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The family of functions
 $S_{\alpha,k}$ and the Liénard Equation

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Abstract

In this paper we study qualitatively the Liénard Equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ with aid of the non-usual family of functions given by

$$S_{\alpha,k}(x, y) = \int_0^{y+F(x)-\alpha G(x)-k} \frac{s}{\alpha s + 1} dx + \int_0^x g(u) du$$

where $F(x) = \int_0^x f(u) du$, $G(x) = \int_0^x g(u) du$ and $\alpha, k \in \mathbb{R}$.

1 Introduction

Throughout this work we consider the equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1}$$

where f and g are functions of \mathbb{R} in \mathbb{R} satisfying the following conditions:

- a) f and g are continuous and ensure uniqueness of solutions.
- b) $x.g(x) > 0$ for $x \neq 0$.

Next, we suppose the above conditions are verified and they will not be mentioned again.

The equation (1) is equivalent to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x) \end{cases} \quad (2)$$

The condition b) ensures the origin (0,0) is the only singular point of (2).

The more natural positive definite function for studying qualitatively the system (2) is the Energy Function

$$E(x, y) = \frac{1}{2}y^2 + \int_0^x g(u) du$$

whose derivative relative to (2) is $\dot{E}(x, y) = -f(x)y^2$.

In [1] we study qualitatively (2) using the family of positive definite functions given by

$$V_\alpha(x, y) = \int_0^y \frac{s}{\alpha s + 1} ds + \int_0^x g(u) du$$

where V_0 is exactly the Energy Function.

The equation (1) is also equivalent to the system

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \quad (3)$$

where $F(x) = \int_0^x f(u) du$. In several works (for example [2], [3] and [4]) the system (3) was studied with aid of the family of functions

$$E_k(x, y) = \frac{1}{2}(y - k)^2 + \int_0^x g(u) du$$

whose derivative relative to the system (3) is $\dot{E}_k(x, y) = -g(x)[F(x) - k]$.

Consider now the function

$$S_{0,k} = \frac{1}{2}[y + F(x) - k]^2 + \int_0^x g(u) du .$$

The derivative of $S_{0,k}$ relative to the system (2) is $\dot{S}_{0,k}(x, y) = -g(x)[F(x) - k]$.

So the function $S_{0,k}$ plays, relatively to the system (2), the same role that E_k relatively to (3). The function $S_{0,k}$ is a member ($\alpha = 0$) of family $S_{\alpha,k} : \Omega_{\alpha,k} \rightarrow \mathbb{R}$ given by

$$S_{\alpha,k}(x, y) = \int_0^{y+F(x)-\alpha G(x)-k} \frac{s}{\alpha s + 1} ds + \int_0^x g(u) du$$

where $F(x) = \int_0^x f(u) du$, $G(x) = \int_0^x g(u) du$ and $\Omega_{\alpha,k}$ is the following open set:

$$\Omega_{\alpha,k} = \mathbb{R}^2 \quad \text{if } \alpha = 0$$

$$\Omega_{\alpha,k} = \left\{ (x, y) \in \mathbb{R}^2 \mid y > -F(x) + \alpha G(x) + k - \frac{1}{\alpha} \right\} \quad \text{if } \alpha > 0$$

and

$$\Omega_{\alpha,k} = \left\{ (x, y) \in \mathbb{R}^2 \mid y < -F(x) + \alpha G(x) + k - \frac{1}{\alpha} \right\} \quad \text{if } \alpha < 0.$$

The derivative of $S_{\alpha,k}$ relative to the system (2) is

$$\dot{S}_{\alpha,k}(x, y) = -\frac{g(x)[F(x) - \alpha G(x) - k]}{\alpha[y + F(x) - \alpha G(x) - k] + 1}. \quad (4)$$

We observe that the sign of $\dot{S}_{\alpha,k}$ is the same of $-g(x)[F(x) - \alpha G(x) - k]$ because $\alpha[y + F(x) - \alpha G(x) - k] + 1 > 0$ on $\Omega_{\alpha,k}$.

In this work we study qualitatively (2) utilizing the family of functions $S_{\alpha,k}$. We observe that the new idea in this paper is only the family $S_{\alpha,k}$. However, as we shall see, the level curves of this family and the relation (4) together suggest to us how to state, in a natural way, several qualitative results about the solutions of the Liénard Equation.

The system (2) can be also studied using the family of positive definite functions given by

$$W_{\alpha,\beta}(x, y) = \int_0^{y/H_\beta(x)} \frac{s}{s^2 + \alpha s + 1} ds + \ln \beta^{-1/2} H_\beta(x)$$

where $H_\beta(x) = [2G(x) + \beta]^{1/2}$, $\beta > 0$ (see [5]).

It is clear that we can also study qualitatively the system (2) combining the functions V_α , $W_{\alpha,\beta}$ and $S_{\alpha,k}$. Many interesting and important works about the Liénard Equation have been published and some are listed in the references. I have a special caress by Theorem 2 in [7], because with aid of it (and of a dream!) I concluded my Doctoral Thesis and indirectly my work [5] was suggested by it.

2 Auxiliary lemmas

Next, we suppose $\alpha \geq 0$ and

$$w(x) = -F(x) + \alpha G(x) + k - \frac{1}{\alpha} \quad \text{if } \alpha > 0 \quad \text{and} \quad w(x) = -\infty \quad \text{if } \alpha = 0.$$

In first place we observe that, for each fixed x , the the function

$$y \mapsto S_{\alpha,k}(x,y)$$

is strictly increasing for $y \geq -F(x) + \alpha G(x) + k$ and strictly decreasing for $w(x) < y < -F(x) + \alpha G(x) + k$. We have also

$$\lim_{y \rightarrow +\infty} S_{\alpha,k}(x,y) = +\infty = \lim_{y \rightarrow w(x)^+} S_{\alpha,k}(x,y).$$

So for each $c > 0$ and for each x , with $G(x) < c$, there is a unique $y_1 > -F(x) + \alpha G(x) + k$ and a unique y_2 , with $w(x) < y_2 < -F(x) + \alpha G(x) + k$ such that

$$S_{\alpha,k}(x,y_1) = S_{\alpha,k}(x,y_2) = c.$$

If there exist $x_1 < 0 < x_2$ such that

$$G(x_1) = G(x_2) = c$$

then the level curve $S_{\alpha,k}(x,y) = c$ is closed and shows, in the case $F(x) \geq \alpha G(x) + k$ and $k < 0$ and $c > \int_0^{-k} \frac{s}{\alpha s + 1} ds$, the following aspect (Figure 1):

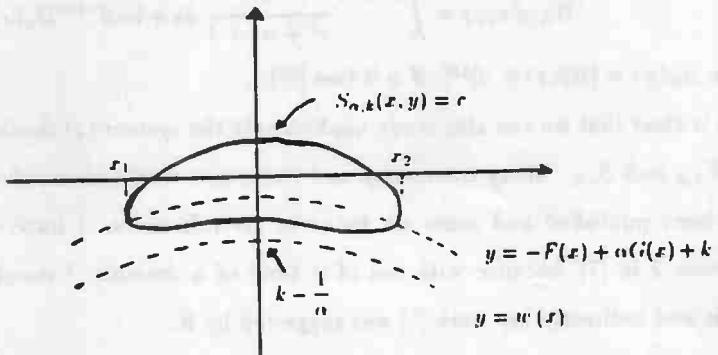


Figure 1

If $F(x) \geq \alpha G(x) + k$ for $x \geq 0$ and there exists $x_2 > 0$ such that $S_{\alpha,k}(x_2, 0) = c$ then the arc

$$S_{\alpha,k}(x, y) = c,$$

with $x > 0$ and $y > -F(x) + \alpha G(x) + k$, crosses the $x > 0$ half-axis at $(x_2, 0)$ and the set

$$\{(x, y) \in \Omega_{\alpha,k} \mid S_{\alpha,k}(x, y) = c, 0 \leq x \leq x_2\}$$

shows, in the case $k < 0$ and $S_{\alpha,k}(x, 0) < c$, for $0 < x < x_2$, the following aspect (Figure 2):

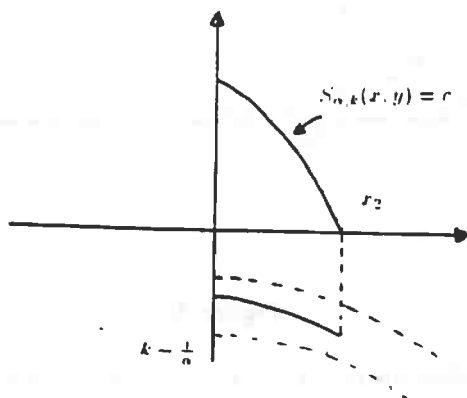


Figure 2

If $F(x) \leq \beta G(x) + k$, for $a \leq x \leq 0$, with $k - \frac{1}{\beta} < 0$, and $w(a) > 0$, then the curve

$$y = w(x) = -F(x) + \beta G(x) + k - \frac{1}{\beta}$$

crosses the $x < 0$ half-axis. Hence, for every $c > 0$, the arc

$$S_{\beta,k}(x, y) = c, \quad x < 0 \text{ and } y < -F(x) + \beta G(x) + k$$

crosses too the $x < 0$ half-axis at a point $(x_1, 0)$, with $a < x_1 < 0$ and the set

$$\{(x, y) \in \Omega_{\beta,k} \mid S_{\beta,k}(x, y) = c, x_1 \leq x \leq 0\}$$

shows, in the case $k > 0$ and $w(x) < 0$ for $x_2 < x < 0$, the following aspect (Figure 3):

If $\alpha = 0$ we have

$$S_{0,k}(x, y) = \frac{1}{2}(y + F(x) - k)^2 + G(x).$$

So, $S_{0,k}(x, y) = c$ is equivalent to

$$y = -F(x) + k + [2c - 2G(x)]^{1/2} \quad \text{or} \quad y = -F(x) + k - [2c - 2G(x)]^{1/2}.$$

The case $\alpha < 0$ can be discussed in a similar way.

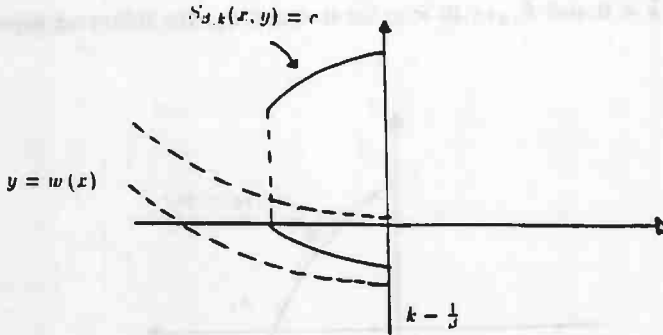


Figure 3

Lemma 2.1 Suppose there are $\alpha > 0$, $b > 0$ and $k \leq 0$ such that

$$F(x) \geq \alpha G(x) + k \quad \text{for } 0 \leq x \leq b. \quad (5)$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (2) with $\gamma(0) = (x_0, 0)$, $0 < x_0 \leq b$, and $t_1 > 0$ such that $0 \leq x(t) \leq b$ for $0 \leq t \leq t_1$. Then, for $0 \leq t \leq t_1$,

$$y(t) > -F(x(t)) + \alpha G(x(t)) + k - \frac{1}{\alpha}.$$

In particular, if $x(t_1) = 0$ then $y(t_1) > k - \frac{1}{\alpha}$.

Proof: From (5), $\dot{S}_{\alpha,k}(x, y) \leq 0$, $0 \leq x \leq b$. It follows that, for each $u \in]0, t_1[$ such that $\gamma(t) \in \Omega_{\alpha,k}$, $0 \leq t \leq u$, we have $\dot{S}_{\alpha,k}(\gamma(t)) \leq 0$ for $0 \leq t \leq u$, and therefore

$$S_{\alpha,k}(\gamma(t)) < c, \quad 0 \leq t \leq u.$$

with $c > S_{\alpha,k}(\gamma(0))$. It follows immediately that the set $\{\gamma(t) \mid 0 \leq t \leq t_1\}$ does not intercept the arc

$$S_{\alpha,k}(x, y) = C', \quad y \leq F(x) + \alpha G(x) + k.$$

Then, for $0 \leq t \leq t_1$ we have

$$y(t) > -F(x(t)) + \alpha G(x(t)) + k - \frac{1}{\alpha}.$$

(This result is intuitive: it is enough to look the Figure 2 with $x_0 < x_2$.) □

Lemma 2.2 Suppose there are $b > 0$, $\alpha \geq 0$, $R > 0$ and $k \leq 0$ such that

$$F(x) \geq \alpha G(x) + k, \quad 0 \leq x \leq b \tag{6}$$

$$F(b) \geq k + [(R - k)^2 - 2G(b)]^{1/2}. \tag{7}$$

Let $\gamma(t)$ be the solution of (2) with $\gamma(0) = (0, y_0)$, $0 < y_0 \leq R$. Then there is $t_1 > 0$ such that

$$\gamma(t_1) = (b_1, 0), \quad 0 < b_1 \leq b.$$

Moreover, if there exists $t_2 > t_1$ such that $0 \leq x(t) \leq b$ for $t_1 \leq t \leq t_2$ and $x(t_2) = 0$ then

$$y(t_2) > k - \frac{1}{\alpha} \quad \text{if } \alpha > 0 \quad \text{and} \quad y(t_2) \geq -R + 2k \quad \text{if } \alpha = 0.$$

Proof: The equation

$$S_{0,k}(x, y) = S_{0,k}(0, R) = \frac{1}{2}(R - k)^2$$

is equivalent to

$$y = -F(x) + k + [(R - k)^2 - 2G(x)]^{1/2} \tag{8}$$

or

$$y = -F(x) + k - [(R - k)^2 - 2G(x)]^{1/2}. \tag{9}$$

The condition (7) ensures the curve (8) intercepts the $x > 0$ half-axis at a point $(b_2, 0)$, $0 < b_2 \leq b$. From (6) we have $F(x) \geq k$, $0 \leq x \leq b$. So

$$S_{0,k}(x, y) \leq 0, \quad 0 \leq x \leq b. \tag{10}$$

Then the solution $\gamma(t)$ of (2) starting at the point $\gamma(0) = (0, y_0)$, $0 < y_0 \leq R$, crosses also the $x > 0$ half-axis at a point $\gamma(t_1) = (b_1, 0)$, $0 < b_1 \leq b_2 \leq b$. From (10) we have for $t_1 \leq t \leq t_2$

$$S_{0,k}(\gamma(t)) \leq S_{0,k}(\gamma(t_1)) \leq S_{0,k}(0, R).$$

Hence and from (9) we have for $t_1 \leq t \leq t_2$

$$y(t) \geq -F(x(t)) + k - [(R - k)^2 - 2G(x(t))]^{1/2}.$$

So, if $\alpha = 0$ and $x(t_2) = 0$ we have $y(t_2) \geq -R + 2k$.

From Lemma 2.1, if $\alpha > 0$,

$$y(t_2) > k - \frac{1}{\alpha}.$$

(See again Figure 2.) □

Lemma 2.3 Suppose there are $a < 0$, $\beta > 0$, $R < 0$ and $k \geq 0$ such that

$$F(x) \leq \beta G(x) + k, \quad a \leq x \leq 0. \quad (11)$$

$$F(a) \leq \beta G(a) + k - \frac{1}{\beta} \quad (12)$$

$$\text{and} \quad R > k - \frac{1}{\beta}. \quad (13)$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (2) with $\gamma(0) = (0, y_0)$, $R \leq y_0 < 0$. Then there is $t_1 > 0$ such that $\gamma(t_1) = (a_1, 0)$ with $a \leq a_1 < 0$, and $y(t) > -F(x(t)) + \beta G(x(t)) + k - \frac{1}{\beta}$, $0 \leq t \leq t_1$.

Proof: The conditions (12) and (13) ensure the curve

$$y = -F(x) + \beta G(x) + k - \frac{1}{\beta}$$

crosses the $x < 0$ half-axis at a point $(a_2, 0)$ with $a \leq a_2 < 0$. From (11)

$$\dot{S}_{\beta,k}(x, y) \leq 0, \quad a \leq x \leq 0.$$

So, the solution $\gamma(t)$ starting at $\gamma(0) = (0, y_0)$, $R \leq y_0 < 0$, can not leave the compact set

$$\left\{ (x, y) \in \Omega_{\beta, k} \mid -F(x) + \beta G(x) + k - \frac{1}{\beta} \leq y \leq 0 \quad \text{and} \quad a_2 \leq x \leq 0 \right\}$$

through the arc

$$y = -F(x) + \beta G(x) + k - \frac{1}{\beta}, \quad a_2 \leq x \leq 0.$$

Then there is $t_1 > 0$ such that the solution $\gamma(t)$ crosses the $x < 0$ half-axis at a point $\gamma(t_1) = (a_1, 0)$, $a_2 \leq a_1 < 0$, and $y(t) > -F(x(t)) + \beta G(x(t)) + k - \frac{1}{\beta}$, for $0 \leq t \leq t_1$. (See Figure 3.)

□

In a similar way we prove the following lemmas.

Lemma 2.4 Suppose there are $a < 0$, $\alpha < 0$ and $k \geq 0$ such that

$$F(x) \leq \alpha G(x) + k, \quad a \leq x \leq 0.$$

Let $\gamma(t)$ be the solution of (2) such that $\gamma(0) = (x_0, 0)$, $a \leq x_0 < 0$ and $t_1 > 0$ such that $a \leq x(t) \leq 0$ for $0 \leq t \leq t_1$. Then for $0 \leq t \leq t_1$

$$y(t) < -F(x(t)) + \alpha G(x(t)) + k - \frac{1}{\alpha}.$$

In particular, if $x(t_1) = 0$, then $y(t_1) < k - \frac{1}{\alpha}$.

Lemma 2.5 Suppose there are $a < 0$, $\alpha \leq 0$, $R < 0$ and $k \geq 0$ such that

$$F(x) \leq \alpha G(x) + k, \quad a \leq x \leq 0 \quad \text{and} \quad F(a) \leq k - [(R - k)^2 - 2G(a)]^{1/2}.$$

Let $\gamma(t)$ be the solution of (2) with $\gamma(0) = (0, y_0)$, $R \leq y_0 < 0$. Then there is $t_1 > 0$ such that

$$\gamma(t_1) = (a_1, 0), \quad a \leq a_1 < 0.$$

Moreover, if there exists $t_2 > t_1$ such that $a \leq x(t) \leq 0$ for $t_1 \leq t \leq t_2$ and $x(t_2) = 0$ then

$$\gamma(t_2) < k - \frac{1}{\alpha} \quad \text{if } \alpha > 0 \quad \text{and} \quad \gamma(t_2) \leq -R + 2k \quad \text{if } \alpha = 0.$$

Lemma 2.6 Suppose there are $b > 0$, $\beta < 0$, $R > 0$ and $k \leq 0$ such that

$$F(x) \geq \beta G(x) + k, \quad 0 \leq x \leq b.$$

$$F(b) \geq \beta G(a) + k - \frac{1}{\beta}$$

and
$$R < k - \frac{1}{\beta}.$$

Let $\gamma(t)$ be the solution of (2) with $\gamma(0) = (0, y_0)$, $0 < y_0 \leq R$. Then there is $t_1 > 0$ such that $\gamma(t_1) = (b_1, 0)$, with $0 < b_1 \leq b$.

To close the section we observe that the solutions of (2) do not admit vertical asymptotes (see [1]).

3 Sufficient conditions for nonexistence of periodic solutions

Theorem 3.1 Suppose there are $a < 0 < b$, $\alpha > 0$ and $k \leq 0$ such that

(i) $g(x)F(x) > 0$ for $a < x < b$ and $x \neq 0$;

(ii) $F(a) \leq -[(k - \frac{1}{\alpha})^2 - 2G(a)]^{1/2}$ and $F(b) \geq [(k - \frac{1}{\alpha})^2 - 2G(b)]^{1/2}$;

(iii) $F(x) \geq \alpha G(x) + k$ for $x > 0$.

Under these conditions the system (2) does not admit non-trivial periodic solution.

Proof: Consider the solution $\gamma(t) = (x(t), y(t))$ starting at $\gamma(0) = (0, y_0)$ with $k - \frac{1}{\alpha} < y_0 < 0$. Suppose there is a smaller $t_2 > 0$ such that $\gamma(t_2) = (0, y_2)$, $y_2 > 0$. From Lemma 2.5 and conditions (i) - (ii) there is $0 < t_1 < t_2$ such that $\gamma(t_1) = (x_1, 0)$, $a \leq x_1 < 0$. It follows that $a \leq x(t) \leq 0$ for $0 \leq t \leq t_2$ and $x(t_2) = 0$. From Lemma 2.5, $y_2 = y(t_2) < \frac{1}{\alpha} - k$. Suppose now there is a smaller $t_4 > t_2$ such that $\gamma(t_4) = (0, y_4)$, $y_4 < 0$. From Lemma 2.2 and conditions (i) - (ii) there is $t_2 < t_3 < t_4$ such that $\gamma(t_3) = (x_3, 0)$,

$0 < x_3 \leq b$. It follows that $0 \leq x(t) \leq b$ for $t_2 \leq t \leq t_4$ and $x(t_4) = 0$. From Lemma 2.1 and condition (iii) we have $y_4 = y(t_4) > k - \frac{1}{\alpha}$. From (i) we have

$$\dot{S}_{0,0}(\gamma(t)) < 0 \quad \text{for } 0 < t < t_4, \quad t \neq t_2.$$

So, $S_{0,0}(\gamma(0)) > S_{0,0}(\gamma(t_4))$ and therefore $\gamma(0) \neq \gamma(t_4)$. It follows that all solution starting at a point $(0, y)$ with $k - \frac{1}{\alpha} < y < 0$ is not periodic.

Consider now the solution $\gamma(t)$ with $\gamma(0) = (x_0, 0)$, $x_0 > 0$ and suppose there is $t_1 > 0$ such that $0 \leq x(t) \leq x_0$ for $0 \leq t \leq t_1$ and $x(t_1) = 0$. From Lemma 2.1 and condition (iii) we have

$$k - \frac{1}{\alpha} < y(t_1) < 0.$$

So, the system (2) does not admit non trivial periodic solution. □

We observe that Theorem 1 in [6] is a particular case of our Theorem 3.1.

Remark 3.1 It can easily be verified that the conditions (i), (ii) and (iii) in Theorem 3.1 ensure all solution starting at $(x_0, 0)$, $x_0 > 0$, approaches the origin as $t \rightarrow +\infty$.

Remark 3.2 From Lemma 2.4 it follows that the condition (iii) can be replaced by: there are $\alpha < 0$, and $k \geq 0$ such that

$$F(x) \leq \alpha G(x) + k \quad \text{for } x < 0.$$

Theorem 3.2 Suppose there are $\alpha > 0$ and $a < 0$ such that

$$(i) \quad g(x)[F(x) - \alpha G(x)] > 0 \quad \text{for } x \geq a \quad \text{and } x \neq 0;$$

$$(ii) \quad F(a) \leq \alpha G(a) - \frac{1}{\alpha}.$$

Under these conditions the system (2) does not admit non-trivial periodic solutions.

Proof: Consider the solution $\gamma(t) = (x(t), y(t))$ with $\gamma(0) = (x_0, 0)$, $x_0 > 0$, and suppose there is $t_1 > 0$ such that $\gamma(t_1) = (x_1, 0)$, $x_1 > 0$. From Lemmas 2.1 - 2.3 we have

$$y(t) > -F(x(t)) + \alpha G(x(t)) - \frac{1}{\alpha} \quad \text{and} \quad x(t) \geq a$$

for $0 \leq t \leq t_1$. Hence and from (i) it follows that for all $t \in [0, t_1]$, with $x(t) \neq 0$, $\dot{S}_{\alpha,0}(\gamma(t)) < 0$. So,

$$S_{\alpha,0}(\gamma(0)) > S_{\alpha,0}(\gamma(t_1))$$

and therefore $\gamma(0) \neq \gamma(t_1)$. It follows that the system (2) does not admit non-trivial periodic solution. □

Remark 3.3 From Lemmas 2.4 - 2.6 it follows that the condition (i) and (ii) can be replaced by: *there are $\alpha < 0$ and $b > 0$ such that*

$$g(x)[F(x) - \alpha G(x)] > 0 \quad \text{for } x \leq b \text{ and } x \neq 0 \quad \text{and} \quad F(b) \geq \alpha G(b) - \frac{1}{\alpha}.$$

Remark 3.4 It can be immediately verified that the conditions (i) and (ii) can be replaced by: *there is $\alpha \in \mathbb{R}$ such that*

$$g(x)[F(x) - \alpha G(x)] > 0 \quad \text{for } x \neq 0.$$

Example 3.1 The equation

$$\ddot{x} + (x^5 - x^4 + 3x^2 + 2x)\dot{x} + x = 0$$

does not admit non-trivial periodic solution.

Solution: $F(x) = \frac{x^6}{6} - \frac{x^5}{5} + x^3 + x^2$ and $G(x) = \frac{x^2}{2}$.

For $x \geq -1$ and $x \neq 0$ we have $[F(x) - \alpha G(x)]g(x) > 0$, with $\alpha = 2$. By other hand,

$$F(-1) < \alpha G(-1) - \frac{1}{\alpha}.$$

From Theorem 3.2 the equation does not admit non-trivial periodic solution.

We observe, in the example above, the Theorem 3.1 can not be applied because $F(x) > 0$ for $x < 0$. Also, the theorem 2.1 in [7] and theorem 1 in [8] can not be applied because

there are $x_1 > 0$ and $x_2 > 0$ such that $F_e(x_1) > \#$ and $F_e(x_2) < 0$, where $F_e(x) = \int_0^1 f_e(s) ds$ and $f_e(x) = -x^4 + 3x^2$.

Example 3.2 The equation

$$\ddot{x} + (x^3 + 6x^2)\dot{x} + \frac{2x}{(2 + 2x + x^2)^2} = 0$$

does not admit non-trivial periodic solution.

Solution: $F(x) = \frac{x^4}{4} + 2x^3$ and $G(x) = \frac{(x+1)^2}{1+(x+1)^2} - \frac{1}{2} - 2 \int_1^{x+1} \frac{1}{(1+u^2)^2} du$

For $\alpha = 1$ and $k = -1$ we have

$$F(x) \geq \alpha G(x) + k \text{ for } x \geq 0 \quad \text{and} \quad F(-1) < - \left[\left(k - \frac{1}{\alpha} \right)^2 - 2G(-1) \right]^{1/2}.$$

We have also

$$g(x)F(x) > 0 \quad \text{for } x \geq -1 \text{ and } x \neq 0.$$

From Theorem 3.1 the equation does not admit non-trivial periodic solution. (Here the condition $F(b) \geq \left[\left(k - \frac{1}{\alpha} \right)^2 - 2G(b) \right]^{1/2}$ is not necessary because $g(x)F(x) > 0$ for all $x \geq -1$ and $x \neq 0$.) The theorems in [7, 8] can not be applied because $g(x)$ is not odd.

4 Sufficient conditions for the origin to be globally asymptotically stable

Theorem 4.1 Suppose the following conditions are verified:

(i) There is $\alpha \in \mathbb{R}$ such that

$$g(x)[F(x) - \alpha G(x)] > 0 \quad \text{for } x \neq 0;$$

(ii) There are $k \leq 0$ and $k_1 \geq 0$ such that $F(x) \geq k$ for $x > 0$ and $F(x) \leq k_1$ for $x < 0$.

(iii) For all $R > 0$ there are $m < 0 < n$ such that

$$F(n) \geq k + [(R - k)^2 - 2G(n)]^{1/2} \quad \text{and} \quad F(m) \leq k_1 - [(R - k_1)^2 - 2G(m)]^{1/2}.$$

Under these conditions the origin is globally asymptotically stable in Liapunov sense.

Proof: Consider the arcs

$$S_{\alpha,0}(x, y) = c \text{ with } y \geq -F(x) + \alpha G(x) \quad (14)$$

$$S_{\alpha,0}(x, y) = c \text{ with } y \leq -F(x) + \alpha G(x) \quad (15)$$

From hypotheses (i) - (iii) the arc (14) intercepts the $x > 0$ half-axis at $(x_1, 0)$, $x_1 > 0$, and (15) crosses the $x < 0$ half-axis at $(x_2, 0)$, $x_2 < 0$. Let K_c be, $c > 0$, the compact set bounded by the arcs (14), (15) and by the lines $x = x_1$ and $x = x_2$. From (i) we have

$$\dot{S}_{\alpha,0}(x, y) < 0 \quad \text{for } x \neq 0. \quad (16)$$

So, K_c is an invariant set for the system (2). Then the condition (16), by La Salle Theorem, ensures that the origin is asymptotically stable and every solution starting at a point in K_c approaches the origin as $t \rightarrow +\infty$. It follows that every solution starting at a point in $\Omega_{\alpha,0}$ approaches the origin as $t \rightarrow \infty$. From Lemmas 2.2 and 2.5 and conditions (ii) - (iii) for every solution $\gamma(t)$ of (2) there is t_1 such that $\gamma(t_1) \in \Omega_{\alpha,0}$. So all solution of (2) approaches the origin as $t \rightarrow +\infty$. Therefore the origin is globally asymptotically stable. □

Remark 4.1 The condition (i) can be replaced by: *There are $\alpha > 0$ ($\alpha < 0$), $a < 0$ ($b > 0$) such that*

$$g(x)[F(x) - \alpha G(x)] > 0 \quad \text{for } x > a \text{ (} x < b \text{) and } x \neq 0$$

and

$$F(a) \leq \alpha G(a) - \frac{1}{\alpha} \quad \left(F(b) \geq \alpha G(b) - \frac{1}{\alpha} \right).$$

Remark 4.2 The condition (i) can be replaced by: *There are $a < 0 < b$, $\alpha > 0$ ($\alpha < 0$), $k \leq 0$ ($k \geq 0$) such that*

$$g(x)F(x) > 0 \quad \text{for } a < x < b \text{ and } x \neq 0;$$

$$F(x) \geq \alpha G(x) + k \text{ for } x > 0 \quad (F(x) \leq \alpha G(x) + k \text{ for } x < 0) ;$$

$$F(a) \leq - \left[\left(k - \frac{1}{\alpha} \right)^2 - 2G(a) \right]^{1/2}$$

$$\text{and} \quad F(b) \geq \left[\left(k - \frac{1}{\alpha} \right)^2 - 2G(b) \right]^{1/2} .$$

Remark 4.3 Suppose that in Theorem 4.1 the following condition is also verified:

(iii) There are $\alpha_1 > 0$, $\alpha_2 < 0$ and $r > 0$ such that

$$F(x) < \alpha_1 G(x) \text{ for } 0 < x < r \quad \text{and} \quad F(x) > \alpha_2 G(x) \text{ for } -r < x < 0 .$$

In this case every non-trivial solution approaches the origin, as $t \rightarrow \infty$, in spiral.

We observe that the condition $x F(x) < 0$ for $0 < |x| < r$ appearing in Theorem 2 in [2] can be replaced by (iii).

Remark 4.4 The condition (iii) is equivalent to the conditions 4.2 and 4.3 appearing in [2].

Example 4.1 For the equation

$$\ddot{x} + (x^4 + 7x^3 + 2x^2 + x)\dot{x} + 5x^3 + x^2 + x = 0$$

the origin is globally asymptotically stable and, for every non-trivial solution $x = x(t)$,

$\gamma(t) = (x(t), \dot{x}(t))$ approaches the origin, in spiral, as $t \rightarrow +\infty$.

$$\text{Solution: } F(x) = \frac{x^5}{5} + \frac{7}{4}x^4 + \frac{2}{3}x^3 + \frac{x^2}{2} \text{ and } G(x) = \frac{5}{4}x^4 + \frac{x^3}{3} + \frac{x^2}{2} .$$

We have, for $\alpha = 1$,

$$g(x)[F(x) - \alpha G(x)] > 0 \quad \text{for } x \neq 0$$

and

$$\lim_{x \rightarrow +\infty} F(x) = +\infty \quad \text{and} \quad \lim_{x \rightarrow -\infty} F(x) = -\infty .$$

We have also there is $r > 0$ such that

$$F(x) < 2G(x) \quad \text{for } 0 < x < r \quad \text{and} \quad F(x) > 0 \quad \text{for } -r < x < 0.$$

The conclusion follows from Theorem 4.1 and Remark 4.3.

Example 4.2 Consider again the equation of the Example 3.2:

$$\ddot{x} + (x^3 + 6x^2)\dot{x} + \frac{2x}{(2 + 2x + x^2)^2} = 0.$$

It can be easily verified that the origin is asymptotically stable and, for every non-trivial solution $x = x(t)$ with $\dot{x}(0) = 0$, $\gamma(t) = (x(t), \dot{x}(t))$ approaches the origin as $t \rightarrow +\infty$.

But the origin is not globally asymptotically stable because there is $a < 0$ such that

$$g(x)[F(x) - G(x)] < 0 \quad \text{for } x \leq a$$

and so the solution $\gamma(t) = (x(t), \dot{x}(t))$, $t \geq 0$, starting at (x_0, y_0) with $x_0 \leq a$ and $y_0 < -F(x_0) + G(x_0) - 1$, does not cross the curve $y = -F(x) + G(x) - 1$.

5 Sufficient conditions for existence of periodic solutions

Theorem 5.1 Suppose that

(i) the origin is repulsive.

Suppose also that there are $\alpha > 0$, $k \leq 0$, $k_1 \geq 0$, and $a < 0 < b$ such that:

$$(ii) \quad F(x) \leq k_1 \quad \text{for } a \leq x \leq 0 \quad \text{and} \quad F(a) \leq k_1 - \left[\left(k - \frac{1}{\alpha} - k_1 \right)^2 - 2G(a) \right]^{1/2};$$

$$(iii) \quad F(x) \geq \alpha G(x) + k \quad \text{for } 0 \leq x \leq b \quad \text{and} \quad F(b) \geq k + \left[\left(\frac{1}{\alpha} + 2k_1 - 2k \right)^2 - 2G(b) \right]^{1/2}.$$

Under these conditions the system (2) admits at least one non-trivial periodic solution located between the lines $x = a$ and $x = b$.

Proof: From Lemma 2.5 and hypotheses (i), (ii), the solution starting at the point $(0, k - \frac{1}{\alpha})$ crosses the $y > 0$ half-axis at $(0, y_1)$ with $0 < y_1 \leq -k + \frac{1}{\alpha} + 2k_1$. From Lemma 2.2 and hypotheses (i), (iii), the solution starting at a point $(0, y_1)$, with $0 < y_1 \leq -k - \frac{1}{\alpha} + 2k_1$ crosses the $y < 0$ half-axis at a point $(0, y_2)$ with $k - \frac{1}{\alpha} < y_2 < 0$. From the Theorem of Poincaré-Bendixon the system (2) admits at least one non-trivial periodic solution. It is clear that this periodic solution is located between the lines $x = a$ and $x = b$. □

We observe that the Theorem 3 in [2] is a particular case of the Theorem 5.1.

Remark 5.1 If there are $\alpha \in \mathbb{R}$ and $r > 0$ such that

$$g(x)[F(x) - \alpha G(x)] < 0 \quad \text{for } x < |x| < r$$

then the origin is repulsive. It is enough to observe that the above condition implies

$$\dot{S}_{\alpha,0}(x, y) > 0 \quad \text{for } 0 < |x| < r$$

and for $c > 0$ sufficiently small the level curve $S_{\alpha,0}(x, y) = c$ is closed.

Corollary 5.1 *Suppose that*

(i) *the origin is repulsive.*

Suppose also that there are $\alpha > 0$, $k \leq 0$ and $a < 0 < b$ such that:

(ii) $F(x) \leq F(a)$ for $a \leq x \leq 0$ and $G(a) \geq \frac{1}{2} \left(k - \frac{1}{\alpha} - F(a) \right)^2$;

(iii) $F(x) \geq \alpha G(x) + k$ for $0 \leq x \leq b$ and $F(b) \geq k + \left[\left(\frac{1}{\alpha} + 2F(a) - 2k \right)^2 - 2G(b) \right]^{1/2}$.

Under these conditions the system (2) admits at least one non-trivial periodic solution located between the lines $x = a$ and $x = b$.

Proof: From (ii) there is $a \leq a_1 < 0$ such that $F(x) \leq F(a)$ for $a_1 \leq x \leq 0$ and $2G(a_1) = \left(k - \frac{1}{\alpha} - F(a) \right)^2$. Now, it is enough to make $k_1 = F(a)$ in Theorem 5.1. □

Theorem 5.2 Suppose that

(i) the origin is repulsive.

Suppose also that there are $\alpha > 0$, $\beta > 0$, $a < 0 < b$, $k \leq 0$ and $k_1 \geq 0$ such that:

(ii) $F(x) \leq \beta G(x) + k_1$, $a \leq x \leq 0$, $F(a) \leq \beta G(a) + k_1 - \frac{1}{\beta}$ and $k - \frac{1}{\alpha} > k_1 - \frac{1}{\beta}$;

(iii) $F(x) \geq \alpha G(x) + k$, $0 \leq x \leq b$ and $F(b) \geq k + [(R - k)^2 - 2G(b)]^{1/2}$

where $R = -k + \frac{1}{\alpha} + 2[\beta G(a) + k_1]$.

Under these conditions the system (2) admits at least one non-trivial periodic solution located between the lines $x = a$ and $x = b$.

Proof: From (ii) we have

$$F(x) \leq \beta G(x) + k_1, \quad a \leq x \leq 0.$$

From hypotheses and Lemmas 2.2, 2.3 and 2.5 the solution starting at $(0, k - \frac{1}{\alpha})$ crosses again the $y < 0$ half-axis at $(0, y_1)$ with $k - \frac{1}{\alpha} < y_1 < 0$. From theorem of Poincaré-Bendixon the system (2) admits at least one non-trivial periodic solution. This solution is evidently located between the lines $x = a$ and $x = b$.

□

Remark 5.2 It can be immediately verified that the condition

$$F(a) \leq \beta G(a) + k_1 - \frac{1}{\beta}$$

can be replaced by

$$S_{\beta, k_1} \left(0, k - \frac{1}{\alpha} \right) \leq S_{\beta, k_1}(a, 0).$$

In a similar way it can be proved the following theorems.

Theorem 5.3 Suppose that

(i) the origin is repulsive.

Suppose also that there are $\alpha < 0$, $k \geq 0$, $k_1 \leq 0$ and $a < 0 < b$ such that:

$$(ii) F(x) \geq k_1 \text{ for } 0 \leq x \leq b \text{ and } F(b) \geq k_1 + \left[\left(k - \frac{1}{\alpha} - k_1 \right) - 2G(b) \right]^{1/2};$$

$$(iii) F(x) \leq \alpha G(x) + k, a \leq x \leq 0 \text{ and } F(a) \leq k - \left[\left(-\frac{1}{\alpha} - 2k_1 + 2k \right)^2 - 2G(a) \right]^{1/2}.$$

Under these conditions the system (2) admits at least one non-trivial periodic solution located between the lines $x = a$ and $x = b$.

Corollary 5.2 Suppose that

(i) the origin is repulsive.

Suppose also that there are $\alpha < 0$, $k \geq 0$ and $a < 0 < b$ such that:

$$(ii) F(x) \geq F(b) \text{ for } 0 \leq x \leq b \text{ and } G(b) \geq \frac{1}{2} \left(k - \frac{1}{\alpha} - F(b) \right)^2;$$

$$(iii) F(x) \leq \alpha G(x) + k \text{ for } a \leq x \leq 0 \text{ and } F(a) \leq k - \left[\left(-\frac{1}{\alpha} - 2F(b) + 2k \right)^2 - 2G(a) \right]^{1/2}.$$

Under these conditions the system (2) admits at least one non-trivial periodic solution located between the lines $x = a$ and $x = b$.

Theorem 5.4 Suppose that

(i) the origin is repulsive.

Suppose also that there are $\alpha < 0$, $\beta < 0$, $a < 0 < b$, $k \geq 0$ and $k_1 \leq 0$ such that:

$$(ii) F(x) \geq \beta G(x) + k_1 \text{ for } 0 \leq x \leq b \text{ and } F(b) \geq \beta G(b) + k_1 - \frac{1}{\beta} \text{ and } k - \frac{1}{\alpha} < k_1 - \frac{1}{\beta};$$

$$(iii) F(x) \leq \alpha G(x) + k \text{ for } a \leq x \leq 0 \text{ and } F(a) \leq k - \left[(R - k)^2 - 2G(a) \right]^{1/2}$$

$$\text{where } R = -k + \frac{1}{\alpha} + 2[\beta G(b) + k_1].$$

Under these conditions the system (2) admits at least one non-trivial periodic solution located between the lines $x = a$ and $x = b$.

Remark 5.4 The condition

$$F(b) \geq \beta G(b) + k_1 - \frac{1}{\beta}$$

in Theorem 5.4 can be replaced by

$$S_{\beta,k} \left(0, k - \frac{1}{\alpha} \right) \leq S_{\beta,k}(b, 0).$$

Example 5.1 The equation

$$\ddot{x} + \left[x^3 + 4x^2 + 3x - \frac{1}{12} \right] \dot{x} + x = 0$$

admits at least one non-trivial periodic solution.

Solution: $F(x) = \frac{x^4}{4} + \frac{4x^3}{3} + \frac{3x^2}{2} - \frac{x}{12}$ and $G(x) = \frac{x^2}{2}$.

We have

$$F(x) \leq F(-1), \quad -1 \leq x \leq 0,$$

$$F(x) > \alpha G(x) + k \quad \text{for all } x > 0$$

where $\alpha = 3$ and $k = -\frac{1}{12}$, and

$$G(-1) > \frac{1}{2} \left[k - \frac{1}{\alpha} - F(-1) \right]^2.$$

From $\lim_{x \rightarrow +\infty} F(x) = +\infty$ it follows that there is $b > 0$ such that

$$F(b) \geq k + \left[\left(\frac{1}{\alpha} + 2F(-1) - 2k \right)^2 - 2G(b) \right]^{1/2}.$$

The origin is repulsive because there is $r > 0$ such that

$$g(x)[F(x) - G(x)] < 0 \quad \text{for } 0 < |x| < r.$$

From Corollary 5.1 the equation admits at least one non-trivial periodic solution located between the lines $x = -1$ and $x = b$, with $b = \frac{1}{\alpha} + 2F(-1) - 2k$.

Example 5.2 The equation

$$\ddot{x} + (x^3 + 4x^2 - 1)\dot{x} + 4x^3 = 0$$

admits at least one non-trivial periodic solution located between the lines $x = -2$ and $x = 2$.

Solution: $F(x) = \frac{x^4}{4} + \frac{4x^3}{3} - x$ and $G(x) = x^4$.

For $\beta = \frac{1}{4}$ and $k_1 = 1$ we have

$$F(x) \leq \beta G(x) + k_1, \quad -2 \leq x \leq 0 \quad \text{and} \quad F(-2) \leq \beta G(-2) + k_1 - \frac{1}{\beta}.$$

For $\alpha = \frac{1}{2}$ and $k = -\frac{2}{3}$ we have

$$F(x) \geq \alpha G(x) + k, \quad 0 \leq x \leq 2 \quad \text{and} \quad k - \frac{1}{\alpha} > k_1 - \frac{1}{\beta}.$$

We have also

$$F(2) > k + [(R - k)^2 - 2G(2)]^{1/2}$$

where $R = -k + \frac{1}{\alpha} + 2[\beta G(-2) + k_1]$.

By other hand the origin is evidently repulsive. From Theorem 5.2 the equation admits at least one non-trivial periodic solution located between the lines $x = -2$ and $x = 2$.

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