Connections between the Sznajd model with general confidence rules and graph theory

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The Sznajd model is a sociophysics model that is used to model opinion propagation and consensus formation in societies. Its main feature is that its rules favor bigger groups of agreeing people. In a previous work, we generalized the bounded confidence rule in order to model biases and prejudices in discrete opinion models. In that work, we applied this modification to the Sznajd model and presented some preliminary results. The present work extends what we did in that paper. We present results linking many of the properties of the mean-field fixed points, with only a few qualitative aspects of the confidence rule (the biases and prejudices modeled), finding an interesting connection with graph theory problems. More precisely, we link the existence of fixed points with the notion of strongly connected graphs and the stability of fixed points with the problem of finding the maximal independent sets of a graph. We state these results and present comparisons between the mean field and simulations in Barabási-Albert networks, followed by the main mathematical ideas and appendices with the rigorous proofs of our claims and some graph theory concepts, together with examples. We also show that there is no qualitative difference in the mean-field results if we require that a group of size $q > 2$, instead of a pair, of agreeing agents be formed before they attempt to convince other sites (for the mean field, this would coincide with the q-voter model).

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I. INTRODUCTION

In the last years, the interest in interdisciplinary problems has increased among physicists, creating many research areas. One of these areas is sociophysics that studies how assumptions about the behavior and social interactions of people in a "microcospic level" creates emerging social behaviors, like opinion propagation, consensus formation, properties of elections, how wealth is distributed in society, among other topics. Typical approaches include modeling using deterministic cellular automata, Monte Carlo simulations of models derived from ferromagnetic models (usually Ising and Potts), mean-field approaches and diffusion-reaction processes [1–8].

The Sznajd model is an opinion propagation model, originally inspired by the Ising model in a linear chain, and is typically used to model consensus formation. It has spawned many variations, including the addition of noise, independent behavior, contrarianlike agents and undecided voters, as well as generalizations to more than two states (opinions) and to arbitrary networks [2,9,10]. In all these variations, the most defining aspect of the Sznajd model is that it gives a greater convincing power to bigger groups of agreeing agents. Even though the importance of this effect has been known by psychologists since the 1950s [11], it is often overlooked in other opinion propagation models, for the sake of simplicity (this happens, for example, in the voter and in the Deffuant models [1,3]).

In a recent work [12], we took the bounded confidence rule (that roughly says that people are only allowed to change opinions in a *smooth* way) that is common to many opinion propagation models [3,4,13], including the Sznajd model, and we generalized it to model biases and prejudices in discrete opinion models (these generalized rules will be called by the umbrella term *confidence rules*). We applied this generalization to the Sznajd model and studied mainly the case with three opinions. In that work, we found a good qualitative (and with the exception of the time scales in which things happened, a quantitative) agreement between the model simulated in Barabási-Albert (BA) networks [14] and the mean-field equations, being able to understand some apparently contradictory results in literature [12]. However, some of the results about the mean field were still rather sketchy.

The present work can be regarded as a unifying effort. In this paper, we recapitulate the Sznajd model with general confidence rules and then write a mean-field equation for a variant of the model that includes as different parameter choices, the model with confidence rules as studied by us in Ref. [12], the usual Sznajd model as defined in Ref. [10], and the versions studied in Ref. [15]. We define a phase space for these equations and state the mean-field results that were found for this variant of the Sznajd model. We find which are the fixed points of the model, how they are organized, and what are their stability properties. This allows us, among other things, to identify which are the static attractors of the mean-field equations and to show that all the different versions of the model have the same behavior. This shows that there is some measure of universality in the qualitative behavior of these models (this is important, as the obvious difficulties in modeling human beings in a reliable and realistic way, show that some degree of universality in human behavior is essential, in order for social modeling to be feasible). We state these results highlighting a connection that was found between them and graph theory problems using a graph derived only from qualitative properties of the confidence rule. The results have some counterintuitive aspects and as such we provide both numerical solutions for the mean-field equations and Monte Carlo simulations for the Sznajd model (more precisely, the version studied in our previous paper [12]) in a BA network.

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Finally, we give rigorous proofs of our claims, with the main mathematical ideas in a separate section and the detailed proofs in the appendices. The paper is structured in a way that a basic understanding of the results is possible without reading the most technical sections.

An Appendix with graph theory concepts and a glossary is provided and we recommend strongly that those not familiar with the specific language and notation used (especially in the results section) read it first.

II. THE SZNAJD MODEL WITH CONFIDENCE RULES

The Sznajd model is an agent-based sociophysics model for opinion propagation. In this model, a society is represented by a network (that is, a collection of nodes linked together by edges), where each node represents an agent (person), each edge is a social connection (friendship, marriage, acquaintances, etc.), and each node i possesses an integer σ_i, between 1 and M, representing its opinion. In our generalization of the Sznajd model, as defined in Ref. [12], we introduce a set of parameters $p_{\sigma\to\sigma'}$ (that are fixed and completely independent with the state of the network), and at each time step the following update rule is used:

(1) A node i is chosen at random, and then a neighbor j of i is chosen.

(2) If they disagree ($\sigma_i \neq \sigma_j$), nothing happens.

(3) If they agree, a neighbor k of j is chosen and is convinced of opinion σ_i with probability $p_{\sigma_k\to\sigma_i}$.

We can interpret the first step as a conversation between two people that know each other, where they discuss some issue. If they disagree, none manages to convince the other. But, if they agree, they may set to convince another person that one of them knows and this person is convinced with a certain probability that depends only of its current point of view and of the opinion the pair is trying to impose.

In the original model the probability weights $p_{\sigma\to\sigma'}$ are not dependent on σ and σ'. The reason why this probability should depend on both opinions is that, usually an opinion includes prejudices about differing points of view (this is strongly related with the idea of cognitive dissonance in psychology [16–18]). This generalization allows for complex interactions among the opinions in an unified way and can be seen as a generalization of the *bounded confidence* rules [3,4], as those rules can be recovered as special cases. Some of the model modifications found in the literature can also be obtained this way, as different parameter choices:

(1) When $p_{\sigma\to\sigma'} = 1\ \forall\ \sigma,\sigma'$ we have the usual model [10].

(2) If $|\sigma - \sigma'| \leqslant \varepsilon \Rightarrow p_{\sigma\to\sigma'} = 1$ and $p_{\sigma\to\sigma'} = 0$ otherwise, we have bounded confidence with threshold ε [19].

(3) Undecided agents can be modeled by a special state σ, such that $p_{\sigma'\to\sigma} = 0\ \forall\ \sigma'$ (undecided agents can only be convinced).

(4) Cyclic interactions, like rock, paper, scissors (A convinces only B, that convinces only C, that convinces only A) [20].

This generalized version of the model has $M(M-1)$ parameters, where M is the number of opinions ($p_{\sigma\to\sigma}$ is irrelevant and can always be taken as 0). These parameters can be thought as the elements of the adjacency matrix of a directed weighted graph, that will be referred to as the *confidence rule*

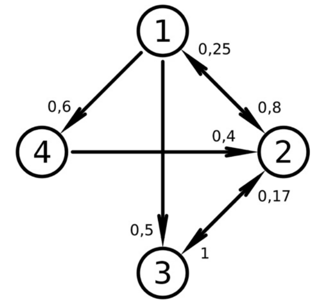


FIG. 1. A confidence rule for four opinions. Here $p_{1\to2} = 0.8$, $p_{1\to3} = 0.5$, $p_{1\to4} = 0.6$, $p_{2\to1} = 0.25$, $p_{2\to3} = 1$, $p_{3\to2} = 0.17$, $p_{4\to2} = 0.4$, and $p_{\sigma\to\sigma'} = 0$ otherwise.

(as the set of parameters $p_{\sigma\to\sigma'}$ and this graph are equivalent, we will refer to both of them as the confidence rule). So the confidence rule is a directed weighted graph, whose nodes are the opinions in the model (so a model with M opinions would have a confidence rule with M nodes) and the arrows represent the ways that opinions are allowed to interact. This graph is useful as a way of schematizing the opinion interactions and as we show in the next sections, it can be used to find the properties of the mean-field fixed points. An example of a confidence rule with four opinions is given in Fig. 1.

We will use this model in our simulations (found in Sec. IV), while for the mean field we will actually use a further generalization of the Sznajd model (that includes the model that was actually used in the simulations as a particular case). In this generalization, at each iteration we choose q agents at random and if they agree (meaning they are on the same state), they attempt to convince r other agents (also chosen at random). If the group of q agents has opinion σ, then each of the targeted r agents is convinced with probability $p_{\sigma'\to\sigma}$ and retains its opinion with probability $1 - p_{\sigma'\to\sigma}$, where σ' is the opinion the targeted agent had before the group attempted to convince it (and hence in general it is different for each of the r agents). Adding up the probabilities of all possible processes we obtain the mean-field equation in the limit of large networks:

$$\dot{\eta}_\sigma = r\sum_{\sigma'}\eta_\sigma\eta_{\sigma'}\left(\eta_\sigma^{q-1}p_{\sigma'\to\sigma} - \eta_{\sigma'}^{q-1}p_{\sigma\to\sigma'}\right), \qquad (1)$$

where η_σ is the proportion of sites with opinion σ (the deduction of this equation from the underlying Markov chain implies that η is actually the expected value of the proportion) and a time unit corresponds to a Monte Carlo sweep (MCS), that is, a number of iterations equal to the number of sites in the network.

III. MEAN-FIELD RESULTS

We now present the mean-field results and some of its consequences. In the following section we will provide simulation results showing that these results are indeed observed in Barabási-Albert networks.

For the analysis of the mean-field case, we built a phase space representation, where the variables are the η_σ. The phase space of this flow is an $(M-1)$-simplex denoted as sim_M (that is embedded in an M dimensional vector space, in order to make the equations more symmetrical), where the vertices correspond to consensus states and the other states are convex combinations of the vertices, with coefficients η_σ:

$$P = \sum_\sigma P_\sigma \eta_\sigma, \tag{2}$$

where P_σ is the coordinate of the vertex corresponding to consensus of opinion σ, and P is the coordinate in phase space of the point representing the state $(\eta_1, \dots, \eta_M)$ (in other words, we are using a barycentric coordinate system).

The results for the mean-field fixed points allow us to find what are the possible configurations of surviving opinions (the attractors), as well as other qualitative properties of the behavior, without us having to explicitly solve nor do the numerical integration of the equations. They can be expressed as problems regarding the existence of groups of nodes satisfying certain conditions in the confidence rule and these results are the same for all $q \geqslant 2$. We will give here these results for a better understanding of the simulations in Sec. IV, leaving the mathematical details for later. Because of the connection of these results with graph theory, a small glossary with examples is provided in Appendix A. For the same reason, we will interchange freely the notion of a set of opinions with the notion of a set of nodes in some graph (like the confidence rule).

In what follows, we will denote by $\mathcal{M}_\Delta$ the manifold with the states where only opinions belonging to a set Δ survive:

$$\mathcal{M}_\Delta = \left\{ \vec{\eta} \in \mathrm{sim}_M \,\middle|\, \sum_{\sigma \in \Delta} \eta_\sigma = 1 \right\}. \tag{3}$$

We will interpret this set of opinions as a set of nodes in the confidence rule and we will denote by Δ_- its predecessor set (the nodes that point to nodes in Δ), by Δ_+ its successor set (the nodes that are pointed by nodes in Δ), by $\overline{\Delta}$ its complement set (the nodes that are not in Δ), and by G_Δ the graph induced in G by Δ (the graph obtained by keeping only the parts of G that are related to Δ). Finally, we will denote by $\mathcal{R}$ the skeleton of the confidence rule, that is, the directed graph obtained by keeping all the arrows with nonzero weight and removing the weights after that. We will also use the concepts of strongly connected graphs and of independent sets (the detailed definitions, together with examples, can be found in Appendix A). The results for the mean-field fixed points are as follows:

(1) Given a fixed point, its stability properties depend only on which opinions survive in it and on the skeleton of the confidence rule.

(2) There exists a fixed point, where all opinions survive, if and only if $\mathcal{R}$ is a union of strongly connected graphs. Moreover, if $\mathcal{R}$ itself is strongly connected, this point is unique and an unstable node (the only exception is the case with one opinion, when the fixed point is the only point in the phase space).

(3) The results concerning only opinions in a set Δ (the fixed points and the stabilities inside $\mathcal{M}_\Delta$) can be found using the model defined by the confidence rule $\mathcal{R}_\Delta$ (in other words, removing opinions from the model leaves us with a model with a different confidence rule that is valid inside of $\mathcal{M}_\Delta$).

(4) If $\mathcal{R}$ is a union of different components $\Delta_1, \dots, \Delta_k$, and $\vec{\eta}_i$ is a fixed point of the model in $\mathcal{M}_{\Delta_i}$, we have that the convex hull $\mathcal{H}$ of the $\vec{\eta}_i$ is constituted entirely of fixed points of the model. The number of stable and unstable directions for these fixed points can be obtained by summing these numbers for each of the $\vec{\eta}_i$ (taking into account only directions parallel to $\mathcal{M}_{\Delta_i}$). For the number of neutral directions, we must take into account that all the directions parallel to $\mathcal{H}$ will be neutral with no movement along them.

(5) A fixed point where only opinions in Δ survive is attractive if and only if $\Delta_- = \overline{\Delta}$. This also implies that Δ is a maximal independent set (see Appendix B1) and hence that $\mathcal{M}_\Delta$ is an attractor (because of the last two items).

(6) If Δ is the set of opinions that survive in a given fixed point and $\mathcal{R}_\Delta$ has k components, the trajectories in a neighborhood of the fixed point are such that

(a) There are $|\Delta| + |(\overline{\Delta} - \Delta_-) \cap \Delta_+| - k$ unstable directions.
(b) There are $|\overline{\Delta} \cap \Delta_-|$ stable directions.
(c) There are $k - 1$ directions along which there is no movement.
(d) There are $|(\overline{\Delta} - \Delta_-) \cap (\overline{\Delta} - \Delta_+)|$ directions that are neither attractive nor repulsive, but along which there is movement.

These results have some interesting consequences and interpretations that should be kept in mind when analyzing the simulation results.

(1) The mean field has no stable situations where two interacting opinions coexist. This means that all possible (static) attractors are of the form $\mathcal{M}_\Delta$, where Δ is a maximal independent set.

(2) The requirement that Δ be maximal for $\mathcal{M}_\Delta$ to be an attractor allows the existence of attractors with surviving opinions that do not convince any opinions at all.

(3) The condition $\Delta_- = \overline{\Delta}$ implies that it is possible to build confidence rules that have no such attractors. These rules display heteroclinic cycles (see Appendix B2), which cause oscillations with diverging period and are heavily affected by finite size effects during simulations. In some cases, these cycles have basins of attraction, even if static attractors are present.

(4) If every opinion can convince any other (that is, $p_{\sigma \to \sigma'} \neq 0$ for all $\sigma \neq \sigma'$), then the consensus states are the only attractors.

IV. SIMULATION RESULTS AND EXAMPLES

For our simulations, we used Barabási-Albert networks with 10^5 sites and minimal connectivity equal to 5 (we used different networks, but always with these same parameters).

In order to compare trajectories obtained by simulations with trajectories obtained by integrating Eq. (1) we recall that η_σ is the expected value of the proportion of sites with opinion σ. Because of this and in order to reduce noise, we take averages over many simulations (one can also reduce noise by choosing a larger network size). More importantly, if the initial condition for the mean-field equations is $(\eta_1, \dots, \eta_M)$, then this means that for the corresponding simulations, each site must have its opinion chosen at random with probability

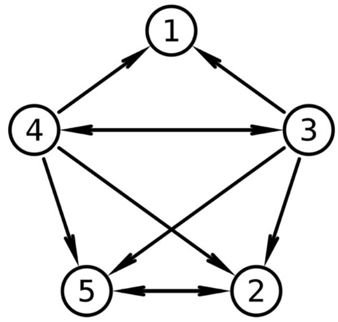


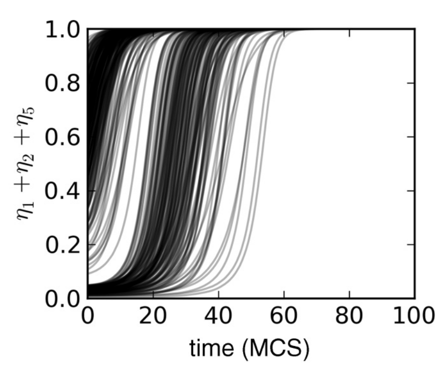
FIG. 2. The skeleton $\mathcal{R}$, of a confidence rule where two of the four maximal independent sets generate attractors. These static attractors can all be obtained by solving $\Delta_- = \overline{\Delta}$ in the rule, as pointed out in Sec. III, and are $\mathcal{M}_{\{1,2\}}$ and $\mathcal{M}_{\{1,5\}}$.

η_σ for opinion σ. We also recall that the model being simulated corresponds to the parameter choice $q = 2$ and $r = 1$.

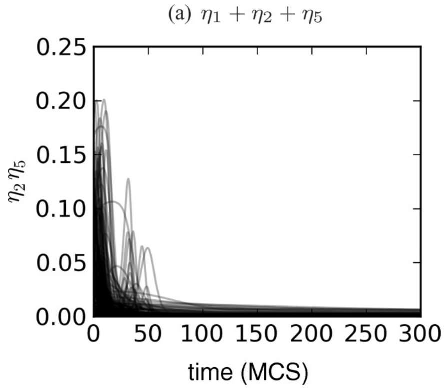
The simulations we will do will be aimed at giving examples of the mean-field results from Sec. III, some of their counterintuitive aspects and some divergences between the simulations and the mean field.

A. Attractors and stability

To illustrate the results about the stability properties of the fixed points, consider the rule $\mathcal{R}$, depicted in Fig. 2 (actually, a family of confidence rules). The maximal independent sets are $\Delta = \{1,2\},\{1,5\},\{3\}$, and $\{4\}$, but only $\{1,2\}$ and $\{1,5\}$ obey $\Delta_- = \overline{\Delta}$, meaning that the only stationary attractors are $\mathcal{M}_{\{1,2\}}$ and $\mathcal{M}_{\{1,5\}}$. After a transient we see one of two situations, the only surviving opinions will be 1 and 2 or they will be 1 and 5. We can see this from the time series of $\eta_1 + \eta_2 + \eta_5$ (it tends to 1) and $\eta_2.\eta_5$ (it tends to 0, although with a longer transient) [Figs. 3(a) and 3(b)].



(a) $\eta_1 + \eta_2 + \eta_5$



(b) $\eta_2.\eta_5$

FIG. 3. Time series for the rule in Fig. 2 with weights either 0 or 1, depicting the attractors. Note that in Fig. 3(a) the ending value is 1 (meaning that opinions 3 and 4 do not survive). In Fig. 3(b) the ending value is 0, showing that opinion 2 and 5 do not survive at the same time.

TABLE I. The fixed points of the rule in Fig. 2 that are not in attractors, denoted by the opinions that survive in them (Δ). The line of fixed points connecting the edge $\overline{P_2P_5}$ to the vertex P_1 is denoted by $1 \times 2,5$. For each fixed point, we list the number of unstable, stable, and neutral directions (u, s, and n, respectively). The relation of these numbers with the sets $\overline{\Delta}$, Δ_-, Δ_+, and the number of components induced by Δ is given in Sec. III.

Δ	$\overline{\Delta}$	Δ_-	Δ_+	u	s	n
3	1,2,4,5	4	1,2,4,5	3	1	0
4	1,2,3,5	3	1,2,3,5	3	1	0
3,4	1,2,5	3,4	1,2,3,4,5	4	0	0
$1 \times 2,5$	3,4	2,3,4,5	2,5	1	2	1

The other fixed points can be found by looking at the other induced graphs that are unions of strongly connected graphs. They are $\mathcal{R}_{\{3\}}$, $\mathcal{R}_{\{4\}}$, $\mathcal{R}_{\{3,4\}}$, and $\mathcal{R}_{\{1,2,5\}}$. Note that $\mathcal{R}_{\{1,2,5\}} = \mathcal{R}_{\{1\}} \cup \mathcal{R}_{\{2,5\}}$ and that both components are strongly connected. This means that we will actually have a line of fixed points connecting some point in the edge $\overline{P_2P_5}$ to the vertex P_1. The stability properties of these points are in Table I. A projection of the phase space (where the weights in the confidence rule were taken as 0 or 1), showing the attractors and the features described in this table can be found in Fig. 4.

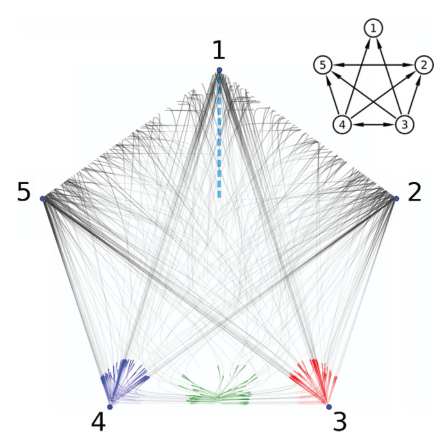


FIG. 4. (Color online) Phase space projection, depicting the structures described in Table I and the attractors. On the top right we see a reordering of the skeleton of the rule making the independent sets more evident. The cyan dashed line shows the location of the saddle points ($1 \times 2,5$; in Table I), the blue shaded trajectories are passing near the point (4), the red ones near the point (3), and the green ones near the point (3,4). For all these trajectories and the gray ones, lighter shades indicate the beginning of the trajectories and darker shades indicate their ending. We can see then the trajectories going to the attractors $\mathcal{M}_{\{1,2\}}$ and $\mathcal{M}_{\{1,5\}}$, with some being at first attracted by the saddles in ($1 \times 2,5$) before being repelled. We can also see the predicted stable direction for the fixed points (3) and (4) and the fact that (3,4) is an unstable node.

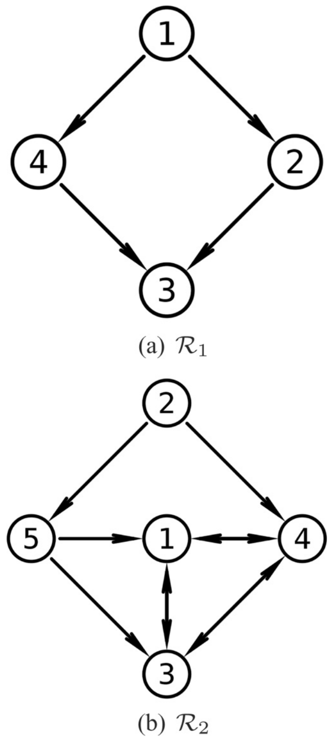


FIG. 5. The skeleton of two confidence rules $\mathcal{R}_1$ and $\mathcal{R}_2$, such that inert opinions are able to survive in the stationary state.

B. Surviving inert opinions

Next, we consider two examples in which we have opinions that survive in an attractor, but do not convince any other opinion (we will call them inert). Consider the rules $\mathcal{R}_1$ and $\mathcal{R}_2$ given in Fig. 5. In $\mathcal{R}_1$, $\mathcal{M}_{\{1,3\}}$ is an attractor, even though opinion 1 is inert (cannot convince any of the others). In $\mathcal{R}_2$, $\mathcal{M}_{\{4,5\}}$, $\mathcal{M}_{\{1,2\}}$, and $\mathcal{M}_{\{2,3\}}$ are attractors, even though opinion 2 is inert.

We now check that this effect is present in the simulations. Time series for the models with confidence rules $\mathcal{R}_1$ and $\mathcal{R}_2$ (once again, the weights are taken as 0 or 1) are given in Figs. 6(a)–6(d).[1]

C. Rules without static attractors

Consider a rule in which all opinions interact (that is, for all pair of distinct opinions σ and σ' either $p_{\sigma\to\sigma'} \neq 0$ or $p_{\sigma'\to\sigma} \neq 0$), but such that every opinion σ has at least one different opinion σ' that it cannot convince. The independent sets of such rule are all unitary, but we have imposed that $\sigma' \notin \{\sigma\}_-$ and $\sigma \neq \sigma'$, so there are no solutions to $\Delta_- = \overline{\Delta}$ for this rule and hence it has no stationary attractors (it is

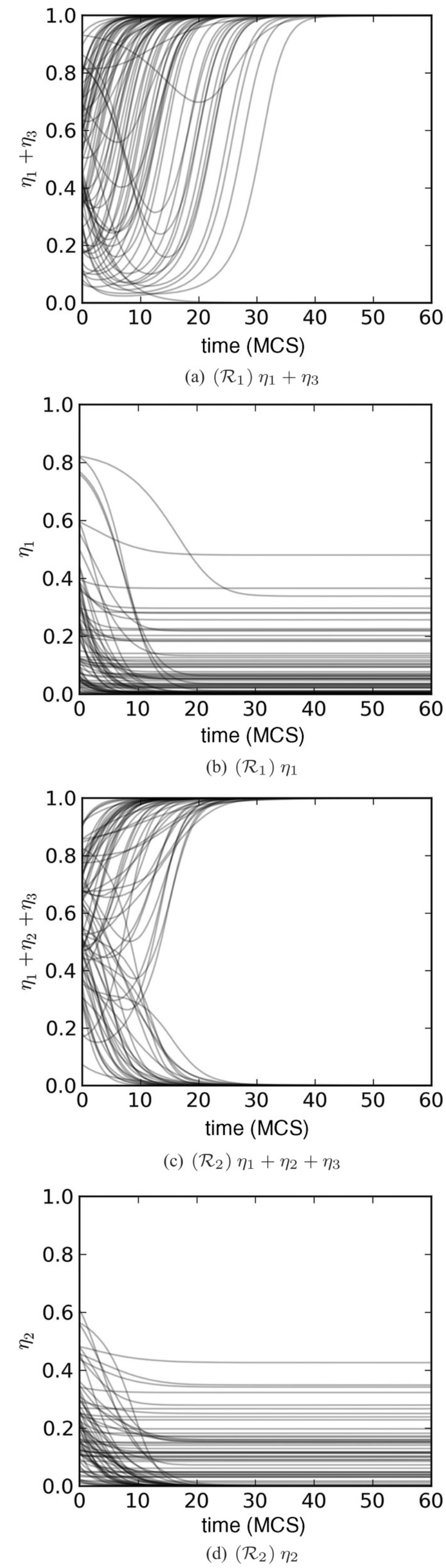


FIG. 6. Time series for the rules $\mathcal{R}_1$ and $\mathcal{R}_2$ in Fig. 5 with weights either 0 or 1. Graphs 6(a) and 6(c) depict time series containing the full attractor (the ending value is either 0 or 1 depending on the attractor reached), while graphs 6(b) and 6(d) focus in the surviving inert opinion (which always decays, but can reach a nonzero stationary value).

[1]In one of the trajectories depicted in Fig. 6(a) the stationary state is not the one predicted by the mean-field theory. In this case, the initial condition had only 17 sites out of 10^5 holding opinion 3, and they were not neighbors. If we remove opinion 3 from the confidence rule $\mathcal{R}_1$, the mean-field attractor becomes $\mathcal{M}_{\{2,4\}}$, which was the one actually reached by this trajectory.

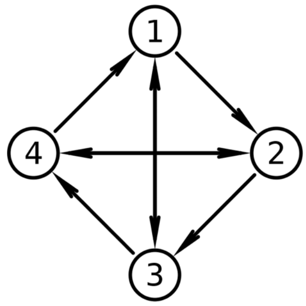


FIG. 7. A rule that has no attractors.

possible to build other types of examples as well). An example for four opinions is given in Fig. 7.

In Appendix B2 we prove that if there are no static attractors (Δ_- is always different from $\overline{\Delta}$), then there exists a directed cycle, where no edges are doubly connected. These cycles in the confidence rule represent heteroclinic cycles in the phase space and the typical behavior in this case is that as time goes by, the trajectories get closer to one of the cycles, causing oscillations with a diverging period (as they pass each time closer to the consensus states, which are fixed points). In simulations, eventually a random fluctuation puts the system in a state where one of the opinions in the cycle gets extinct, leading the system to a stationary state.

We measured the time τ needed to reach consensus for the confidence rule in Fig. 7. As this seems to be a finite size effect, we made simulations for various network sizes (ranging from 10^3 to 10^5) and compared them with each other. The initial condition that we used was a small perturbation of the one where all opinions are drawn uniformly. The cumulative distribution for τ as a function of the network size N is in Fig. 8, while the average time to reach consensus is presented in Fig. 9. We note that the average time (when measured as Monte Carlo sweeps) seems to scale as $\langle\tau\rangle \simeq \log N$, indicating that the consensus attractors are finite size effects.

D. Long transients and stationary states

In many simulations, there are situations in which the trajectories get stuck for long times in states that are not attractors. In some of these cases, the simulation got to a stationary state where there are no active connections between the agents (that is, a connection between a pair of agreeing sites and a neighbor that they can convince, according to the confidence rule). In other cases there are active connections, but some opinions appear in negligible amounts and the set Δ of opinions that are not negligible forms an independent set, but not a solution to $\Delta_- = \overline{\Delta}$. In the latter cases, the fixed points in $\mathcal{M}_\Delta$ are saddle points meaning that (recalling the number of unstable directions of a fixed point, as stated in Sec. III) $\widetilde{\Delta} \equiv (\overline{\Delta} - \Delta_-) \cap \Delta_+ \neq \varnothing$ and usually, one (or more) of the negligible opinions will be able to rise again, causing long transients.

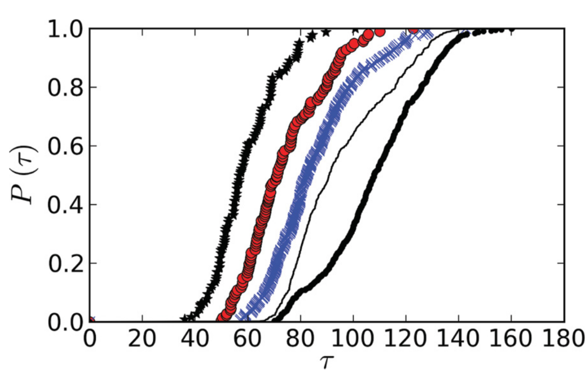


FIG. 8. (Color online) The cumulative distribution of the time to reach consensus, measured in Monte Carlo sweeps, for the rule in Fig. 7, using network sizes 1000 (black stars), 3160 (red circles), 10 000 (blue X's), 31 600 (continuous black line), and 100 000 (black circles). For each network size, 500 simulations were done.

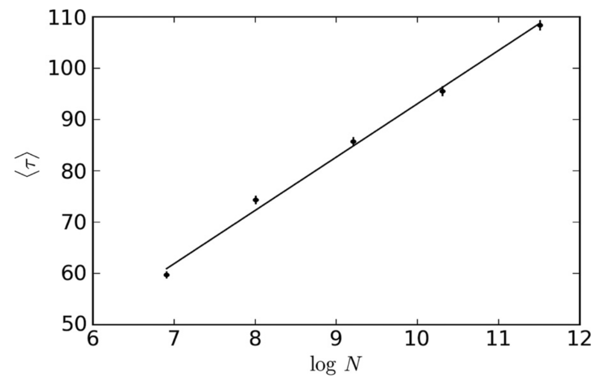


FIG. 9. The average time to reach consensus measured in Monte Carlo sweeps, for the rule in Fig. 7, using network sizes 1000, 3160, 10 000, 31 600, and 100 000. For each network size, 500 simulations were done. (the logarithm base is e).

Considering the mean-field equations, if $\sigma \in \widetilde{\Delta}$ (meaning that σ is negligible), then it evolves according to (see Sec. V B for further explanations)

$$\eta_\sigma(t) = \frac{\eta_{\sigma 0}}{1 - \eta_{\sigma 0} t \sum_{\sigma' \in \Delta} \eta_{\sigma' 0} p_{\sigma' \to \sigma}}, \tag{4}$$

as long as the opinions in $\overline{\Delta}$ are negligible. This implies that the time the trajectories spend close to these saddle points can be estimated, considering the time it takes for some of the opinions in $\widetilde{\Delta}$ to duplicate its proportion of sites in the network (all the other opinions in $\overline{\Delta}$ will remain negligible for much longer times; see Appendix D). Solving (4) we get

$$\tau \simeq \min_{\sigma \in \widetilde{\Delta}} \left(\frac{1}{2\eta_{\sigma 0} \sum_{\sigma' \in \Delta} \eta_{\sigma' 0} p_{\sigma' \to \sigma}} \right). \tag{5}$$

We now verify this relation for the integration of the mean-field equations and compare these results with the simulations. We will use the rule in Fig. 10, with $\Delta = \{3,5\}$.

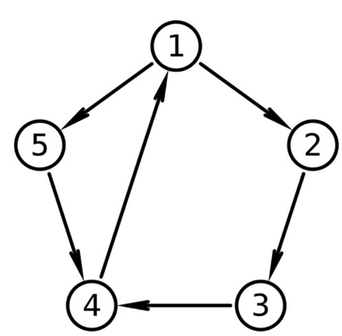
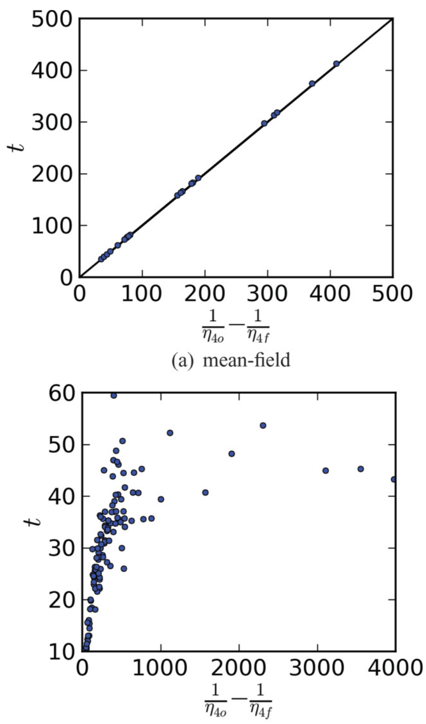


FIG. 10. A rule particularly prone to display long transients.



(a) mean-field

(b) Simulations in Barabási-Albert networks

FIG. 11. (Color online) Trapping times measured in Monte Carlo sweeps for $\Delta = \{3,5\}$ (more precisely the time the trajectory took since it crossed the surface $\eta_3 + \eta_5 > 1 - \varepsilon$, until it crossed the surface $\eta_4 > 2\varepsilon$, with $\varepsilon = 0.025$). We can see that simulations behave very differently than the mean field. Particularly, trapping times are smaller than predicted by Eq. (6) and the relationship between the two variables is not linear. The blue points correspond to the measured values and the black line corresponds to the prediction of Eq. (6) (that holds only for the integration of the mean-field equations).

For this choice of confidence rule and opinion set, we have $\widetilde{\Delta} = \{4\}$ and we can approximate Eq. (4) with

$$\frac{1}{\eta_{4o}} - \frac{1}{\eta_4} \simeq t. \tag{6}$$

The graphs for the mean field and the simulations can be found in Figs. 11(a) and 11(b) and show that if a simulation does not go to a stationary state, then it undergoes a transient much faster than what is expected from the mean-field equations. On the other hand, there is the possibility of a simulation reaching a stationary state, where the simulation is no longer in a transient, but has not reached, a mean-field attractor.

V. ANALYTICAL RESULTS

Our goal in this section is to study the mean-field equation [Eq. (1)], in order to find the fixed points and their stability properties. We will derive here the results that were given earlier in Sec. III, showing the main mathematical ideas. Some of the most technical bits are in separate appendices, in order to keep the focus in the reasonings that we deem more important.

We recall that the mean-field equation is

$$\dot{\eta}_\sigma = r \sum_{\sigma'} \eta_\sigma \eta_{\sigma'} \left(\eta_\sigma^{q-1} p_{\sigma' \to \sigma} - \eta_{\sigma'}^{q-1} p_{\sigma \to \sigma'} \right), \tag{7}$$

with $q \geqslant 2$ and $r \geqslant 1$, where η_σ is the proportion of sites holding opinion σ. We will denote the number of opinions by M and the confidence rule that is being used by $\mathcal{R}$. The phase space of this equation is the simplex defined by

$$\sum_\sigma \eta_\sigma = 1 \quad \text{and} \quad \eta_\sigma \geqslant 0 \,\forall\, \sigma. \tag{8}$$

It is easy to show that the trajectories never leave the phase space. To see that, we first note that the sum of the variables does not change with time:

$$\begin{aligned} \frac{d}{dt} \sum_\sigma \eta_\sigma &= \sum_\sigma \dot{\eta}_\sigma \\ &= r \sum_{\sigma,\sigma'} \eta_\sigma \eta_{\sigma'} \left(\eta_\sigma^{q-1} p_{\sigma' \to \sigma} - \eta_{\sigma'}^{q-1} p_{\sigma \to \sigma'} \right) = 0, \end{aligned}$$

because the term being summed is antisymmetric by a change between σ and σ'. Then, we look at the derivative of $\log(\eta_\sigma)$:

$$\begin{aligned} \left| \frac{d}{dt} \log(\eta_\sigma) \right| &= \left| \frac{\dot{\eta}_\sigma}{\eta_\sigma} \right| \\ &\leqslant r \sum_{\sigma'} \eta_{\sigma'} \left(\eta_\sigma^{q-1} p_{\sigma' \to \sigma} + \eta_{\sigma'}^{q-1} p_{\sigma \to \sigma'} \right) \\ &\leqslant r \sum_{\sigma'} 2\eta_{\sigma'} = 2r. \end{aligned}$$

This means that if we start in the interior of the phase space, $|\log(\eta_\sigma)|$ does not diverge in a finite amount of time an so η_σ does not become 0 in a finite amount of time, keeping the trajectories inside the phase space.

A. The fixed point equations

In order to find the fixed points we must solve the equations $\dot{\eta}_\sigma = 0$. In other words, defining the vector field $\vec{F} \equiv \dot{\vec{\eta}}$, we must find the roots of $\vec{F}$ that are inside the phase space. This means that a fixed point must obey

$$\begin{aligned} &\eta_\sigma = 0 \quad \text{or} \\ &\sum_{\sigma'} \left(\eta_\sigma^{q-1} \eta_{\sigma'} p_{\sigma' \to \sigma} - \eta_{\sigma'}^q p_{\sigma \to \sigma'} \right) = 0 \end{aligned} \tag{9}$$

for each opinion σ. Given the form of Eq. (9), it is natural to separate the solutions according to which opinions survive (meaning they are held by a proportion of agents different from 0) and which opinions do not (we will call them extinct). We will denote the set of surviving opinions by Δ and the set of extinct opinions by Ω, that is, the fixed points of the model are the solutions (such that $\eta_\sigma \neq 0 \forall \sigma \in \Delta$) of

$$\sum_{\sigma'} \left(\eta_\sigma^{q-1} \eta_{\sigma'} p_{\sigma' \to \sigma} - \eta_{\sigma'}^q p_{\sigma \to \sigma'} \right) = 0 \forall \sigma \in \Delta, \tag{10}$$

for each of the possible sets of surviving opinions Δ, setting $\eta_\sigma = 0$ for all opinions σ in the corresponding Ω. Note that, *a priori*, we must solve the system (10) for all the $2^M - 1$ possibilities for Δ that are different from an empty set. However, if we substitute $\eta_\sigma = 0 \forall \sigma \in \Omega$ in Eq. (10), we get

$$\sum_{\sigma' \in \Delta} \left(\eta_\sigma^{q-1} \eta_{\sigma'} p_{\sigma' \to \sigma} - \eta_{\sigma'}^q p_{\sigma \to \sigma'} \right) = 0 \forall \sigma \in \Delta, \tag{11}$$

which is the system (10) for the model with confidence rule $\mathcal{R}_\Delta$ and in the case where all opinions survive. This means that we can reduce the understanding of the general case to the understanding of the case $\Omega = \varnothing$ (as far as finding the fixed points is concerned). Moreover, if $\mathcal{R}_\Delta$ has k components $\Delta_1, \dots, \Delta_k$, this means that $p_{\sigma\to\sigma'} = 0$ whenever $\sigma \in \Delta_i, \sigma' \in \Delta_j$, and $i \neq j$. We can then rewrite the system of Eq. (11) as

$$\sum_j \sum_{\sigma' \in \Delta_j} \left(\eta_\sigma^{q-1}\eta_{\sigma'} p_{\sigma'\to\sigma} - \eta_{\sigma'}^q p_{\sigma\to\sigma'}\right) = 0 \forall \sigma \in \Delta_i,$$

but we already have

$$\sum_{\sigma' \in \Delta_j} \left(\eta_\sigma^{q-1}\eta_{\sigma'} p_{\sigma'\to\sigma} - \eta_{\sigma'}^q p_{\sigma\to\sigma'}\right) = 0 \forall \sigma \in \Delta_i$$

whenever $i \neq j$, no matter the values of the η_σ, because all of the p's in this case are equal to 0. So we arrive at the system of equations,

$$\sum_{\sigma' \in \Delta_i} \left(\eta_\sigma^{q-1}\eta_{\sigma'} p_{\sigma'\to\sigma} - \eta_{\sigma'}^q p_{\sigma\to\sigma'}\right) = 0 \forall \sigma \in \Delta_i, \quad (12)$$

that is, the equation for the fixed points of the model with rule $\mathcal{R}_{\Delta_i}$, meaning that we have a fixed point where all opinions survive if and only if the variables corresponding to opinions in $\Delta_1, \dots, \Delta_k$ are themselves fixed points (after we adjust their normalization) where all opinions survive, but for the models with confidence rules $\mathcal{R}_{\Delta_1}, \dots, \mathcal{R}_{\Delta_k}$, respectively.

More precisely, recalling that we are looking for solutions for $\vec{F} = \vec{0}$, where

$$F_\sigma(\vec{\eta}) = r \sum_{\sigma'} \eta_\sigma \eta_{\sigma'} \left(\eta_\sigma^{q-1} p_{\sigma'\to\sigma} - \eta_{\sigma'}^{q-1} p_{\sigma\to\sigma'}\right),$$

then if we define $\vec{\eta}_i$ and $\vec{F}_i$ as the vectors containing only the coordinates of $\vec{\eta}$ and $\vec{F}$ that are in Δ_i, that is,

$$\vec{\eta} = [\vec{\eta}_1\, \vec{\eta}_2 \dots \vec{\eta}_k] \quad \text{and} \quad \vec{F} = [\vec{F}_1\, \vec{F}_2 \dots \vec{F}_k], \quad (13)$$

then for every i, $\vec{F}_i$ is a homogeneous function of $\vec{\eta}_i$ only. So $\vec{F}_i$ being $\vec{0}$ depends exclusively on $\vec{\eta}_i$ being a fixed point of the model with rule $\mathcal{R}_{\Delta_i}$ (after changing the normalization, so that the sum of all variables is 1). Moreover, if all the $\vec{\eta}_i$ are fixed points where all opinions survive (which is equivalent to saying that $\vec{\eta}$ is a fixed point where all opinions survive), then defining

$$\eta_{\Delta_i} = \sum_{\sigma \in \Delta_i} \eta_\sigma \quad \text{and}$$
$$\vec{\zeta}_1 = \left[\frac{\vec{\eta}_1}{\eta_{\Delta_1}}\, \vec{0} \dots \vec{0}\right], \quad \vec{\zeta}_2 = \left[\vec{0}\, \frac{\vec{\eta}_2}{\eta_{\Delta_2}}\, \vec{0} \dots \vec{0}\right], \dots,$$
$$\vec{\zeta}_k = \left[\vec{0} \dots \frac{\vec{\eta}_k}{\eta_{\Delta_k}}\right], \quad (14)$$

it follows that all the points given by

$$\sum_{i=1}^k \alpha_i \vec{\zeta}_i \ \text{such that} \ \sum_{i=1}^k \alpha_i = 1 \quad \text{and} \quad \alpha_i \geqslant 0 \forall i \quad (15)$$

are fixed points of the model. The set of points defined by Eq. (15) is the convex hull of the points defined by the $\vec{\zeta}_i$ and as these vectors are linearly independent, it means the convex hull must have $k-1$ dimensions. We will denote this set of points by $\mathcal{H}(\vec{\zeta}_1, \dots, \vec{\zeta}_k)$.

These arguments lead us to the conclusion that the most important is to understand which are the fixed points in the case where all opinions survive and the model has only one component. Unfortunately, in this case we are still left with a qth degree system, with M variables and M equations (actually, only $M-1$ of the equations are independent and the last variable is determined by the constraint $\sum_\sigma \eta_\sigma = 1$). It is clear then that, in the general case, trying to find the fixed points exactly is a hopeless task. Because of this we will aim to get a qualitative understanding of the solutions.

Instead of trying to find directly which are the possible solutions, we will suppose at first that we have a hypothetical solution $\vec{\eta}^*$ where only opinions in Δ survive. We will make the stability analysis of this hypothetical solution and then using the stability results, together with continuity arguments, we will be able to determine when there actually is a fixed point where all opinions survive, as well as how many of these points exist. Finally, using the reasonings from this subsection we arrive at the full qualitative picture of the structure and stability of the fixed points.

B. Stability of the fixed points

Let $\vec{\eta}^*$ be the hypothetical solution we are studying and suppose that our confidence rule $\mathcal{R}$ has one component. We will start with a linear stability analysis, so we will first need to find the Jacobian $\mathcal{J}$ of $\vec{F}$:

$$\mathcal{J}_{\sigma,\sigma'} = \frac{\partial F_\sigma}{\partial \eta_{\sigma'}} = r\left(\eta_\sigma^q p_{\sigma'\to\sigma} - q\eta_\sigma \eta_{\sigma'}^{q-1} p_{\sigma\to\sigma'}\right) + r\delta_{\sigma,\sigma'} \sum_{\sigma''} \left(q\eta_\sigma^{q-1}\eta_{\sigma''} p_{\sigma''\to\sigma} - \eta_{\sigma''}^q p_{\sigma\to\sigma''}\right). \quad (16)$$

We then need to substitute our hypothetical solution and find the signs of the real parts of the eigenvalues of the resulting matrix. We will denote the Jacobian evaluated at $\vec{\eta}^*$ by $\mathcal{J}^*$. We first note that $\eta_\sigma = 0$ for all σ in Ω and that if $\mathcal{R}_\Delta$ has k components, $\Delta_1, \dots, \Delta_k$, then $p_{\sigma\to\sigma'} = 0$ whenever $\sigma \in \Delta_i, \sigma' \in \Delta_j$, and $i \neq j$. Substituting in Eq. (16) and using a reasoning similar to the one used to find Eq. (12) we get

$$\begin{aligned}
\mathcal{J}^*_{\sigma,\sigma'} &= -r\delta_{\sigma,\sigma'} \sum_{\sigma'' \in \Delta} \left(\eta_{\sigma''}^q p_{\sigma\to\sigma''}\right), \quad \text{if} \quad \sigma \in \Omega,\\
\mathcal{J}^*_{\sigma,\sigma'} &= r\eta_\sigma^q p_{\sigma'\to\sigma}, \quad \text{if} \quad \sigma \in \Delta, \sigma' \in \Omega,\\
\mathcal{J}^*_{\sigma,\sigma'} &= r\left(\eta_\sigma^q p_{\sigma'\to\sigma} - q\eta_\sigma\eta_{\sigma'}^{q-1} p_{\sigma\to\sigma'}\right)\\
&\quad + r\delta_{\sigma,\sigma'} \sum_{\sigma'' \in \Delta_i} \left(q\eta_\sigma^{q-1}\eta_{\sigma''} p_{\sigma''\to\sigma} - \eta_{\sigma''}^q p_{\sigma\to\sigma''}\right), \quad \text{if}\\
&\sigma,\sigma' \in \Delta_i, \quad \text{and}\\
\mathcal{J}^*_{\sigma,\sigma'} &= 0, \quad \text{if } \sigma \in \Delta_i, \sigma' \in \Delta_j, \text{ and } i \neq j.
\end{aligned} \quad (17)$$

This means that $\mathcal{J}^*$ can be permutated to

$$\mathcal{J}^* \sim \begin{bmatrix} \mathcal{J}_1^* & 0 & \dots & 0 & \mathcal{N}_1 \\ 0 & \mathcal{J}_2^* & \dots & 0 & \mathcal{N}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathcal{J}_k^* & \mathcal{N}_k \\ 0 & 0 & \dots & 0 & \mathcal{J}_\Omega^* \end{bmatrix}, \quad (18)$$

by writing first the lines and columns corresponding to opinions in Δ_1, followed by the ones corresponding to the opinions in Δ_2 and so on, until the opinions in Δ_k and, in the end, writing the lines and columns corresponding to opinions in Ω. It follows that the elements in the block $\mathcal{J}^*_\Omega$ are described by the first case in Eq. (17) ($\sigma \in \Omega$), the blocks $\mathcal{N}_i$ are described by the second case ($\sigma \in \Delta_i, \sigma' \in \Omega$) and the blocks $\mathcal{J}^*_i$ are described by the third case ($\sigma, \sigma' \in \Delta_i$). The permutation of the matrix $\mathcal{J}^*$ in Eq. (18) is block triangular, so we can find the eigenvalues of $\mathcal{J}^*$ by putting together the eigenvalues of $\mathcal{J}^*_1, \ldots, \mathcal{J}^*_k$ and $\mathcal{J}^*_\Omega$, meaning the matrices $\mathcal{N}_i$ are not relevant in our analysis.

Comparing Eqs. (17) and (16), together with the description of the fixed points in this case (confidence rule with k components), given by Eq. (15), we can verify that $\mathcal{J}^*_i/\eta^q_{\Delta_i}$ is the Jacobian, for the case in which the model has $\mathcal{R}_{\Delta_i}$ as its confidence rule, evaluated in a fixed point where all opinions survive. This means that we need only study the fixed points where all opinions survive for a rule with one component, in order to understand the eigenvalues of the blocks $\mathcal{J}^*_i$. The analysis of this case is rather technical and is left, together with a partial analysis of the eigenvectors, to Appendix C, but the result is quite simple. If we have a rule $\mathcal{R}'$, with only one component, then the Jacobian of $\vec{F}$, evaluated in a fixed point where all opinions survive has $M-1$ eigenvalues with positive real part and 1 eigenvalue that is always equal to 0. Moreover, the coordinates of the fixed point form the eigenvector with null eigenvalue. Going back to the general case, this means that the blocks $\mathcal{J}^*_1, \ldots, \mathcal{J}^*_k$ are responsible for $|\Delta| - k$ eigenvalues with positive real part and k eigenvalues equal to 0. However, the result for the eigenvectors means that these null eigenvalues are not relevant to the stability analysis, as there is no movement along these directions in a neighborhood of the fixed point. The reason for this is that one of the corresponding eigenvectors is $\vec{\eta}^*$ itself, that points outwards of the phase space (it can be regarded as an artifact of having embedded an $M-1$ phase space in M dimensions), while the other $k-1$ eigenvectors are $\vec{\zeta}_i - \vec{\eta}^*$ (as each of the $\vec{\zeta}_i$ will be an eigenvector). The vectors $\vec{\zeta}_i - \vec{\eta}^*$ generate the hyperplane where $\mathcal{H}(\vec{\zeta}_1, \ldots, \vec{\zeta}_k)$, defined in Eq. (15), is located, which means that if we start in the fixed point $\vec{\eta}^*$ and go along any of the directions corresponding to one of these $k-1$ null eigenvalues, we will only find other fixed points.

The remaining eigenvalues of $\mathcal{J}^*$ are the eigenvalues of $\mathcal{J}^*_\Omega$. Because of Eq. (17), this is a diagonal matrix and the eigenvalues are trivial:

$$\lambda_\sigma = -r \sum_{\sigma' \in \Delta} \eta^{*q}_{\sigma'} p_{\sigma \to \sigma'} \leqslant 0, \tag{19}$$

for each $\sigma \in \Omega$. The dual eigenvector (that is, the eigenvector of $\mathcal{J}^{*T}$) corresponding to λ_σ is also trivial, and is given by the vector with all coordinates equal to 0, except for the one corresponding to σ. This indicates that λ_σ is responsible for telling us if the trajectories near the fixed point are attracted or repelled to the manifold $\eta_\sigma = 0$.

$\lambda_\sigma \neq 0$ if and only if there exists $\sigma' \in \Delta$ such that $p_{\sigma \to \sigma'} \neq 0$, which can be translated to $\sigma \in \Delta_-$. As $\Omega = \overline{\Delta}$ by definition, then this means that $\mathcal{J}^*_\Omega$ contributes with $|\overline{\Delta} \cap \Delta_-|$ eigenvalues that have negative real part. All the others are null eigenvalues and in this case we must go beyond a linear stability analysis.

Suppose that $\lambda_\sigma = 0$. The lower order term for $\dot{\eta}_\sigma$ that is different from 0 in a neighborhood of the fixed point is

$$\dot{\eta}_\sigma \simeq r \eta^q_\sigma \sum_{\sigma' \in \Delta} \eta^*_{\sigma'} p_{\sigma' \to \sigma}, \tag{20}$$

meaning that the trajectories are repelled from the manifold $\eta_\sigma = 0$, unless $p_{\sigma' \to \sigma} = 0$ for all σ' in Δ. This translates to $\sigma \notin \Delta_+$ when looking at the confidence rule and hence we have $|(\overline{\Delta} - \Delta_-) \cap \Delta_+|$ of these unstable directions (opinions in Ω that satisfy $\sigma \in \Delta_+$, but not $\sigma \in \Delta_-$). In particular, for $q = 2$ and $r = 1$, the solution of Eq. (20) reads

$$\eta_\sigma(t) = \frac{\eta_{\sigma 0}}{1 - \eta_{\sigma 0} t \sum_{\sigma'} \eta^*_{\sigma'} p_{\sigma' \to \sigma}}, \tag{21}$$

as stated in Sec. IV D [Eq. (4)].

All that is left is to study the stabilities of the manifolds $\eta_\sigma = 0$, when $p_{\sigma \to \sigma'} = p_{\sigma' \to \sigma} = 0$ for all σ' in Δ. These are related to opinions that are extinct in the fixed point and only interact with other opinions that get extinct in this point. This causes the dynamics to be extremely slow along these directions and so if we have a repulsive direction ($(\overline{\Delta} - \Delta_-) \cap \Delta_+ \neq \varnothing$) this part of the dynamics is irrelevant (these directions have neutral stability), as the trajectory would be repelled away from the fixed point, before the slower dynamics could play any role. Let ω be the set of opinions such that $\lambda_\sigma = 0$ and suppose that we are in a situation where either $\lambda_\sigma < 0$ or $\sigma \notin \Delta_+$ for all opinions in Ω. In this case, $\omega \neq \Omega$, because otherwise there would be no connections among Δ and Ω, contradicting our assumption that $\mathcal{R}$ has only one component. We can then define

$$\Lambda \equiv \max_{\sigma \in \Omega - \omega} \lambda_\sigma < 0 \quad \text{and} \quad \eta_\omega \equiv \sum_{\sigma \in \omega} \eta_\sigma. \tag{22}$$

We show in Appendix D that starting in a sufficiently close neighborhood of $\eta_\omega = 0$ the following inequality holds:

$$\eta_{\omega 0} e^{|\Omega - \omega| r / q \Lambda} \leqslant \eta_\omega \leqslant \eta_{\omega 0} e^{-|\Omega - \omega| r / \Lambda}, \tag{23}$$

and so trajectories are neither attracted to nor repelled from $\mathcal{M}_\Delta$, meaning that the opinions that get extinct and only interact with other opinions that get extinct are responsible for neutral directions (but along which there is movement).

We can now put all these results together. If $\vec{\eta}^*$ is a fixed point, such that $\mathcal{R}_\Delta$ has k components, then the trajectories in a neighborhood of $\vec{\eta}^*$ are such that (remembering that $\Omega = \overline{\Delta}$):

(1) There are $|\Delta| + |(\overline{\Delta} - \Delta_-) \cap \Delta_+| - k$ unstable directions.

(2) There are $|\overline{\Delta} \cap \Delta_-|$ stable directions.

(3) There are $k - 1$ directions along which there is no movement.

(4) There are $|(\overline{\Delta} - \Delta_-) \cap (\overline{\Delta} - \Delta_+)|$ directions that are neither attractive nor repulsive, but along which there is movement.

Finally, we can use this result to find the static attractors of the model. For a fixed point to be attractive, all of its directions must be either stable, or neutral, without movement, which means

$$|\Delta| + |(\overline{\Delta} - \Delta_-) \cap \Delta_+| - k = 0, \quad \text{and}$$
$$|(\overline{\Delta} - \Delta_-) \cap (\overline{\Delta} - \Delta_+)| = 0. \tag{24}$$

As $|\Delta| \geqslant k$, then the first of these equations means that we must have $|\Delta| = k$ and $|(\overline{\Delta} - \Delta_-) \cap \Delta_+| = 0$. $|\Delta| = k$ is the same as saying that Δ is an independent set (that is, $\mathcal{R}_\Delta$ has no arrows), which implies $\Delta_-,\Delta_+ \subseteq \overline{\Delta}$. The remaining equations read

$$(\overline{\Delta} - \Delta_-) \cap \Delta_+ = \varnothing \quad \text{and} \quad (\overline{\Delta} - \Delta_-) \cap (\overline{\Delta} - \Delta_+) = \varnothing. \tag{25}$$

This implies that

$$((\overline{\Delta} - \Delta_-) \cap \Delta_+) \cup ((\overline{\Delta} - \Delta_-) \cap (\overline{\Delta} - \Delta_+)) = \varnothing \\ \Leftrightarrow (\overline{\Delta} - \Delta_-) \cap (\Delta_+ \cup (\overline{\Delta} - \Delta_+)) = \varnothing. \tag{26}$$

As Δ is independent, $\Delta_+ \subseteq \overline{\Delta}$ and so $\Delta_+ \cup (\overline{\Delta} - \Delta_+) = \overline{\Delta}$, implying

$$(\overline{\Delta} - \Delta_-) \cap \overline{\Delta} = \varnothing \Leftrightarrow \overline{\Delta} - \Delta_- = \varnothing \Leftrightarrow \overline{\Delta} \subseteq \Delta_-. \tag{27}$$

Again, Δ is independent, meaning that we have $\Delta_- \subseteq \overline{\Delta} \Rightarrow \overline{\Delta} = \Delta_-$. This means that if Δ is an independent set and a solution to Eq. (25) then it must obey $\overline{\Delta} = \Delta_-$. On the other hand, if Δ' is some solution to $\overline{\Delta}' = \Delta'_-$, then $\overline{\Delta}' - \Delta'_- = \varnothing$ and we can substitute this in Eq. (25) to verify that Δ' is a solution. Moreover, we show in Appendix B1 that $\overline{\Delta}' = \Delta'_-$ alone implies that Δ' is a maximal independent set. Putting it all together, we have that the equation,

$$\overline{\Delta} = \Delta_-, \tag{28}$$

is equivalent to Eq. (24). Δ being independent also means that all points in $\mathcal{M}_\Delta$ are fixed points, leading us to the conclusion that the static attractors of this model are all of the form $\mathcal{M}_\Delta$, where $\overline{\Delta} = \Delta_-$ and that to find the solutions of this equation, it suffices to check the maximal independent sets of the confidence rule being used. Finally, all the stability properties depend only on Δ and on what are the connections in the confidence rule $\mathcal{R}$ and are therefore completely defined by the skeleton of $\mathcal{R}$ and by the opinions that survive in the point being analyzed.

C. Existence of the fixed points

In the last section we made the stability analysis of a hypothetical fixed point, showing the relation between the stability properties, the skeleton of the confidence rule, and the opinions that survive in the fixed point. We will now use those results to determine which are the fixed points that actually exist. Our discussion in Sec. V A shows that it is enough to study when a fixed point where all opinions survive exists, for a model where the confidence rule has only one component.

In this case, a fixed point where all opinions survive must be an unstable node (in case it exists) and so, if we had embedded our phase space in $M-1$ instead of M dimensions, (substituting η_M by $1 - \sum_{\sigma\neq M} \eta_\sigma$, for example), the Jacobian $\widetilde{\mathcal{J}}$ of the corresponding flux would be a real matrix that is positive definite when evaluated in such a fixed point, implying that $\det(\widetilde{\mathcal{J}}) > 0$. In Appendix E, we use this information, together with the implicit function theorem and the Poincaré-Hopf theorem, to show that if all the parameters $p_{\sigma\to\sigma'}$ are different from 0 (which is the same as saying that the skeleton of the confidence rule is a complete directed graph), then there is exactly one fixed point where all opinions survive, which we will call the coexistence fixed point.

The general case can be obtained using a continuity argument. Suppose that we have a confidence rule $\mathcal{R}$ that has no fixed points where all opinions survive. Because of the result from Appendix E, this means that some of the p's must be equal to 0. On the other hand, $\mathcal{R}$ can be regarded as a point in a parameter space (where the $p_{\sigma\to\sigma'}$ are the coordinates) and the implicit function theorem tells us that in the region where all the p's are different from 0, both the coexistence fixed point and the eigenvalues of $\widetilde{\mathcal{J}}$ change continuously with a continuous change of the parameters. It is also possible to make an infinitesimal perturbation of the parameters in $\mathcal{R}$, to get a confidence rule where all of the parameters are different from 0. This means that we can build a continuous path in the parameter space, that only goes through $\mathcal{R}$ and confidence rules such that all the p's are different from 0 (with the exception of $\mathcal{R}$). If we follow the coexistence fixed point in the phase space, as the parameters are changed (that is, as we walk along the path we have built in the parameter space), we see that the fixed point must collide in the border, as the parameters approach the rule $\mathcal{R}$, becoming a fixed point $\vec{\eta}^*$, where not all opinions survive. Following the eigenvalues of $\widetilde{\mathcal{J}}$ we get to the conclusion that all of their real parts are bigger than or equal to 0. But $\Omega \neq \varnothing$ for the fixed point $\vec{\eta}^*$ and all the eigenvalues originating from $\mathcal{J}^*_\Omega$ (that are also eigenvalues of $\widetilde{\mathcal{J}}$) have real parts smaller than or equal to 0. This can be reconciled only if $\lambda_\sigma = 0\ \forall\ \sigma \in \Omega$ for the fixed point $\vec{\eta}^*$, which translates to $\Delta_- \cap \overline{\Delta} = \varnothing$.

This means that it is a necessary condition for a rule that does not display a coexistence fixed point to possess a set $\Delta \neq \varnothing, V$ (V is the set of all opinions) such that $\Delta_- \cap \overline{\Delta} = \varnothing$. On the other hand, suppose that we have a rule with one component that has a set $\Delta \neq \varnothing, V$ such that $\Delta_- \cap \overline{\Delta} = \varnothing$. As we have only one component, we must have $\Delta_+ \cap \overline{\Delta} \neq \varnothing$ (otherwise there would be no connections between Δ and $\overline{\Delta}$), and so if we define $\eta_\Delta = \sum_{\sigma\in\Delta} \eta_\sigma$, then

$$\dot{\eta}_\Delta = -r \sum_{\sigma\in\Delta} \sum_{\sigma'\in\overline{\Delta}\cap\Delta_-} \eta_\sigma \eta^q_{\sigma'} p_{\sigma\to\sigma'}, \tag{29}$$

meaning $\dot{\eta}_\Delta < 0$ in the whole region of the phase space where all opinions coexist. This implies that there exists a fixed point where all opinions survive if and only if there is no set of opinions $\Delta \neq \varnothing, V$, obeying $\Delta_- \cap \overline{\Delta} = \varnothing$. We show in Appendix B3, that this is equivalent to saying the graph of the confidence rule is strongly connected (that is, we can start in any node and get to any other node. For a rule with one component, this is also equivalent to saying that any node is part of some cycle). Finally, when the coexistence fixed point is present, it must always be unique (this happens because all rules are a perturbation of rules where all the p's are different from 0, and such confidence rules always have unique coexistence fixed points).

Going back to our results from Sec. V A, it follows that there exists a fixed point where only the opinions in Δ survive if and only if Δ induces an union of strongly connected graphs (as such a fixed point exists if and only if it exists for each of the components separately).

D. Heteroclinic cycles and nonstatic attractors

We now consider a confidence rule $\mathcal{R}$ such that there exists a cycle in $\mathcal{R}$, $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_r \to \sigma_1$, where none of the edges is doubly connected (σ_i points to σ_{i+1}, but the opposite is

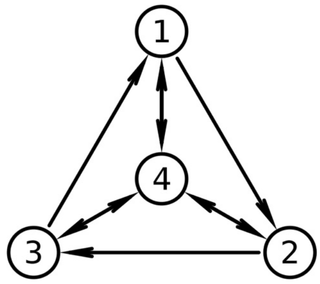


FIG. 12. A rule with a static attractor and a heteroclinic cycle that has a basin of attraction.

not true). In the phase space, these cycles manifest themselves as heteroclinic cycles. These cycles will always be polygonal curves connecting the vertices of the simplex that correspond to the nodes the cycle in the graph goes through, that is, $\overline{P_{\sigma_1} P_{\sigma_2} \ldots P_{\sigma_r} P_{\sigma_1}}$. Moreover if Δ is the set of nodes the cycle goes through, it means that Δ induces in the confidence rule a strongly connected graph with one component and as such, there exists an unstable fixed point where all the opinions in Δ coexist. Hence, the heteroclinic cycle is fully contained in the border of $\mathcal{M}_\Delta$ and there is a fixed point in the bulk of $\mathcal{M}_\Delta$ that leads the trajectories to its border.

A consequence is that if there is a neighborhood of $\mathcal{M}_\Delta$, where this manifold is attractive, then these trajectories will eventually be attracted by the heteroclinic cycle, causing oscillations with diverging period. In Appendix B2 we prove that if a graph has no solutions to $\Delta_- = \overline{\Delta}$, then it has at least one directed cycle, where no edge is doubly connected, but it is also possible to build a confidence rule, where there are static attractors and a heteroclinic cycle that has a basin of attraction. An example of such a rule is given by Fig. 12, where both P_4 and the cycle $\overline{P_1 P_2 P_3 P_1}$ have basins of attraction.

VI. CONCLUSIONS

In this work, we expanded our previous results about the Sznajd model with general confidence rules (interpreted here as biases and prejudices), giving analytical results about the existence, structure, and stability properties of the fixed points in the mean-field case, finding a very rich behavior. We gave simulation results in Barabási-Albert networks that show examples of this mean-field behavior and showed some of the discrepancies between the model simulated in these networks and the integration of the mean-field equations.

Even though neither the equations for the fixed points can be solved analytically, nor can the exact eigenvalues of the Jacobian be all determined, our dynamical systems approach was still able to determine the sign of the real parts of these eigenvalues and the higher order behaviors, when these were needed. Surprisingly, this analysis showed us that the various properties of the fixed points depend only on a few qualitative properties of the confidence rule (the directed skeleton). This, in turn, allowed us to make a connection between the mean-field results and graph theory problems and this connection can even be used to study more complex behaviors, like the heteroclinic cycles in the phase space that always appear in the absence of attractors.

In regard to the simulations, most of the discrepancies with the mean field seem to come from the existence of frozen states that do not correspond to mean-field attractors, but that can be reached by the model in a network. It is not entirely clear if these are purely finite size effects, but their origin suggests that they should be more common as the number of opinions increases and that the introduction of a random noise, in which opinions change randomly with a given probability, should destroy this effect. A curious finding in the confidence rule studied in Sec. IV D is that when simulations got close to the frozen states, but managed to get away from them, they took much less time than would be expected from the mean-field results. The same can be said about the confidence rule studied in Sec. IV C.

These results seem to indicate that there is some measure of universality in models that give a greater convincing power to groups of agreeing people, like the variants of the Sznajd model that were studied and in this sense this work can be seen as a unifying effort. The present results can be put together with a previous work [21], in which we made simulations of the Sznajd model in a Watts-Strogatz network and obtained similar results in the small-world regime, but completely different results in the non-small-world regime. In our opinion, this indicates that the most important aspects in these models are the confidence rule and the network that are being used. In particular, we believe that the conclusions from this paper should hold for complex networks (not only Barabási-Albert networks), but this may not be the case in regular lattices.

Given the simple conclusions that were reached and the generality of our model (we would like to stress that the mean-field results are valid not only for the Sznajd model but for the q-voter model with $q \geqslant 2$), we believe that similar approaches might be fruitful in other models where asymmetrical interactions exist, way beyond opinion propagation and sociophysics, like infection spreading and ecology models. It would also be interesting to see if similar connections with graph theory problems exist in other models and, if they do, how rich they are.

ACKNOWLEDGMENTS

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APPENDIX A: GRAPH THEORY CONCEPTS AND GLOSSARY

If G is a weighted graph, with adjacency matrix $G_{i\to j}$ (that is, the matrix containing the weights of the graph) and $G_{i\to j} \geqslant 0$, one can define its directed skeleton $\mathrm{Sk}_{\mathrm{dir}}(G)$ as the directed graph with adjacency matrix:

$$S_{i\to j} = \begin{cases} 0, & \text{if} \quad G_{i\to j} = 0, \\ 1, & \text{if} \quad G_{i\to j} \neq 0. \end{cases} \tag{A1}$$

An example of skeleton is given in Fig. 13.

Let now Δ be a set of nodes in a directed graph $\mathcal{S}$ (typically in our problems, Δ will be a set of opinions and $\mathcal{S}$ will be the skeleton of the confidence rule). We define the following

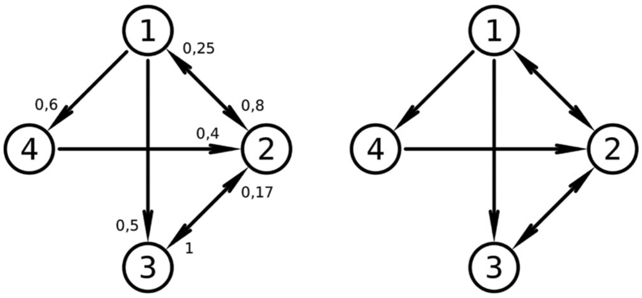

FIG. 13. An example. The graph to the right is the directed skeleton of the weighted graph to the left.

terms (we will use in the examples $\mathcal{S}$ equal to the skeleton in Fig. 13):

(1) The predecessor set of Δ, denoted by Δ_-, is the set of nodes in $\mathcal{S}$ that point to some node in Δ. For example, $\{3\}_- = \{1,4\}_- = \{1,2\}$ and $\{2\}_- = \{1,3,4\}$.

(2) Analogously, the successor of Δ, denoted by Δ_+, is the set of nodes in $\mathcal{S}$ that are pointed by nodes in Δ. For example, $\{2,3\}_+ = \{1,2,3\}$.

(3) The complement of Δ, $\overline{\Delta}$ is the set of nodes in $\mathcal{S}$ that are not in Δ. For example, $\overline{\{2,3\}} = \{1,4\}$.

(4) Δ is an independent set if and only if $\mathcal{S}$ has no connections among nodes in Δ. $\{1\}$ and $\{3,4\}$ are independent sets. Note that if Δ is independent, it follows that $\Delta_-,\Delta_+ \subseteq \overline{\Delta}$.

(5) An independent set Δ is maximal if it contains all the nodes in the graph or if the addition of any node not in Δ destroys independence. $\{3,4\}$ is a maximal independent set, while $\{3\}$ is independent but not maximal.

(6) If Δ is a set of nodes from $\mathcal{S}$, then the graph induced by Δ, $\mathcal{S}_\Delta$ is the graph whose set of nodes is Δ and whose connections are the connections between the elements of Δ that existed in $\mathcal{S}$. The graph $\mathcal{S}_{\{2,3,4\}}$ can be found in Fig. 14.

(7) The union of two graphs G and H, denoted $G \cup H$ is the graph with all the nodes of G and H, but only connections that already existed between G and H (in short it means referring to two unrelated graphs as parts of the same graph, without changing anything else). The graph $\mathcal{S}_{\{1,2,3\}} \cup \mathcal{S}_{\{4\}}$ is shown in Fig. 15. In addition, the more familiar concept of component can be defined as a graph that is not the union of any smaller parts and is also not part of a larger graph with the same property.

(8) A graph is strongly connected if we can start at any node and get to any other node, respecting the directions of the arcs. $\mathcal{S}$, $\mathcal{S}_{\{4\}}$, and $\mathcal{S}_{\{1,2\}}$ are strongly connected, but $\mathcal{S}_{\{1,3\}}$ is not because there is no path from 3 to 1 in it.

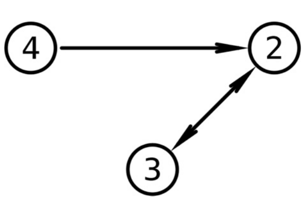

FIG. 14. The graph induced in $\mathcal{S}$ by the set $\{2,3,4\}$.

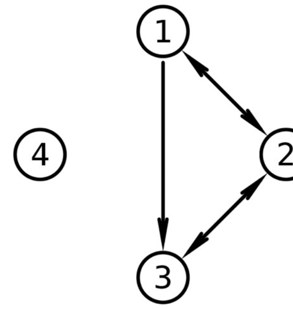

FIG. 15. An example of graph union. The graph $\mathcal{S}_{\{1,2,3\}} \cup \mathcal{S}_{\{4\}}$.

APPENDIX B: GRAPH THEORY THEOREMS WITH APPLICATIONS TO OUR MODEL

1. $\overline{\Delta} = \Delta_-$ implies maximal independence

Theorem 1. Let G be a graph and let Δ be a set of nodes in it such that $\overline{\Delta} = \Delta_-$. This implies that Δ is a maximal independent set.

Proof. To see this, suppose that $\Delta_- = \overline{\Delta}$ but Δ is not independent, then it follows that there exists $\sigma,\sigma' \in \Delta$ such that $\sigma \in \{\sigma'\}_+ \Leftrightarrow \sigma' \in \{\sigma\}_- \Rightarrow \sigma' \in \Delta \cap \Delta_- \neq \varnothing \Rightarrow \Delta \cap \overline{\Delta} \neq \varnothing$, which is a contradiction.

So if $\Delta_- = \overline{\Delta}$ then Δ is independent. If $\overline{\Delta} = \varnothing$ then it is trivial that Δ is maximal. If $\overline{\Delta} \neq \varnothing$, take $\sigma \in \overline{\Delta}$. It follows that

$$\begin{aligned}(\Delta \cup \{\sigma\})_- &= \Delta_- \cup \{\sigma\}_- = \overline{\Delta} \cup \{\sigma\}_- \\ &\Rightarrow (\Delta \cup \{\sigma\}) \cap (\Delta \cup \{\sigma\})_- \\ &= (\Delta \cup \{\sigma\}) \cap (\overline{\Delta} \cup \{\sigma\}_-) \\ &= ((\Delta \cup \{\sigma\}) \cap \overline{\Delta}) \cup ((\Delta \cup \{\sigma\}) \cap \{\sigma\}_-) \\ &\supseteq (\Delta \cup \{\sigma\}) \cap \overline{\Delta} = \{\sigma\} \neq \varnothing \\ &\Rightarrow (\Delta \cup \{\sigma\}) \cap (\Delta \cup \{\sigma\})_- \neq \varnothing, \end{aligned} \tag{B1}$$

which implies that $\Delta \cup \{\sigma\}$ is not independent and hence, Δ is maximal. ■

2. Relation between the absence of attractors and heteroclinic cycles

Theorem 2. Let G be a directed graph such that no set of nodes obeys $\overline{\Delta} = \Delta_-$, then there exists a directed cycle in G that does not use any of the doubly linked edges.

Proof. Suppose that there is no such cycle in G and let G' be the graph G after removing all the doubly linked edges. By hypothesis, G' is a directed acyclic graph and so a topological ordering in G' is possible. This means that we can define a strict partial order in $V(G)$:

$$i \prec j \text{ if and only if there is a path from } j \text{ to } i \text{ in } G'.$$

We can also restrict this order to a subset Ω of $V(G)$, such that $i,j \in \Omega \Rightarrow i \prec_\Omega j$ if and only if $i \prec j$ (note that this is not the same thing as saying the path exists in G'_Ω). Consider now the set Δ built from the following algorithm:

(1) Attribute $\Delta \leftarrow \varnothing$, $\Xi \leftarrow \varnothing$, and $\Omega \leftarrow V(G)$.

(2) If Ω equals $\varnothing$ stop, else let i be a minimal element of $\prec_\Omega$.

(3) Remove i from Ω and add it to the set Δ.

(4) Remove the elements from the predecessor of i with respect to G that are in Ω, $(i_-(G) \cap \Omega)$, from Ω and add them to Ξ.

(5) Go to 2.

By construction the set Ξ obeys $\overline{\Delta} = \Xi \subseteq \Delta_-$ (the predecessor with respect to G). Moreover, the set Δ is independent. To see this, suppose that at some time during the construction of Δ, there are no connections in G between nodes in Δ and nodes in Ω when we reach step 2 (this is trivially true for the starting iteration) and let i be the minimal element of $\prec_\Omega$ chosen in this step. As i is minimal, there are no nodes in Ω such that $j \prec_\Omega i$ and hence there are no paths from i to any other element in Ω in the graph G' and hence $i_+(G) \cap \Omega$ contains only nodes that are connected to i through doubly connected edges, implying $i_+(G) \cap \Omega \subseteq i_-(G) \cap \Omega$ and so in step 4 we are transferring all the nodes in Ω, that had any connection with Δ after step 3, to the set Ξ. So after an iteration of the algorithm there are still no connections between nodes in Δ and Ω when we reach step 2 again and so by induction, it holds during the whole construction of Δ that there are no connections between nodes that are in Δ and nodes that are in Ω. But as the nodes are added to Δ from Ω one at a time, each node that is added does not add connections between nodes in Δ, implying that Δ remains independent during its whole construction. On the other hand, this implies $\Delta_- \subseteq \overline{\Delta}$. Recalling that by the construction of Ξ we have $\overline{\Delta} \subseteq \Delta_-$, it follows that $\Delta_- = \overline{\Delta}$. ■

The relevance of this theorem to our problem is that a solution to $\overline{\Delta} = \Delta_-$ in the skeleton of the rule is equivalent to a static attractor in the phase space and saying the cycle $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_1$ in this skeleton has no doubly connected edges is equivalent to saying that the polygonal curve $\overline{P_{\sigma_1} P_{\sigma_2} \ldots P_{\sigma_1}}$ is a heteroclinic cycle, meaning that every rule that has no static attractors must have at least one heteroclinic cycle.

3. Necessary and sufficient condition for a graph to be strongly connected

Let G be a graph and $\Delta \neq \varnothing$ a set of nodes.

Definition 1. Δ is a sink (source) if and only if it obeys $\Delta_+ \cap \overline{\Delta} = \varnothing$ ($\Delta_- \cap \overline{\Delta} = \varnothing$). In both cases, Δ is called minimal if there is no nonempty proper subset of it with the same property.

Definition 2. The span of a node i, i_{span} is the set of all nodes j in G, such that j can be reached from i. The span of a set of nodes Δ is defined as the union of the span of each of its nodes. (For the purposes of this definition, a node always reaches itself, and so i always belongs to i_{span}).

Corollary 1. Every span is a sink and if Δ is a sink, then $\Delta = \Delta_{\text{span}}$.

Corollary 2. As no arc leaves a sink, if X is a sink and $Y \subseteq X$ then $Y_{\text{span}} = Y_{\text{span}}(G_X)$, the span of Y in G_X.

Theorem 3. A sink (source) is minimal if and only if it induces a strongly connected graph.

Proof. Let Δ be a sink that induces a strongly connected graph in G. Suppose by absurd that Δ in not minimal, then there exists $\Gamma \subset \Delta$, such that Γ is also a sink and $\Gamma \neq \varnothing$. Let $\omega \in \Delta - \Gamma$. As G_Δ is strongly connected, then for all $i, j \in \Delta$ we have $i \in j_{\text{span}}(G_\Delta)$ and hence $\omega \in \Gamma_{\text{span}}(G_\Delta)$. As $\Gamma \subset \Delta$ and Δ is a sink in G, it follows that $\Gamma_{\text{span}}(G_\Delta) = \Gamma_{\text{span}}(G)$ and as Γ is also a sink in G we have $\Gamma_{\text{span}}(G) = \Gamma$. But this implies $\omega \in \Gamma$, which is a contradiction and so Δ must be minimal.

On the other hand, if Δ is a minimal sink in G and we suppose by absurd that G_Δ is not strongly connected, there exists $i, j \in \Delta$ such that $i \notin j_{\text{span}}(G_\Delta)$. Δ is a sink, so this means $j_{\text{span}}(G_\Delta) = j_{\text{span}}(G)$. In addition, $j_{\text{span}}(G) \subseteq \Delta_{\text{span}}(G) = \Delta$. But then $j_{\text{span}}(G) \subseteq \Delta - \{i\}$ and so $j_{\text{span}}(G)$ is a nonempty proper subset of Δ that is a sink, contradicting the assumption that Δ was minimal. Hence, G_Δ must be strongly connected.

The proof for sources is obtained considering the graph G', obtained by switching the orientation of all the arcs of G (which transforms sinks in sources and vice versa, but keeps the same induced graphs strongly connected) ■

The relevance of this to our problem is that when the confidence rule has only one component, the condition that we found for the existence of a coexistence fixed point can be rephrased as saying that the set of all nodes is a minimal source. This theorem shows then that this is equivalent to saying the confidence rule is strongly connected, which makes it more easy to see what the result for many components is.

APPENDIX C: SPECTRUM OF THE JACOBIAN FOR A COEXISTENCE POINT IN A RULE WITH ONLY ONE COMPONENT

In this Appendix we introduce some matrix theory theorems and apply them to find the signs of the real parts of the eigenvalues of the Jacobian of the model, for a coexistence fixed point of a rule with one component.

Theorem 4 (Gershgorin). Let $M \in \mathbb{M}_n(\mathbb{C})$ be a square matrix whose general term is $m_{i,j}$. So if λ is an eigenvalue of M, then there exists an i such that

$$|\lambda - m_{i,i}| \leqslant \sum_{j \neq i} |m_{i,j}|.$$

Theorem 5 (Levy-Desplanques). Let $M \in \mathbb{M}_n(\mathbb{C})$ be an irreducible square matrix whose general term is $m_{i,j}$. If

$$|m_{i,i}| \geqslant \sum_{j \neq i} |m_{i,j}| \, \forall \, i$$

and there exists an i such that

$$|m_{i,i}| > \sum_{j \neq i} |m_{i,j}|,$$

then $\det(M) \neq 0$.

Theorem 6. Let $M \in \mathbb{M}_n(\mathbb{C})$ be a symmetrical irreducible square matrix whose general term is $m_{i,j}$. If $m_{i,i} \neq 0$ for some i, then for all k such that $1 \leqslant k < n$, there exists an irreducible principal submatrix of M with order k.

Proof. The case $n = 1$ follows from our assumption that some $m_{i,i}$ is different from 0. The irreducibility of a matrix depends only on which of its terms are equal to 0 and which of them are different from 0. So we only need to prove the case where we have a binary matrix (all terms are either 0 or 1).

Let X be a set of indexes of the matrix M (integers between 1 and n). We will denote by M_X the principal submatrix whose lines and columns are the ones corresponding to the indexes in X. As M is symmetric, all of its principal submatrices and all of their permutations are also symmetric. So suppose that

M_X is a principal submatrix that is not irreducible. This means that there exists a permutation that takes it not only to a block triangular form, but to a block diagonal form, that is,

$$M_X \sim \begin{bmatrix} M_Y & 0 \\ 0 & M_Z \end{bmatrix}, \tag{C1}$$

where $X = Y \cup Z$. Finally, M is the adjacency matrix of an undirected graph G, while M_X is the adjacency matrix of the induced subgraph G_X. This implies that Eq. (C1) is the same as the equation $G_X = G_Y \cup G_Z$, meaning that G_X is connected if and only if M_X is irreducible.

It suffices then to prove that given a connected graph with $n > 1$ nodes, we can always remove one of its nodes to get a new connected graph. Let i be any node in G. We will define the sets $i_{(n)}$ by the following recursion:

$$i_{(k+1)} = i_{(k)} \cup (i_{(k)})_+,$$

with $i_{(1)} = \{i\}$. As G is connected and has n nodes, it follows that $i_{(n)} = V(G)$. Moreover, as we have more than one node in G, there exists an $m < n$, such that

$$i_{(m)} \neq V(G) \quad \text{and} \quad i_{(m+1)} = V(G).$$

So if we define $\delta = i_{(m+1)} - i_{(m)}$, then all the nodes in $V(G) - \delta$ can be reached from i, without going through nodes in δ and if $j \in \delta$, there exists a path between i and j, such that j is the only node from δ in this path. It follows that removing any of the nodes in δ gives a new connected graph, finishing the proof. ■

The next theorem is a strengthening of a theorem found in Ref. [22].

Theorem 7. Let $M \in \mathbb{M}_n(\mathbb{C})$ be a square matrix and let its Hermitian part be $H = (M + M^\dagger)/2$. If H is positive semidefinite, with the multiplicity of 0 equal to μ, then for all $T \in \mathbb{M}_n(\mathbb{C})$, such that T is Hermitian positive definite, then MT (and TM) is positive semidefinite and the sum of the geometric multiplicities of its eigenvalues with null real part is smaller than or equal to μ.

Proof. By our hypothesis, the eigenvalues of H are nonnegative real numbers and its eigenvectors can be arranged as an orthonormal basis. We can split this basis in two parts, $\{u_i\}$, with the eigenvectors with eigenvalue 0 and $\{v_i\}$, for the others. Define $\lambda_i > 0$, the eigenvalue such that $Hv_i = \lambda_i v_i$ and define U, the linear span of $\{u_i\}$. Let x be a column vector and $x^\dagger$ its conjugate transpose, so

$$\begin{aligned}
x' &= \sum_i \alpha_i u_i, \quad x'' = \sum_j \beta_j v_j, \quad \text{and} \\
x &= x' + x'' \Rightarrow x^\dagger H x = (x'^\dagger + x''^\dagger) H (x' + x'') \\
&= (Hx')^\dagger (x' + x'') + x''^\dagger (Hx') + x''^\dagger H x'' = x''^\dagger H x'' \\
&= \sum_{i,j} \beta_i^* \beta_j v_i^\dagger H v_j = \sum_{i,j} \beta_i^* \beta_j \lambda_j \delta_{i,j} = \sum_i |\beta_i|^2 \lambda_i.
\end{aligned} \tag{C2}$$

Hence, $x^\dagger H x > 0 \Leftrightarrow x \notin U$. On the other hand $2\mathrm{Re}(x^\dagger M x) = x^\dagger M x + (x^\dagger M x)^* = x^\dagger (M + M^\dagger) x = 2 x^\dagger H x$. Let S be a nonsingular matrix and w, a normalized eigenvector of $S^\dagger M S$, with eigenvalue γ. Taking $x = Sw$, it follows,

$$\begin{aligned}
\mathrm{Re}(\gamma) &= \mathrm{Re}(\gamma w^\dagger w) = \mathrm{Re}(w^\dagger S^\dagger M S w) \\
&= \mathrm{Re}(x^\dagger M x) = x^\dagger H x.
\end{aligned} \tag{C3}$$

Define $W = S^{-1}U = \{y \in W \Leftrightarrow Sy \in U\}$, so $\mathrm{Re}(\gamma) > 0 \Leftrightarrow w \notin W$. As the dimension of U is μ, it follows that W also has dimension μ. The sum σ of the geometric multiplicities of the eigenvalues of $S^\dagger M S$ with null real part is the dimension of the linear span N of the corresponding eigenvectors. As all these eigenvectors belong to W and W is a linear subspace, then it follows that N is a subspace of W and hence $\sigma \leqslant \mu$.

All the properties of the spectrum of a matrix (including algebraic and geometric multiplicities) are encoded in its Jordan canonical form, and this form is invariant by similarity transformations, so $S^\dagger M S$, $MSS^\dagger$, and $SS^\dagger M$ have the same spectral properties. This proves the theorem, as any Hermitian positive definite matrix T can be written as $SS^\dagger$, with a nonsingular S using a Cholesky decomposition. ■

Theorem 8 (Euler). If $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable homogeneous function with order k and Jacobian $\mathcal{J}(\vec{x})$, then

$$\mathcal{J}(\vec{x}).\vec{x} = k\vec{F}(\vec{x}).$$

Proof. By hypothesis $\vec{F}(\lambda\vec{x}) = \lambda^k \vec{F}(\vec{x})$. Deriving with respect to λ yields $\mathcal{J}(\lambda\vec{x}).\vec{x} = k\lambda^{k-1}\vec{F}(\vec{x})$, so substituting $\lambda = 1$ gives the relation we want to prove. ■

We now apply these theorems to the analysis of the Jacobian that arose in Sec. V B. Let $\vec{\eta}^{\,*}$ be a coexistence fixed point (that is, all opinions survive) in a model with a rule that has only one component and at least two opinions. We can apply Euler's theorem (theorem 8) together with the homogeneity of $\vec{F}$ [as seen in Eq. (1)] to get $\mathcal{J}\vec{\eta} = (q+1)\vec{F}$ and so in the fixed point we have $\mathcal{J}^*\vec{\eta}^{\,*} = \vec{0}$.

In addition, because the sum of all variables is a constant, this leads to $\vec{1}.\vec{F} = 0$, where $\vec{1} = (1, \ldots, 1)$, so deriving with respect to η_σ gives the equation $\vec{1}\mathcal{J} = \vec{0}$. Let then D be the diagonal matrix whose diagonal terms are the coordinates of $\vec{\eta}^{\,*}$ [in other words $D = \mathrm{diag}(\vec{\eta}^{\,*})$], then the symmetric matrix A, defined as

$$A = \mathcal{J}^* D + (\mathcal{J}^* D)^T, \tag{C4}$$

has off-diagonal terms given by

$$A_{\sigma,\sigma'} = r(1-q)\left(\eta_\sigma^{*q}\eta_{\sigma'}^* p_{\sigma'\to\sigma} + \eta_\sigma^*\eta_{\sigma'}^{*q} p_{\sigma\to\sigma'}\right) \leqslant 0, \tag{C5}$$

and each of the rows (columns) of A sum 0. Moreover, as the confidence rule has only one component, A is irreducible, implying that at least one off-diagonal term in each row is different from 0 and hence all the diagonal terms are positive (since each of the rows must have a sum equal to 0). Finally, we use this information to apply Gershgorin's theorem (theorem 4) and find that A is positive semidefinite.

We will now show that the 0 is an eigenvalue of A, with multiplicity equal to 1. This is equivalent to saying that

$$\det(A) = 0 \quad \text{and} \quad \sum_\sigma \det(A^{(\sigma)}) > 0, \tag{C6}$$

where $X^{(\sigma)}$ denotes the principal submatrix of X, obtained by removing row and column σ. First we note that as each of the rows sums 0, this already implies that $\det(A) = 0$. Then we note that if $\sigma \neq \sigma'$ and $A_{\sigma,\sigma'} \neq 0$, then both $A^{(\sigma)}$ and $A^{(\sigma')}$ are such that one of the rows has a positive sum. However, every row has an off-diagonal term different from 0, meaning that for all the $A^{(\sigma)}$, at least one of the rows has a positive sum and no rows have negative sums. So if $A^{(\sigma)}$

is an irreducible matrix, we can apply the Levy-Desplanques theorem (theorem 5) to show that $\det(A^{(\sigma)}) \neq 0$. The existence of this submatrix is guaranteed by theorem 6, meaning that it suffices to prove that $\det(A^{(\sigma)}) \geqslant 0$ for all σ. This can be done applying Gershgorin's theorem again, to find that all the $A^{(\sigma)}$ are positive semidefinite.

We will now use theorem 7 to translate the results about the matrix A to the Jacobian $\mathcal{J}^*$. We first apply this theorem using $M = 2\mathcal{J}^* D$, implying that $H = A$, and using $T = D^{-1}/2$, to find that $\mathcal{J}^*$ is positive semidefinite and the sum of the geometric multiplicities of the eigenvalues with null real part is 1. As $\mathcal{J}^*\vec{\eta}^* = \vec{0}$ it follows that 0 has a geometric multiplicity equal to 1 and all the other eigenvalues of $\mathcal{J}^*$ have positive real part. Finally, we show that the algebraic multiplicity of 0 is 1. Once again, this is equivalent to showing that

$$\sum_\sigma \det(\mathcal{J}^{*(\sigma)}) \neq 0.$$

We start noting that $A^{(\sigma)} = \mathcal{J}^{*(\sigma)} D^{(\sigma)} + (\mathcal{J}^{*(\sigma)} D^{(\sigma)})^T$ (because D is diagonal). We then apply theorem 7 once more, but now using $M = 2\mathcal{J}^{*(\sigma)} D^{(\sigma)}$, so $H = A^{(\sigma)}$ and $T = (D^{(\sigma)})^{-1}/2$. As all the $A^{(\sigma)}$ are positive semidefinite and at least one of them is positive definite, this implies that all the $\mathcal{J}^{*(\sigma)}$ are positive semidefinite, with at least one of them being positive definite. When looking at their determinants this means that $\det(\mathcal{J}^{*(\sigma)}) \geqslant 0$ for all σ and the determinant is positive for at least one σ, implying that $\sum_\sigma \det(\mathcal{J}^{*(\sigma)}) > 0$, completing the proof that the algebraic multiplicity of 0 is 1.

Finally, we show the eigenvector corresponding to the eigenvalue 0 is the only one that is not parallel to the phase space. Suppose that $\mathcal{J}^*\vec{v} = \lambda\vec{v}$, where $\vec{v} \neq \vec{0}$ and $\lambda \neq 0$, it follows that $\lambda\vec{1}.\vec{v} = \vec{1}\mathcal{J}^*\vec{v} = \vec{0}.\vec{v} \Rightarrow \vec{1}.\vec{v} = 0$, implying that $\vec{v}$ is parallel to the phase space, while on the other hand $\vec{1}.\vec{\eta}^* = 1$.

Putting these results together, all the eigenvalues have positive real part and the corresponding eigenvectors are parallel to the phase space, with the exception of the eigenvector $\vec{\eta}^*$, that is not parallel and has eigenvalue 0 (with multiplicity 1).

APPENDIX D: HIGH ORDER STABILITY ANALYSIS FOR FIXED POINTS IN WHICH OPINIONS GET EXTINCT

In this section we do the detailed higher order analysis that lead us to the inequalities (23). Suppose that we have a fixed point of Eq. (1) in which only opinions in Δ survive and let $\Omega = \overline{\Delta}$. For each $\sigma \in \Omega$, we define λ_σ as

$$\lambda_\sigma = -r \sum_{\sigma' \in \Delta} \eta_{\sigma'}^{*q} p_{\sigma\to\sigma'} \leqslant 0. \tag{D1}$$

Suppose that either $\lambda_\sigma < 0$ or $\sigma \notin \Delta_+$ for all opinions in Ω and let ω be the set of opinions such that $\lambda_\sigma = 0$ (we will assume that $\Omega \neq \omega$). We define

$$\Lambda \equiv \max_{\sigma\in\Omega-\omega} \lambda_\sigma < 0 \quad \text{and} \quad \eta_\omega \equiv \sum_{\sigma\in\omega} \eta_\sigma. \tag{D2}$$

It follows from the first order analysis we did in Sec. V B, that if $\sigma \in \Omega - \omega$ and the initial value of η_σ, $\eta_{\sigma o}$ is sufficiently close to 0, then η_σ evolves as

$$\eta_\sigma(t) = \eta_{\sigma o} e^{\lambda_\sigma t}, \tag{D3}$$

as long as all opinions in Ω remain negligible. It follows from the mean-field equations that

$$\dot{\eta}_\omega = r \sum_{\sigma\in\omega} \sum_{\sigma'\in\Omega-\omega} \eta_\sigma \eta_{\sigma'} \left(\eta_\sigma^{q-1} p_{\sigma'\to\sigma} - \eta_{\sigma'}^{q-1} p_{\sigma\to\sigma'}\right). \tag{D4}$$

So if the opinions in Ω appear in a sufficiently small amount, we have

$$\eta_\sigma \simeq \eta_{\sigma o} e^{\lambda_\sigma t} \forall \sigma \in \Omega - \omega \Rightarrow$$

and substituting in Eq. (D4),

$$\dot{\eta}_\omega \simeq r \sum_{\sigma\in\omega} \sum_{\sigma'\in\Omega-\omega} \left(\eta_{\sigma' o} e^{\lambda_{\sigma'} t} \eta_\sigma^q p_{\sigma'\to\sigma} - \eta_{\sigma' o}^q e^{q\lambda_{\sigma'} t} \eta_\sigma p_{\sigma\to\sigma'}\right), \tag{D5}$$

yielding the following inequalities:

$$\begin{aligned}
&-r \sum_{\sigma\in\omega} \sum_{\sigma'\in\Omega-\omega} \eta_{\sigma' o}^q e^{q\lambda_{\sigma'} t} \eta_\sigma p_{\sigma\to\sigma'} \\
&\quad \leqslant \dot{\eta}_\omega \leqslant r \sum_{\sigma\in\omega} \sum_{\sigma'\in\Omega-\omega} \eta_{\sigma' o} e^{\lambda_{\sigma'} t} \eta_\sigma^q p_{\sigma'\to\sigma} \\
&\quad \Rightarrow -r \sum_{\sigma\in\omega} \sum_{\sigma'\in\Omega-\omega} e^{q\lambda_{\sigma'} t} \eta_\sigma \leqslant \dot{\eta}_\omega \leqslant r \sum_{\sigma\in\omega} \sum_{\sigma'\in\Omega-\omega} e^{\lambda_{\sigma'} t} \eta_\sigma \\
&\quad \Rightarrow -r \sum_{\sigma\in\omega} \sum_{\sigma'\in\Omega-\omega} e^{q\Lambda t} \eta_\sigma \leqslant \dot{\eta}_\omega \leqslant r \sum_{\sigma\in\omega} \sum_{\sigma'\in\Omega-\omega} e^{\Lambda t} \eta_\sigma \\
&\quad \Rightarrow -r|\Omega-\omega| e^{q\Lambda t} \eta_\omega \leqslant \dot{\eta}_\omega \leqslant r|\Omega-\omega| e^{\Lambda t} \eta_\omega \\
&\quad \Rightarrow -r|\Omega-\omega| e^{q\Lambda t} \leqslant \frac{d}{dt}\ln(\eta_\omega) \leqslant r|\Omega-\omega| e^{\Lambda t}.
\end{aligned} \tag{D6}$$

Integrating in time and taking the limit $t \to \infty$ gives the inequalities (23):

$$\eta_{\omega o} e^{|\Omega-\omega| r/q\Lambda} \leqslant \eta_\omega \leqslant \eta_{\omega o} e^{-|\Omega-\omega| r/\Lambda}, \tag{D7}$$

and so trajectories are neither attracted to nor repelled from $\mathcal{M}_\Delta$. This ensures that the whole reasoning is consistent, as it is always possible to make $\eta_{\omega o}$ sufficiently small, so that the hypothesis that the opinions in Ω appear in a suficiently small amount, always holds.

APPENDIX E: APPLYING THE POINCARÉ-HOPF THEOREM TO THE CASE OF A CONFIDENCE RULE WITH COMPLETE DIRECTED SKELETON

In this Appendix we use the following theorem,

Theorem 9 (Poincaré-Hopf). Let $\mathcal{M}$ be a compact, orientable, and differentiable manifold and let $\vec{F}$ be a vector field defined in $\mathcal{M}$, such that it has only isolated zeros (every zero has an open neighborhood in which it is unique). If either $\mathcal{M}$ has no border or if $\vec{F}$ points outwards (according to the orientation of $\mathcal{M}$) along all points of the border, then the sum of the indices [2] of all the zeros of $\vec{F}$ in the interior of $\mathcal{M}$ equals the Euler characteristic of $\mathcal{M}$.

to show that a rule with complete directed skeleton always has exactly one fixed point where all opinions coexist. In the

[2]The index in the case when the Jacobian is not singular equals the sign of its determinant. More information about indices and their meaning can be found in most textbooks about differential geometry.

case that is going to be used, the theorem is the same as saying that if a vector field is defined in a sphere, in a way that it points outwards along the surface and only sources are possible inside the sphere, then there exists exactly one source in the inside (generalized for a hypersphere).

Consider a rule with a skeleton corresponding to a complete directed graph and define

$$p_{\min} = \min_{\sigma \neq \sigma'}\{p_{\sigma\to\sigma'}\}, \quad \text{and} \quad p_{\max} = \max_{\sigma \neq \sigma'}\{p_{\sigma\to\sigma'}\}. \tag{E1}$$

In order to apply the Poincaré-Hopf theorem, we build a family of manifolds that includes the phase space:

$$V_\epsilon = \{\vec{\eta} \in \mathrm{sim}_M | \eta_\sigma \geqslant \epsilon \forall \sigma\}. \tag{E2}$$

These manifolds all satisfy the hypothesis of the theorem and the borders of V_ϵ are given by the facets $\eta_\sigma = \epsilon$ (that is, we are using M dimensions to define our manifolds, but we are embedding them in $M-1$ dimensions). The fixed points we obtain for the flow in the mean-field equation are not isolated when we look at the problem in M dimensions (because of the homogeneity of the equations), but our results about the Jacobian show that embedding the phase space in $M-1$ dimensions instead of M is enough to isolate the zeros (this follows from applying the implicit function theorem. Another way of isolating the zeros would be to add a term in the equation that is 0 inside the phase space, but is different from 0 outside, but this has the downside of making the hypothesis to be checked more complicated). It also follows from the spectrum of this Jacobian that the indices of any fixed points in the interior of any of the manifolds V_ϵ would be 1 (for a nonsingular Jacobian, the index of the fixed point equals the sign of the determinant of the Jacobian).

The last hypothesis to be checked is then that the vector field $\vec{F}$ points outside along the border. If $\eta_\sigma = \epsilon$ then

$$\begin{aligned} F_\sigma &= r\epsilon^q \sum_{\sigma'} \eta_{\sigma'} p_{\sigma'\to\sigma} - r\epsilon \sum_{\sigma'} \eta_{\sigma'}^q p_{\sigma\to\sigma'} \\ &\leqslant r\epsilon^q p_{\max} \sum_{\sigma'} \eta_{\sigma'} - r\epsilon p_{\min} \min_{\vec{\eta}\in V_\epsilon} \left\{\sum_{\sigma'} \eta_{\sigma'}^q\right\}. \end{aligned}$$

As

$$\min_{\vec{\eta}\in V_\epsilon} \left\{\sum_{\sigma'} \eta_{\sigma'}^q\right\} = \frac{1}{M^{q-1}},$$

it follows that

$$F_\sigma \leqslant r\epsilon^q p_{\max} - \frac{r\epsilon p_{\min}}{M^{q-1}}.$$

So it suffices to take

$$\epsilon < \frac{1}{M}\left(\frac{p_{\min}}{p_{\max}}\right)^{1/(q-1)} \tag{E3}$$

in order to get $F_\sigma < 0$ for all σ, implying that we can apply the theorem in the manifold V_ϵ.

The Euler characteristic of all of the V_ϵ is 1, meaning that if we can apply the theorem, there exists exactly one fixed point in its interior. Together with Eq. (E3), this means that there exists exactly one coexistence fixed point and it obeys

$$\eta_\sigma \geqslant \frac{1}{M}\left(\frac{p_{\min}}{p_{\max}}\right)^{1/(q-1)} \forall\sigma. \tag{E4}$$

[1] R. A. Holley and T. M. Liggett, Ann. Probab. **3**, 643 (1975).
[2] K. Sznajd-Weron and J. Sznajd, Int. J. Mod. Phys. C **11**, 1157 (2000).
[3] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, Adv. Complex Syst. **3**, 87 (2000).
[4] R. Hegselmann and U. Krause, J. Artificial Societies Social Simulation **5**(3), 1 (2002).
[5] R. Ochrombel, Int. J. Mod. Phys. C **12**, 1091 (2001).
[6] R. Axelrod, J. Conflict Resolution **41**, 203 (1997).
[7] S. Galam, Physica A **336**, 56 (2004).
[8] C. Castellano, S. Fortunato, and V. Loreto, Rev. Mod. Phys. **81**, 591 (2009).
[9] K. Sznajd-Weron, M. Tabiszewski, and A. M. Timpanaro, Europhys. Lett. **96**, 48002 (2011).
[10] A. T. Bernardes, D. Stauffer, and J. Kertész, Eur. Phys. J. B: Conden. Matter Complex Syst. **25**, 123 (2002).
[11] S. E. Asch, in *Groups, Leadership and Men*, edited by H. Guetzkow (Carnegie Press, Pittsburgh, 1951).
[12] A. M. Timpanaro and C. P. C. Prado, Phys. Rev. E **80**, 021119 (2009).
[13] D. Stauffer, Int. J. Mod. Phys. C **13**, 315 (2002).
[14] A. L. Barabási and R. Albert, Science **286**, 509 (1999).
[15] P. Przybyła, K. Sznajd-Weron, and M. Tabiszewski, Phys. Rev. E **84**, 031117 (2011).
[16] L. Festinger, *A Theory of Cognitive Dissonance* (Stanford University Press, Palo Alto, 1957).
[17] J. Elster, *Sour Grapes: Studies in the Subversion of Rationality* (Cambridge University Press, Cambridge, 1985).
[18] S. W. S. Lee and N. Schwarz, Science **328**, 709 (2010).
[19] D. Stauffer, Int. J. Mod. Phys. C **13**, 315 (2002).
[20] T. Reichenbach, M. Mobilia, and E. Frey, Nature (London) **448**, 1046 (2007).
[21] A. M. Timpanaro and C. P. C. Prado, Phys. Rev. E **84**, 027101 (2011).
[22] C. R. Johnson, J. Economic Theory **9**, 53 (1974).