A note on Banach spaces failing Schroeder-Bernstein property

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We use a Banach space introduced by Gowers and Maurey and the Figiel' space to present a Banach space W failing the Schroeder-Bernstein property and also isomorphic to each of its hyperplanes such that W^m is isomorphic to some complemented subspace of W^n, with $m, n \in \mathbb{N}$ if and only if $m = n$, moreover $(W^*)^n$ is isomorphic to W^*, $\forall n \in \mathbb{N}$, $n \geq 1$.

1. Introduction

Let X be a Banach space over real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, X is said to satisfy the Schroeder-Bernstein property (SBP) if for any Banach space Y, if X and Y are isomorphic to complemented subspace of one another ($X \overset{c}{\hookrightarrow} Y$ and $Y \overset{c}{\hookrightarrow} X$), then X and Y are isomorphic $(X \sim Y)$ [3].

Gowers [7] gave the first solution to Schroeder-Bernstein problem for real Banach spaces. that is to say, there exists a Banach space Z which fails SBP, moreover this happens because (*) $Z^3 \sim Z$ and $Z^2 \not\sim Z$.

Aftewards Gowers and Maurey [8] introduced a complex Banach space V not isomorphic to each of its hyperplanes and failing SBP because $V \oplus \mathbb{C} \overset{c}{\hookrightarrow} V$. Moreover, in the same paper (page 563), (*) was generalized in the following way: $\forall k \in \mathbb{N}, k \geq 2$, there exists a complex Banach space Z_k such that $Z_k^m \sim Z_k^n$, with $m, n \in \mathbb{N}$ if and only if m and n are equal modulo k. So for every i, $1 \leq i < k$, $Z_k^i \sim (Z_k^i)^k \oplus Z_k^i$ and $Z_k^i \not\sim (Z_k^i)^k$, therefore Z_k^i also fails SBP .

In short, if X is any of these Banach spaces, then it fails SBP because (**) either $X \oplus \mathbb{C} \overset{c}{\hookrightarrow} X$ and $X \oplus \mathbb{C} \not\sim X$ or there exists $m \in \mathbb{N}$ such that $X^m \overset{c}{\hookrightarrow} X$ and $X^m \not\sim X$.

In theorem 2.2 we present a Banach space X failing SBP which does satisfy (**), because X is isomorphic to each of its hyperplanes and $X^m \overset{c}{\hookrightarrow} X^n$, with $m, n \in \mathbb{N}$ if and only if $m \leq n$.

2. The Banach space W

In order to constructed the Banach space W we will need to recall some definitions of [5].

Let X be a Banach space and ξ an ordinal number, X^ξ will indicate the Banach space of continuous X-valued functions defined on interval of ordinals $[1, \xi]$ and equipped with the supremum norm.

Let γ be an ordinal. A γ-sequence in a set A is the image of a function $f : [1, \gamma[\rightarrow A$ and will be denoted by $(x_\theta)_{\theta<\gamma}$. If A is a topological space and β is an ordinal, we will say that the γ-sequence is β-continuous if for every β-sequence of ordinals $(\theta_\xi)_{\xi<\beta}$ of $[1, \gamma]$ that converges to θ_β when ξ converges to β, we have that x_{θ_ξ} converges to x_{θ_β}.

Let α be a nondenumerable regular ordinal, φ any ordinal and X a Banach space. By X_α^φ we will denote the set of $x^{**} \in X^{**}$ having the following property: for every limit ordinal $\beta < \alpha$ and for every φ-sequence $X^\eta = (x_\xi^*(\eta))_{\xi<\beta}$ of β-sequence of X^* such that there exists $K \in \mathbb{R}$

with $\|x^*_\xi(\eta)\| \leq K$, $\forall \eta < \varphi$, $\forall \xi < \beta$ and such that $x^*_\xi(\eta)(x) \overset{\xi\to\beta}{\longrightarrow} 0$, $\forall x \in X$, uniformly in η, we have $x^{**}(x^*_\xi(\eta)) \overset{\xi\to\beta}{\longrightarrow} 0$ unformly in η.

The density character dens X of a Banach space X is the smallest cardinal number δ such that there exists a set of cardinality δ dense in X and the cardinality of an ordinal number ξ will be denoted by $\overline{\xi}$.

cX will denote the canonical image of the Banach space X into X^{**} and if Γ is any set by $c_0(\Gamma, X)$ we denote the Banach space of X-valued function defined on Γ such that for any positive ϵ the set $\{\gamma \in \Gamma : \|f(\gamma)\| \geq \epsilon\}$ is finite, with the supremum norm. $\ell_1(\Gamma, X^*)$ will be its dual and $\ell_\infty(\Gamma, X^{**})$ its bidual.

The Banach space X is said to have the Mazur's property if every weak* sequentially continuous functional in X^{**} belongs to X.

Let $L(X,Y)$ be the set of all continuous linear operators from the Banach space X into the Banach space Y. An operator $T \in L(X,Y)$ is Fredholm if its Kernel is finite dimensional and its range is finite codimensional. T is inessential ($T \in In(X,Y)$) if $I_X - ST$ is Fredholm for every $S \in L(Y,X)$. If $L(X,Y) = In(X,Y)$, X and Y are said essentially incomparable [1].

Finally, we note that the results of [5] are also true to complex Banach spaces.

Lemma 2.1. *Let X be a separable reflexive Banach space, Y a Banach space having the Mazur's property, φ the initial ordinal such that dim $Y^* = \overline{\varphi}$, α and β nondenumerable regular ordinals with $\overline{\varphi} < \overline{\alpha} < \overline{\beta}$ and $n \in \mathbb{N}, n \geq 1$, then*

a) $\dfrac{(X^\alpha \oplus Y^\beta)^\varphi_\beta}{c(X^\alpha \oplus Y^\beta)} \sim Y$

b) $\dfrac{(X^{\alpha n} \oplus Y^{\beta n})^\varphi_\alpha}{c(X^{\alpha n} \oplus Y^{\beta n})} \sim X^n \oplus C_0(\Gamma, Y)$ *for some* $\Gamma \neq \phi$.

Proof. At first, we note that if M and N are Banach spaces and θ is a nondenumerable regular ordinal, then it is not difficult to verify that

$$\frac{(M \oplus N)^\varphi_\theta}{c(M \oplus N)} \sim \frac{M^\varphi_\theta}{cM} \oplus \frac{N^\varphi_\theta}{cN} (***)$$

a) Let L be the usual isomorphism of $\ell_1([1,\alpha], X^*)$ onto $(X^\alpha)^*$, by proposition 2.6 of [5], $(X^\alpha)^\varphi_\beta = (L^*)^{-1}(m^\varphi_\beta([1,\alpha],X)$, where $m^\varphi_\beta([1,\alpha],X]$ is the closed subspace of $\ell_\infty([1,\alpha],X^{**})$ consisting of $\alpha+1$-sequences $(x^{**}_\theta)_{\theta<\alpha+1}$ of X^{**} which are ξ-continuous $\forall\xi$, $\xi < \beta$ and such that $x^{**}_\theta \in X^\varphi_\beta$, $\forall\theta$, $\theta \leq \alpha$.

Since $X^{**} = cX$ and $\overline{\alpha} < \overline{\beta}$, it follows that $(X^\alpha)^\varphi_\beta = cX^\alpha$, thus it suffices to use (***) having in mind that $\dfrac{(Y^\beta)^\varphi_\beta}{cY^\beta} \sim Y$ [5, corollary 2.8].

b) Again it suffices to use (***) having in mind that $\dfrac{(X^{\alpha n})^\varphi_\alpha}{cX^{\alpha n}} \sim X^n$ and $\dfrac{(Y^{\beta n})^\varphi_\alpha}{cY^{\beta n}} \sim C_0(\Gamma, Y)$ for some $\Gamma \neq \phi$ [5, corollary 2.8].

Let $p \in \mathbb{R}$ be, $1 \leq p < +\infty$. F will denote the complex version of the Banach space considered in [4] and G indicate a complex separable Banach space failing SBP which does not contain a complemented subspace isomorphic to ℓ_p. The Banach space V constructed in 4.3 of

[8] has these properties, because it is not isomorphic to each of its hyperplanes, see theorem 19 of [8].

Theorem 2.2. *Let α and β be nondenumerable regular ordinals with $\overline{\alpha} < \overline{\beta}$ and dim $G^* < \overline{\beta}$. Then $W = F^\alpha \oplus G^\beta$ fails SBP, W^m does not contain a complemented subspace isomorphic to W^n, with $m, n \in \mathbb{N}$ and $m < n$ and $(W^*)^n \sim W^*$, $\forall n \in \mathbb{N},\ n \geq 1$.*

Proof. Since G fails SBP, there exists a Banach space H non-isomorphic to G such that G and H are isomorphic to complemented subspaces of one another, so the same happens with $F^\alpha \oplus G^\beta$ and $F^\alpha \oplus H^\beta$. If $F^\alpha \oplus G^\beta \sim F^\alpha \oplus H^\beta$, then using the remark 2.3 of [6] we have

$$\frac{(F^\alpha \oplus G^\beta)^\varphi_\beta}{c(F^\alpha \oplus G^\beta)} \sim \frac{(F^\alpha \oplus H^\beta)^\varphi_\beta}{c(F^\alpha \oplus H^\beta)}$$

Thus lemma 2.1.a implies that $G \sim H$, which is an absurd. So $F^\alpha \oplus G^\beta$ fails SBP.

If $W^n \overset{c}{\hookrightarrow} W^m$ with $m, n \in \mathbb{N}$, then by (***) and lemma 2.1.b we have $F^n \oplus C_0(\Gamma_1, G) \overset{c}{\hookrightarrow} F^m \oplus C_0(\Gamma_2, G)$ for some Γ_1, Γ_2 non-empty sets.

If F and $C_0(\Gamma_2, G)$ are not essentially incomparable, then having in mind that every closed infinite-dimensional subspace of F contains a complemented subspace isomorphic to ℓ_p, it follows from theorem 4.3 of [1] that $\ell_p \overset{c}{\hookrightarrow} C_0(\Gamma_2, G)$ and by corollary 2.5 of [2] we have $\ell_p \overset{c}{\hookrightarrow} G$, which is an absurd.

So F and $C_0(\Gamma_2, G)$ are essentially incomparable and therefore the same happens with F^m and $C_0(\Gamma_2, G)$ [6, page 622]

Now, from theorem 3 of [6], there are complemented subspaces M of F^m and N of $C_0(\Gamma_2, G)$ so that $F^n \sim M \oplus N$. Since that F^n and $C_0(\Gamma_2, G)$ are essentially incomparable, it follows that N is finite dimensional space, so $F^n \sim M$, because F^n is isomorphic to each of its hyperplanes. In particular F^n is isomorphic to a subspace of F^m, consequently $m \geq n$.

To finish the proof we remark that $(W^*)^n \sim \ell_1(\Gamma_1, F^*) \oplus \ell_1(\Gamma_2, G^*)$, $\forall n \in \mathbb{N},\ n \geq 1$, where the cardinality of Γ_1 and Γ_2 are respectively $\overline{\alpha}$ and $\overline{\beta}$, so $(W^*)^n \sim W^*$, $\forall n \in \mathbb{N},\ n \geq 1$.

Remark 2.3. If X is any of the Banach spaces mentioned in this note and failing SBP, then there exists a Banach Y such that $X \overset{c}{\hookrightarrow} Y$ and $Y \overset{c}{\hookrightarrow} X$, $X \not\sim Y$ and $X^n \sim Y^n$ for some $n \in \mathbb{N}$, $n \geq 2$. Indeed, it suffices to have in mind that $Z^2 \sim (Z^2)^2$, $V^2 \sim (V \oplus \mathbb{C})^2$ and $(Z^i_k)^k \sim (Z^k_k)^k$, for every i, $1 \leq i < k$. This suggests the following:

Question 2.4. Let X and Y be Banach spaces. If $X \overset{c}{\hookrightarrow} Y$ and $Y \overset{c}{\hookrightarrow} X$, then is it true that there exists $n \in \mathbb{N}$, $n \geq 1$ such that $X^n \sim Y^n$?

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