Big\}.$$

Then, $\overline{L}$ is a Lagrangian on M as in Definition 2.2.1; we denote by $\overline{\mathcal{L}}$ the corresponding action functional in $\Omega_{P,Q}([a,b],M;\overline{U})$, defined as in (2.2.1).

Let θ and A be as in Proposition 3.4.2, $\hat{A}$ be as in (3.2.1) and Θ as in (3.4.1). Then, γ is a critical point of $\mathcal{L}$ in $\Omega_{P,Q}([a,b],M,\mathcal{D};U)$ if and only if it is a critical point of $\overline{\mathcal{L}}|_{\Theta^{-1}(0)}$.

By Corollary 3.4.4 and Proposition 4.2.1, this is equivalent to the existence of $\lambda \in C^0([a,b],I\!R^{n-k})^*$ such that γ is a critical point of $\overline{\mathcal{L}}_\lambda = \overline{\mathcal{L}} - \lambda \circ \Theta$ in $\Omega_{P,Q}([a,b],M;\hat{A} \cap \overline{U})$.

We will prove in Lemma 4.2.2 that the Lagrange multiplier λ is of class C^1, i.e., that it is given by:

$$\lambda(\alpha) = \int_a^b \lambda_0(t)\,\alpha(t)\,\mathrm{d}t, \quad \forall\,\alpha \in C^0([a,b],I\!R^{n-k}), \tag{4.2.2}$$

for some C^1 map $\lambda_0 : [a,b] \mapsto (I\!R^{n-k})^*$. Therefore, $\overline{\mathcal{L}}_\lambda$ is the action functional corresponding to the Lagrangian $\overline{L}_\lambda$ in M defined by:

$$\overline{L}_\lambda(t,v) = \overline{L}(t,v) - \lambda_0(t)\,\theta_{(t,m)}(v), \quad (t,v) \in \hat{A} \cap \overline{U}, \tag{4.2.3}$$

where $m = \pi(v)$.

We now prove that $\overline{L}$ and $\overline{L}_\lambda$ are hyper-regular and we compute their Legendre transforms. The fiber derivatives $\mathbf{F}\overline{L}$ and $\mathbf{F}\overline{L}_\lambda$ are easily computed as:

$$\begin{aligned} \mathbf{F}\overline{L}(t,v) &= \mathbf{F}L(t,\pi_{\mathcal{D}}(v)) \circ \pi_{\mathcal{D}} + g\big(\pi_{\mathcal{D}'}(v),\pi_{\mathcal{D}'}(\cdot)\big), \\ \mathbf{F}\overline{L}_\lambda(t,v) &= \mathbf{F}\overline{L}(t,v) - \lambda_0(t)\,\theta_{(t,m)}. \end{aligned} \tag{4.2.4}$$

The hyper-regularity is proven by exhibiting explicit inverses:

$$\begin{aligned} \mathbf{F}\overline{L}^{-1}(t,p) &= \mathbf{F}L^{-1}(t,p|_{\mathcal{D}}) + g^{-1}(p|_{\mathcal{D}'}), \\ \mathbf{F}\overline{L}_\lambda^{-1}(t,p) &= \mathbf{F}\overline{L}^{-1}(t,p + \lambda_0(t)\,\theta_{(t,m)}). \end{aligned} \tag{4.2.5}$$

By g^{-1} in the above formula we mean the inverse of g seen as a linear map from $\mathcal{D}_m$ to $\mathcal{D}_m^*$.

We now compute the Legendre transforms $\overline{H}$ and $\overline{H}_\lambda$ of $\overline{L}$ and $\overline{L}_\lambda$ respectively. Using Definition 2.1.1 and (4.2.4), we compute easily:

$$E_{\overline{L}_\lambda}(t,v) = E_{\overline{L}}(t,v) = E_L(t,\pi_{\mathcal{D}}(v)) + \frac{1}{2}\, g(\pi_{\mathcal{D}'}(v),\pi_{\mathcal{D}'}(v)); \tag{4.2.6}$$

and, using (4.2.5), we therefore obtain:
(4.2.7)

$$\overline{H}(t,p) = H(t,p) + \frac{1}{2}\, g^{-1}(\pi_{\mathcal{D}'},\pi_{\mathcal{D}'}),$$

$$\overline{H}_\lambda(t,p) = \overline{H}(t,p+\lambda_0(t)\,\theta_{(t,m)}) = $$

$$= H(t,p) + \frac{1}{2}\, g^{-1}\big((p+\lambda_0(t)\,\theta_{(t,m)})|_{\mathcal{D}'},(p+\lambda_0(t)\,\theta_{(t,m)})|_{\mathcal{D}'}\big).$$

We now compute the Hamilton equations of the Hamiltonian $\overline{H}_\lambda$ with the help of local coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ in TM^* and of a local g-orthonormal referential $X_1,\ldots,X_{n-k}$ of $\mathcal{D}'$.

We write:

$$\text{(4.2.8)} \qquad \overline{H}_\lambda(t,p) = H(t,p) + \frac{1}{2}\sum_{i=1}^{n-k}\big(p+\lambda_0(t)\,\theta_{(t,m)}\big)(X_i)^2,$$

and, using (2.2.6), the Hamilton equations of $\overline{H}_\lambda$ are given by:
(4.2.9)

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{\partial H}{\partial p} + \displaystyle\sum_{i=1}^{n-k}(p+\lambda_0\,\theta)(X_i)\,X_i, \\[2ex] \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\dfrac{\partial H}{\partial q} - \displaystyle\sum_{i=1}^{n-k}(p+\lambda_0\,\theta)(X_i)\left[\lambda_0\,\dfrac{\partial\theta}{\partial q}(X_i) + (p+\lambda_0\,\theta)\left(\dfrac{\partial X_i}{\partial q}\right)\right]. \end{cases}$$

By Theorem 2.2.4, γ is a critical point of $\overline{\mathcal{L}}_\lambda$ if and only if it admits a lift $\Gamma : [a,b] \mapsto TM^*$ satisfying (4.2.9) with $\Gamma(a) \in T_{\gamma(a)}P^o$ and $\Gamma(b) \in T_{\gamma(b)}Q^o$.

Now, it follows easily from (4.0.2) that $\frac{\partial H}{\partial p}$ is in $\mathcal{D}$; since γ is horizontal, i.e., $\frac{\mathrm{d}q}{\mathrm{d}t} \in \mathcal{D}$, from the first equation of (4.2.9) it follows that $(p+\lambda_0\,\theta)(X_i) = 0$ for all $i = 1,\ldots,n-k$. Setting $(p+\lambda_0\,\theta)(X_i) = 0$ in (4.2.9) we obtain the Hamilton equations of H, which concludes the proof. □

We are left with the proof of the *regularity* of the Lagrange multiplier λ. We will use the generalized functional calculus developed in Subsection 4.1.

Lemma 4.2.2. *Under the assumptions of Theorem 4.0.5, using the notations adopted in its proof, if γ is horizontal and it is a critical point of $\overline{\mathcal{L}} - \lambda \circ \Theta$ for some $\lambda \in C^0([a,b],\mathbb{R}^{n-k})^*$, then there exists a C^1 map $\lambda_0 : [a,b] \mapsto (\mathbb{R}^{n-k})^*$ such that (4.2.2) holds.*

Proof. We set

$$\lambda_0 = \lambda|_{C^0_0([a,b],\mathbb{R}^{n-k})} \in D^0([a,b],(\mathbb{R}^{n-k})^*);$$

we first prove the regularity of the generalized function λ_0. To this aim, we *localize* the problem by considering variational vector fields along γ having support in the domain of a local chart $q = (q_1, \ldots, q_n)$ in M.

Let $[c, d] \subset [a, b]$ be such that $\gamma([c, d])$ is contained in the domain of the local chart; we still denote by λ_0 the restriction of λ_0 to $[c, d]$.

Since γ is a critical point of $\overline{\mathcal{L}} - \lambda \circ \Theta$, by standard computations it follows that the following equality holds:

$$\begin{aligned}(4.2.10)\qquad &\int_c^d \frac{\partial \overline{L}}{\partial q}(t, q(t), \dot q(t))\, v(t) + \frac{\partial \overline{L}}{\partial \dot q}(t, q(t), \dot q(t))\, \dot v(t)\, \mathrm{d}t \\ &\qquad - \left\langle \lambda_0, \frac{\partial \theta}{\partial q}\bigg|_{(t,q(t))} \big(v(t), \dot q(t)\big) + \theta_{(t,q(t))}\, \dot v(t) \right\rangle = 0,\end{aligned}$$

for every vector field v of class C^1 along γ having support in $]c, d[$. In terms of the local coordinates, the maps θ, $\frac{\partial \theta}{\partial q}(\cdot, \dot q)$, $\frac{\partial \overline{L}}{\partial q}$ and $\frac{\partial \overline{L}}{\partial \dot q}$ evaluated along γ will be interpreted as follows:

- $\theta \in C^1\big([c, d], \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^{n-k})\big)$;
- $\dfrac{\partial \theta}{\partial q}(\cdot, \dot q) \in C^0\big([c, d], \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^{n-k})\big)$;
- $\dfrac{\partial \overline{L}}{\partial q}, \dfrac{\partial \overline{L}}{\partial \dot q} \in C^0\big([c, d], \mathbb{R}^{n*}\big)$,

where $\mathrm{Lin}(\cdot, \cdot)$ denotes the space of linear maps between two vector spaces.

Using the definition of derivative for generalized functions, from (4.2.10) we get:

$$(4.2.11)\qquad \left\langle -\left(\frac{\partial \overline{L}}{\partial \dot q}\right)' + \frac{\partial \overline{L}}{\partial q} - \lambda_0 \frac{\partial \theta}{\partial q}(\cdot, \dot q) + (\lambda_0\, \theta)', v \right\rangle = 0,$$

for every C^1 map $v : [c, d] \mapsto \mathbb{R}^n$ having support in $]c, d[$, and, by density, for every $v \in C^1_0([c, d], \mathbb{R}^n)$. It follows:

$$(4.2.12)\qquad -\left(\frac{\partial \overline{L}}{\partial \dot q}\right)' + \frac{\partial \overline{L}}{\partial q} - \lambda_0 \frac{\partial \theta}{\partial q}(\cdot, \dot q) + \lambda_0'\, \theta + \lambda_0\, \theta' = 0.$$

Let $X_1, \ldots, X_{n-k}$ be a referential of $\mathcal{D}'$ along γ; in terms of the local coordinates the X_i's will be thought as elements of $C^1([c, d], \mathbb{R}^n)$; moreover, we set

$$X = (X_1, \ldots, X_{n-k}) \in C^1\big([c, d], \mathrm{Lin}(\mathbb{R}^{n-k}, \mathbb{R}^n)\big).$$

Multiplying (4.2.12) by X, we obtain:

$$(4.2.13)\qquad \lambda_0'\, \theta(X) + \lambda_0\, \theta'(X) - \lambda_0 \frac{\partial \theta}{\partial q}(X, \dot q) + \frac{\partial \overline{L}}{\partial q} X - \left(\frac{\partial \overline{L}}{\partial \dot q}\right)' X = 0.$$

Evaluating the first equation of (4.2.4) at X_i, by the horizontality of γ we get:

$$\frac{\partial \overline{L}}{\partial \dot q} X_i = 0, \quad \forall i = 1, \ldots, n-k, \tag{4.2.14}$$

hence:

$$\left(\frac{\partial \overline{L}}{\partial \dot q} X\right)' = -\frac{\partial \overline{L}}{\partial \dot q} X' \in C^0\big([c,d], (\mathbb{R}^{n-k})^*\big). \tag{4.2.15}$$

Now, considering that $\theta(X)$ is invertible, by (4.2.15) we can write (4.2.12) in the form:

$$\lambda_0' = \lambda_0 h_1 + h_2, \tag{4.2.16}$$

with $h_1 \in C^0\big([c,d], \mathrm{Lin}(\mathbb{R}^{n-k}, \mathbb{R}^{n-k})\big)$ and $h_2 \in C^0\big([c,d], (\mathbb{R}^{n-k})^*\big)$.

Applying three times Corollary 4.1.3, from (4.2.16) we conclude that λ_0 belongs to $C^1\big([c,d], (\mathbb{R}^{n-k})^*\big)$. By Lemma 4.1.4, $\lambda_0 \in C^1\big([a,b], (\mathbb{R}^{n-k})^*\big)$.

By Lemma 4.1.5, there exist $\sigma_a, \sigma_b \in (\mathbb{R}^{n-k})^*$ such that:

$$\lambda(\alpha) = \int_a^b \lambda_0 \, \alpha \, \mathrm{d}t + \sigma_a \, \alpha(a) + \sigma_b \, \alpha(b), \quad \forall \alpha \in C^0([a,b], \mathbb{R}^{n-k}). \tag{4.2.17}$$

To conclude the proof we show that $\sigma_a = \sigma_b = 0$. Let's show for instance that $\sigma_a = 0$; the equality $\sigma_b = 0$ is totally analogous.

Using local charts around $\gamma([a,d])$, for d close to a, we consider variational vector fields v of class C^1 supported in $[a,d[$, with $v(a) \in T_{\gamma(a)}P$. Arguing as in the deduction of formula (4.2.10), we get the following equality:

$$\begin{aligned}
&\int_a^d \frac{\partial \overline{L}}{\partial q}(t, q(t), \dot q(t))\, v(t) + \frac{\partial \overline{L}}{\partial \dot q}(t, q(t), \dot q(t))\, \dot v(t) \, \mathrm{d}t \\
&- \int_a^d \lambda_0(t) \Big[\frac{\partial \theta}{\partial q}\Big|_{(t,q(t))} \big(v(t), \dot q(t)\big) + \theta_{(t,q(t))}\, \dot v(t) \Big] \mathrm{d}t \\
&- \sigma_a \Big[\frac{\partial \theta}{\partial q}\Big|_{(a,q(a))} \big(v(a), \dot q(a)\big) + \theta_{(a,q(a))}\, \dot v(a) \Big] = 0.
\end{aligned} \tag{4.2.18}$$

From Lemma 4.1.3 and formula (4.2.12) it follows that $\frac{\partial \overline{L}}{\partial \dot q}$ is of class C^1, and we can use integration by parts in (4.2.18) to obtain an equality of the form:

$$\int_a^d u(t)\, v(t) \, \mathrm{d}t + \sigma_a \, \theta_{(a,q(a))} \, \dot v(a) = 0, \tag{4.2.19}$$

for some $u \in C^0([a,d], \mathbb{R}^{n*})$, whenever v is chosen such that $v(a) = 0$. By considering arbitrary v supported in $]a,d[$, from (4.2.19) we obtain that $u \equiv 0$ in $[a,d]$, so that the integral in (4.2.19) vanishes for all v. Now, we can choose v with $v(a) = 0$

and $\dot{v}(a)$ arbitrary, and from (4.2.19) we obtain that $\sigma_a = 0$, because $\theta_{(a,q(a))}$ is surjective. This concludes the proof. □

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