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A residual inaccuracy measure for coherent systems under nonhomogeneous Poisson processes

by

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Abstract
Inaccuracy and information measures based on cumulative residual entropy are useful in various fields and attracting attention in Statistics, Probability Theory and, particularly, in Reliability Theory. Using a point process martingale approach and a compensator version of Taneja and Kumar generalized inaccuracy measure of two nonnegative continuous random variables we define an inaccuracy measure between two coherent systems whose components are subject to failures according to a double stochastic Poisson processes.

Keywords: Joint signature point process; cumulative residual inaccuracy measure; non homogeneous Poisson process; minimal repair; coherent system.

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1 Introduction

An alternate measure of entropy based on distribution function rather than the density function of a random variable, called cumulative residual entropy (CRE), was proposed in Rao et al. (2004) which has been extended to cumulative residual inaccuracy measure by Taneja and Kumar (2012). Observing the common component lifetimes of two coherent systems and using a point process martingale approach [8] extended the definition to a symmetric inaccuracy measure. In this framework we define an inaccuracy measure between two coherent systems whose components are subject to failures according to a double stochastic Poisson processes.

The main inaccuracy measure for the uncertainty of two positive and absolutely continuous random variables, S and T, defined in a complete probability space $(\Omega, \Im, P)$ is the Kerridge (1961) inaccuracy measure, given as

$$H(S,T) = E[-\log g(T)] = -\int_0^\infty \log g(x) f(x) dx,$$

where f and g are probability density functions of T and S, respectively.
In the case when S and T are identically distributed, the Kerridge inaccuracy measure gives the well-known Shanon (1948) entropy defined as

$$H(T) = E[-\log f(T)] = -\int_0^\infty \log f(x) f(x) dx.$$

Rao et al. (2004) and Rao (2005) provided an extension by using survival functions of S and T instead of probability density functions. By a similar way the Kerridge measure of inaccuracy can be extended by using survival functions as in Kumar and Taneja (2015) and Kundu et al. (2016).
The Taneja and Kumar cumulative residual inaccuracy measure between S and T is defined as

$$\varepsilon(S,T) = -\int_0^\infty \overline{F}(t) \log \overline{G}(t) dt == E[\int_0^T \bigwedge_S (s) ds].$$

where $\overline{F} = 1 - F$, $\overline{G} = 1 - G$ are the reliability functions of T and S, respectively, F, G their distribution functions and $\bigwedge_S(t) = -\log \overline{G}(t)$ is the lifetime S hazard function. It is important to note that the expression makes sense in the set $\{t > S \wedge T\}$ where $S \wedge T = \min\{S, T\}$ and we set, by convention, $0 \log 0 = 0$.
Indeed, $\varepsilon(S, T)$ represents the information content when using $\overline{G}(t)$, the survival function asserted by the experimenter, due to missing/incorrect information, instead of the true survival function $\overline{F}(t)$. Some transformation of the measure can be seen at Psarrakos and Di Crecenzo (2018).

Bueno (2019) observes two component lifetimes T and S, which are finite positive absolutely continuous random variables defined in a complete probability space $(\Omega, \Im, P)$, with $P(S \neq T) = 1$, through the family of sub σ-algebras $(\Im_t)_{t\geq 0}$ of $\Im$, where

$$\Im_t = \sigma\{1_{\{S>s\}}, 1_{\{T>s\}}, 0 \leq s < t\}$$

satisfies Dellacherie's conditions of right continuity and completeness. Considers, through the Doob Meyer decomposition, the unique predictable compensator processes $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ such that $1_{\{T\leq t\}} - A_t$ and $1_{\{S\leq t\}} - B_t$ are 0 means $\Im_t$-martingales.
The compensator process is expressed in terms of conditional probabilities given the available information and generalizes the classical notion of hazard. Intuitively corresponds to produce whether the failure is going to occur now, on the basis of all observations available up to, but not including the present.
Follows, by the well-known equivalence results between distribution functions and compensator processes, see Arjas and Yashin (1988), that $A_t = -\log \overline{F}(t \bigwedge T)$ and $B_t = -\log \overline{G}(t \bigwedge S)$. Identifying $\bigwedge_S(t)$ and B_t, in the set $\{S > t\}$ Bueno proves that

$$\varepsilon(S, T) = E[\int_0^T B_s ds] = E[1_{\{S\leq T\}}|T - S|].$$

Also, using the same arguments as above we have

$$\varepsilon(T, S) = E[\int_0^S A_s ds] = E[1_{\{T\leq S\}}(S - T)] = E[1_{\{T\leq S\}}|S - T|].$$

Bueno considers the following definition which is a symmetric generalization of the Taneja and Kumar inaccuracy measure:

Definition 1.1
If S and T are continuous positive random variables defined in a complete probability space $(\Omega, \Im, P)$, we define the cumulative residual inaccuracy measure as

$$CRI_{S,T} = CRI_{T,S} = \varepsilon(S, T) + \varepsilon(T, S) = E[\int_0^T B_s ds] + E[\int_0^S A_s ds] =$$

$$E[1_{\{S\leq T\}}|T - S|] + E[1_{\{T\leq S\}}|S - T|] = E[|T - S|].$$

$CRI_{T,S}$ can be seen as a dispersion measure when using a lifetime S asserted by the experimenter information of the true lifetime T. Provide that we identify random variables that are equal almost everywhere, $CRI_{S,T}$ is a metric in the L^1 space of random variables.

In Section 2 of this article we define a joint signature point process of two coherent systems, a subject important by it self, which will be useful to calculate the residual cumulative inaccuracy measure between coherent systems. We also gives reasons to asymptotically extend the inaccuracy measure for a double stochastic Poisson processes. In section 3 we define the cumulative inaccuracy measure for coherent system for coherent systems under a double stochastic Poisson processes and specialize in nonhomogeneous Poisson processes to model a minimal repair coherent system and give some examples calculating the inaccuracy measure between minimal repair point processes.

2. A joint signature point process and asymptotic reliability

2.1. A joint signature point process

In our general setup we consider coherent systems lifetimes T and S with component lifetimes $T_1, ..., T_n$ and $S_1, ..., S_m$, respectively, see Barlow and Proschan (1981), which are finite and positive random variables defined in a complete probability space $(\Omega, \Im, P)$. We assume that $P(T_i = T_j) = P(S_i = S_j) = 0$ for all $1 \leq i \leq n, \quad 1 \leq j \leq m$ and , also, that relations between random variables and measurable sets, respectively, always hold with probability one, which means that the term P-a.s., is suppressed. Therefore the lifetimes can be dependent but simultaneous failures are ruled out.

The mathematical description of our observations, the complete information level, is given by a family of sub σ-algebras of $\Im$, denoted by $(\Im_t)_{t\geq 0}$, where

$$\Im_t = \sigma\{1_{\{T_{(i)}>s\}}, 1_{\{S_{(j)}>s\}}, 1 \leq i, j \leq n, 0 < s < t\},$$

satisfies the Dellacherie conditions of right continuity and completeness. Intuitively, at each time t the observer knows if the event $\{T_{(i)} \leq t\}, (\{S(j) \leq t\})$ have either occurred or not and if it had, he knows exactly the value $T_{(i)}(S_{(j)})$.

For a mathematical basis of stochastic processes applied to reliability theory see the book of Aven and Jensen (1999) and Bremaud (1981). In particular, an extended and positive random variable τ is an $\Im_t$-stopping time if, and only if, $\{\tau \leq t\} \in \Im_t$, for all $t \geq 0$; an $\Im_t$-stopping time τ is called predictable if an increasing sequence $(\tau_n)_{n\geq 0}$ of $\Im_t$-stopping time, $\tau_n < \tau$, exists such that $\lim_{n\to\infty} \tau_n = \tau$; an $\Im_t$-stopping time τ is totally inaccessible if $P(\tau = \sigma < \infty) = 0$ for all predictable $\Im_t$-stopping time σ.
Follows that components and system lifetimes are $\Im_t$ stopping times. We assume that $T_1, ..T_n, S_1, ..., S_m$ are totally inaccessible $\Im_t$-stopping times. In a certain way, absolutely continuous lifetimes are totally inaccessible $\Im_t$-stopping time.

The evolution of components on time define point processes given through the failure times: we denote by $T_{(1)} < T_{(2)} < ... < T_{(n)}$ $(S_{(1)} < S_{(2)}, ... < S_{(m)})$ the ordered lifetimes $T_1, T_2, ..., T_n$ $(S_1, S_2, , ..., S_m)$ as they appear in time. As a convention we define $T_{(n+1)} =$

$T_{(n+2)} = ... = S_{(m+1)} = S_{(m+2)} = ... = \infty$ indicating that the sequences $(T_{(n)})_{n\geq 1}$ and as in Bremaud, $(S_{(m)})_{m\geq 1}$ define non explosives point processes.

Under the above hypothesis, it is well-known that a coherent system fails at failure of one of their components. This fact motivates the following Theorem:

Theorem 2.1.1
Under the above hypothesis and notation we have:

$$P(T \leq t, S \leq s|\Im_{t\vee s}) = \sum_{i=1}^{n}\sum_{j=1}^{m} 1_{\{T=T_{(k)},S=S_{(j)}\}} 1_{\{T_{(k)}\leq t, S_{(j)}\leq s\}}.$$

Proof
As $\{T = T_{(k)}, S = S_{(j)}\}$ defines a partition of Ω follows from the total probability law that

$$P(T \leq t, S \leq s|\Im_{t\vee s}) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(T \leq t, S \leq s, T = T_{(k)}, S = S_{(j)}|\Im_{t\vee s}) =$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} E[[1_{\{T_{(k)}\leq t, S_{(j)}\leq s\}} 1_{\{T=T_{(k)},S=S_{(j)}\}}|\Im_{t\vee s}].$$

However, see Dellacherie (1972),

$$\{T = T_{(k)}\} \in \Im_{T_{(k)}} = \{A \in \Im_{\infty}\{A \cap \{T_{(k)} \leq t\}\} \in \Im_t, \forall t > 0\}$$

$$\{S = S_{(j)}\} \in \Im_{S_{(j)}} = \{A \in \Im_{\infty}\{A \cap \{S_{(j)} \leq s\}\} \in \Im_s, \forall s > 0\}$$

and we have

$$\{T = T_{(k)}\} \cap \{T_{(k)} \leq t\} \in \Im_t \subseteq \Im_{t\vee s}$$

and

$$\{S = S_{(j)}\} \cap \{S_{(j)} \leq s\} \in \Im_s \subseteq \Im_{t\vee s}.$$

Therefore

$$\{T = T_{(k)}\} \cap \{T_{(k)} \leq t\} \cap \{S = S_{(j)}\} \cap \{S_{(j)} \leq s\} \in \Im_{t\vee s}$$

is $\Im_{t\vee s}$- measurable set implying

$$P(T \leq t, S \leq s|\Im_{t\vee s}) = \sum_{i=1}^{n}\sum_{j=1}^{m} 1_{\{T_{(k)}\leq t, S_{(j)}\leq s\}} 1_{\{T=T_{(k)},S_{(j)}=S\}}.$$

The above decomposition allows us to define the joint signature point process as:

Definition 2.1.2
The vector $(1_{\{T=T_{(k)},S=S_{(j)}\}}, 1 \leq j \leq n, 1 \leq k \leq m)$ is defined as the joint signature point process of the bivariate lifetime (T, S).

Remark 2.1.3
As $P(T_i = T_j) = P(S_i = S_j) = 0$ for all $1 \leq i \leq n, \quad 1 \leq j \leq m$, the collection $\{\{T = T_{(i)}\}, 1 \leq i \leq n\}$ is a partition of Ω and $\sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} = 1$. Also, the collection $\{\{S = S_{(j)}\}, 1 \leq j \leq m\}$ is a partition of Ω with $\sum_{j=1}^{m} 1_{\{S=S_{(j)}\}} = 1$. Therefore

$$\sum_{j=1}^{m} 1_{\{T=T_{(k)},S=S_{(j)}\}} = \sum_{j=1}^{m} 1_{\{T=T_{(k)}\}} 1_{\{S=S_{(j)}\}} = 1_{\{T=T_{(k)}\}}.$$

As in Bueno (2013), the vector $(1_{\{T=T_{(k)}\}}, 1 \leq k \leq n)$ is defined as the marginal signature point process of the coherent system with lifetime T. We also have

$$P(T \leq t|\Im_t) = \sum_{i=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)} \leq t\}}.$$

The joint conditional reliability function of (S, T) defined as $P(T > t, S > s|\Im_{t\vee s})$ is:

Theorem 2.1.4
Under the above hypothesis and notation we have:

$$P(T > t, S > s|\Im_{t\vee s}) = \sum_{i=1}^{m} \sum_{j=1}^{n} 1_{\{T=T_{(k)},S=S_{(j)}\}} 1_{\{T_{(k)}>t,S_{(j)}>s\}}.$$

Proof
Observe the equality

$$1\{T > t, S > s\} = 1 - 1\{T \leq t\} - 1\{S \leq s\} + 1\{T \leq t, S \leq s\}.$$

Therefore

$$E[1\{T > t, S > s\}|\Im_{t\vee s}] = 1 - E[1\{T \leq t\}|\Im_{t\vee s}] -$$
$$E[1\{S \leq s\}|\Im_{t\vee s}] + E[1\{T \leq t, S \leq s\}|\Im_{t\vee s}].$$

As $\{T = T_{(k)}\} \in \Im_{T_{(k)}}$, $\{T = T_{(k)}\} \cap \{T_{(k)} \leq t\} \in \Im_t \subseteq \Im_{t\vee s}, \forall t > 0$ and we have

$$E[1\{T \leq t\}|\Im_{t\vee s}] = E[\sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)} \leq t\}}] =$$

$$E[\sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} (\sum_{j=1}^{m} 1_{\{S=S_{(j)}\}}) 1_{\{T_{(k)} \leq t\}}] =$$

$$E[\sum_{k=1}^{n} \sum_{j=1}^{m} 1_{\{T=T_{(k)},S=S_{(j)}\}} 1_{\{T_{(k)} \leq t\}}].$$

With the same argument we get

$$E[1\{S \leq s\}|\Im_{t\vee s}] = E[\sum_{k=1}^{n} \sum_{j=1}^{m} 1_{\{T=T_{(k)},S=S_{(j)}\}} 1_{\{S_{(j)} \leq s\}}].$$

Using Theorem 2.1.1 we have

$$E[1\{T>t,S>s\}|\Im_{t\vee s}]=E[\sum_{k=1}^{n}\sum_{j=1}^{n}1_{\{T=T_{(k)},S=S_{(j)}\}}-$$

$$\sum_{k=1}^{n}\sum_{j=1}^{m}1_{\{T=T_{(k)},S=S_{(j)}\}}1_{\{T_{(k)}\leq t\}}-\sum_{k=1}^{n}\sum_{j=1}^{m}1_{\{T=T_{(k)},S=S_{(j)}\}}1_{\{S_{(j)}\leq s\}}+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}1_{\{T_{(k)}\leq t,S_{(j)}\leq s\}}1_{\{T=T_{(k)},S_{(j)}=S\}}]=$$

$$E[\sum_{k=1}^{n}\sum_{j=1}^{m}1_{\{T=T_{(k)},S=S_{(j)}\}}\{1-1_{\{T_{(k)}\leq t\}}-1_{\{S_{(j)}\leq s\}}+1_{\{T_{(k)}\leq t,S_{(j)}\leq s\}}\}]=$$

$$E[\sum_{k=1}^{n}\sum_{j=1}^{m}1_{\{T=T_{(k)},S=S_{(j)}\}}1_{\{T_{(k)}>t,S_{(j)}>s\}}].$$

Remark 2.1.5
Using Theorem 2.1.4 we can calculate the "systems joint reliability" as

$$P(T>t,S>s)=E[P(T>t,S>s|\Im_{t\vee s})]=E[\sum_{k=1}^{n}\sum_{j=1}^{m}1_{\{T=T_{(k)},S=S_{(j)}\}}1_{\{T_{(k)}>t,S_{(j)}>s\}}]=$$

$$\sum_{k=1}^{n}\sum_{j=1}^{m}P(\{T=T_{(k)},S=S_{(j)}\}\cap\{T_{(k)}>t,S_{(j)}>s\})$$

.

As in Randles and Wolfe (1979), if the components $T_1,T_2,...,T_n,S_1,S_2,...,S_n$ are independent and identically distributed with continuous distribution F, the events $\{T=T_{(k)},S=S_{(j)}\}$ and $\{T_{(k)}\leq t,S_{(j)}\leq s\}$ are independent and we have

$$P(T>t,S>s)=\sum_{k=1}^{n}\sum_{j=1}^{n}P(T=T_{(k)},S=S_{(j)})P(T_{(k)}>t,S_{(j)}>s).$$

Remark 2.1.6
Navarro, et al.(2013), consider coherent systems with shared common independent and identically distributed component lifetimes $T_1, ..., T_n$ with continuous distribution function F to define the "bivariate signature matrix".
For the sake of clarity, one can think of the two system of interest as being on n_1 and n_2 components and having $n_{1,2}$ components in common, reaching over $n = n_1 + n_2 - n_{1,2}$ components. If $n_{1,2} = 0$ the lifetimes of the two systems are independent and the joint distribution of their lifetimes is simply the product of their marginal distributions. It is also considered the case in which these systems are based just on some of these component lifetimes and not on all of them. So these two system might share all, some, or none of these components.

Navarro et al. define the the random vector $\mathbf{I} = (I_1, I_2)$ by

$$\mathbf{I} = (k, j) \ \ whenever \ \ T = T_{(k)} \ \ and \ \ S = T_{(j)}.$$

The bivariate probability function of $\mathbf{I}$ is denoted by $p_{k,j} = P(\mathbf{I} = (k, j)), \quad i, j = 1, ...n$ with

$$p_{k,j} = \frac{|A_{k,j}|}{n!},$$

where $|A_{k,j}|$ is the size of the set

$$A_{k,j} = \{\sigma \in \wp_n : T = T_{(k)} \ \ and \ \ S = T_{(j)} \ \ whenever \ \ T_{\sigma(1)} < .... < T_{\sigma(n)}\}$$

and $\wp_n$ is the set of permutations of the set $\{1, ..., n\}$

The matrix $\mathbf{P} = (p_{k,j})$ is called the bivariate signature matrix (BSM) associated with (S, T). Also $s_j^S = \sum_{k=1}^n p_{k,j}$ define the univariate (marginal) coherent system signature corresponding to lifetime S and $s_k^T = \sum_{j=1}^n p_{k,j}$ define the univariate (marginal) coherent system signature corresponding to lifetime T.
If we consider the systems component lifetimes immersed in $\{T_1, ..., T_n\}$ we can write

$$G(t, s) = P(T \leq t, S \leq s) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{k,j} F_{k,j}(t, s),$$

where $p_{k,j} = P(T = T_{(k)} S = T_{(j)})$, $F_{k,j}(t, s) = P(T_{(k)} \leq t, T_{(j)} \leq s)$ and $G(t, s)$ is the joint distribution function of the system lifetimes. G can have a singular part in the set $\{T = S\}$, in which case we have

$$F_{i,i} = P(T_{(i)} \leq t, T_{(i)} \leq s) = F_{(i)}(t \wedge s)$$

and we can continue to use the above decomposition.

2.2. Compensator process and asymptotic reliability.

The point process $N_t((i)) = 1_{\{T_{(i)} \leq t\}}$ is an $\Im_t$-sub-martingale, that is, $T_{(i)}$ is $\Im_t$-measurable and $E[N_t((i))|\Im_s] \geq N_s((i))$ for all $0 \leq s \leq t$.

From Doob-Meyer decomposition, there exists an unique $\Im_t$-predictable process, denoted $(A_t((i))_{t\geq 0}$, called the $\Im_t$-compensator of $N_t((i))$, with $A_0((i)) = 0$ and such that $M_t((i)) = N_t((i)) - A_t((i))$ is a zero mean uniformly integrable $\Im_t$-martingale. We assume that $T_i, 1 \leq i \leq n$ are totally inaccessible $\Im_t$-stopping time and, under this assumption, $A_t((i))$ is continuous.
As $N_t((i))$ can only count on the time interval $(T_{(i-1)}, T_{(i)}]$, the corresponding compensator differential $dA_t((i))$ must vanish outside this interval.

The $\Im_t$-compensator of $P(T \leq t|\Im_t)$, where T is the system lifetime is set in the following Theorem:

Theorem 2.2.1
Let $T_1, T_2, ..., T_n$, be the components lifetimes of a coherent system with lifetime T. Under the hypothesis and notation of Section 2.1, the $\Im_t$-submartingale $P(T \leq t|\Im_t)$, $t < u$, has the $\Im_t$-compensator

$$\sum_{k=1}^{n} \int_0^t 1_{\{T=T_{(k)}\}} dA_s((k)).$$

Proof
We consider the deterministic process

$$1_{\{T=T_{(k)}\}}(w, s) = 1_{\{T=T_{(k)}\}}(w).$$

It is left continuous and, therefore, $\Im_t$-predictable, implying that, see Bremaud (1981),

$$\int_0^t 1_{\{T=T_{(k)}\}}(s) dM_s((k))$$

is an $\Im_t$-martingale.
As a finite sum of $\Im_t$-martingales is an $\Im_t$-martingale, we have

$$\sum_{k=1}^{n} \int_0^t 1_{\{T=T_{(k)}\}} dM_s((k)) =$$

$$\sum_{k=1}^{n} \int_0^t 1_{\{T=T_{(k)}\}} d1_{\{T_{(k)} \leq s\}} - \sum_{k=1}^{n} \int_0^t 1_{\{T=T_{(k)}\}} dA_s((k)).$$

is an $\Im_t$-martingale. As the compensator is unique we finish the proof.

We consider the definition of a $\Im_t$-doubly stochastic Poisson process.

Definition 2.2.2
A point process N_t, adapted to a history $(\Im_t)_{t\geq 0}$, is called a $\Im_t$-doubly stochastic Poisson process, directed by A_t if, for all $t \geq s \geq 0$ and all $u \in \Re$,

$$E\{\exp[iu(N_t - N_s)|\Im_s\} = \exp[(e^{iu} - 1)(A_t - A_s)],$$

where $(A_t)_{t\geq 0}$ is a finite, non negative $\Im_0$-measurable process. Also, the above expression yields, for all $t \geq s \geq 0$ and all $k \geq 0$,

$$P(N_t - N_s = k|\Im_s) = \exp[-(A_t - A_s)]\frac{[A_t - A_s]^k}{k!}$$

and N_t is called a $\Im_t$-doubly stochastic Poisson process or a $\Im_t$-conditional Poisson process. If we have $A_t = \int_0^t \lambda_s ds$, where λ_t is a nonnegative $\Im_0$-measurable process with $\int_0^t \lambda_s ds < \infty$, λ_t is called intensity process.

If $A_t = A$ where A is some nonnegative random $\Im_0$-measurable random variable , N_t is called a homogeneous doubly stochastic Poisson process.

If $A_t = A(t)$ where $A(t)$ is a deterministic function of time, $N_t = N(t)$ is called a non homogeneous Poisson process

To continue we apply Brown Theorem in the signature point process representation of a coherent system

Theorem 2.2.3

Brown, (1982). Let $(\Im_t^n)_{n\geq 1}$ be a sequence of histories defined on a common probability space $(\Omega, \Im, P)$, $(N_t^n)_{n\geq 1}$ be a sequence of a simple point processes $\Im_t^n$-adapted, for each n, and $(A_t^n)_{n\geq 1}$ the sequence of $\Im_t^n$-compensator of $(N_t^n)_{n\geq 1}$. Let $(A_t)_{t\geq 0}$ be a cumulative process defined on $(\Omega, \Im, P)$, with continuous trajectories and such that for each $t > 0$

i) A_t is $\Im_0^n$-measurable for every $n = 1, 2, 3, ...$

ii) $A_t^n \to A_t$, in probability when $n \to \infty$.

Then N_t^n converges weakly to a doubly stochastic Poisson process directed by A_t.

In the following we apply Theorem 2.2.3 to calculate the asymptotic reliability of a coherent system.

Corollary 2.2.4

Let $T_1, T_2, ..., T_n, ...$ be component lifetimes of a coherent system with lifetime T. Consider a component level filtration given by

$$\Im_t^n = \sigma\{1_{\{T_{(i)}>s\}}, 1 \leq i \leq n, 0 < s < t\},$$

and the point process

$$N_t^n = P(T \leq t|\Im_t^n) = \sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)}\leq t\}}$$

with $\Im_t^n$-compensator

$$A_t^n = \sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} A_t((k)).$$

If, for all $t \geq 0$, $A_t^n \to A_t$, in probability when $n \to \infty$, where A_t has continuous sample path and is $\Im_0^n$-measurable, for each n, then N_t^n converges weakly to a doubly stochastic Poisson processdirected by A_t.

Proof

As, for $k \leq n$ $\{T_{(k)} = T\} \in \Im^n_{T_{(k)}}, \{T_{(k)} = T\} \cap \{T_{(k)} \leq t\} \in \Im^n_t, \forall t \geq 0$, N^n_t is $\Im^n_t$-adapted and the proof follows from Brown theorem. We denote this limit by

$$A^T_t = \sum_{k=1}^{\infty} 1_{\{T=T_{(k)}\}} A_t((k)).$$

$(A^T_t)_{t\geq 0}$ is the $\Im_t$-compensator of $(N^T_t)_{t\geq 0}$, where

$$N^T_t = \lim_{n\to\infty} N^n_t = \sum_{k=1}^{\infty} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)}\leq t\}},$$

also denoted by $(T_n)_{n\geq 1}$.

Remark 2.2.5

To give means and consistency to system signature of "infinite order" we report to the following result from the paper Navarro et al. (2008): Given an arbitrary coherent system with lifetime T and distribution function $F_T(t)$, in n i.i.d. components, there exists, for any integer $m > n$, an equivalent (equal in law) coherent system in m i.i.d. components with the same distribution function $F_T(t)$. Formally proves:

Theorem 2.2.6 (Navarro et al.(2008))

Let $\mathbf{s} = (s_1, ..., s_k)$ be the signature of an arbitrary coherent system of order k. Then, for any integer $n > k$, the system with signature $\mathbf{s}$ is equivalent to the n component system with signature $\mathbf{s}^* = (s^*_1, ..., s^*_n)$ given by

$$\mathbf{s}^* = \sum_{i=1}^{k} s_i \sum_{j=i}^{n+i-k} \frac{\binom{j-1}{i-1}\binom{n-j}{k-i}}{\binom{n}{k}} \mathbf{s_{j:n}},$$

where $\mathbf{s_{j:n}} = (0, ..., 0, 1, 0, ..., 0)$ is the signature vector of a j-out-of-n:F system. It is important to note that

$$\sum_{j=i}^{n+i-k} \frac{\binom{j-1}{i-1}\binom{n-j}{k-i}}{\binom{n}{k}} = 1$$

and

$$\lim_{n\to\infty} \frac{\binom{j-1}{i-1}\binom{n-j}{k-i}}{\binom{n}{k}} = 0.$$

Example 2.2.7

As in Navarro et al. (2013), let T_1, T_2, T_3, and T_4 be independent and identically distributed component lifetimes with distribution function F. Le $S$and T be the lifetimes of the following coherent systems with a single shared component: $S = \wedge\{T_1 \vee T_2, T_1 \vee T_2, T_2 \vee T_3\}$ and $T = T_3 \wedge T_4$. [15] calculate the probability distribution of the random pair (I_1, I_2) by using the definition in Remark 2.1.7. The matrix $\mathbf{P}$ is given by

0	0	0	0
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0
$\frac{1}{3}$	$\frac{1}{6}$	0	0
0	0	0	0

As a two component system,the system signature corresponding to the lifetime T is $(1, 0)$. As a three component system, its signature is $(\frac{2}{3}, \frac{1}{3}, 0)$ and as a four component system have it signature identified by $\mathbf{P}$, equal to $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0)$.
As a two component system,the system signature corresponding to the lifetime T is $(0, 1, 0)$. As a four component system have it signature determined by $\mathbf{P}$, equal to $(0, \frac{1}{2}, \frac{1}{2}, 0)$.

Example 2.2.8

To exemplify the asyntotic procedure we consider the well-known Cesaro Summability Condition:

(I) If $0 < p(n, k) < 1$, for all n and $1 \leq k \leq n$, then the sum $\sum_{k=1}^{n} p(n, k)$ and the product $\pi_{k=1}^{n}(1 - p(n, k)$, either converges or diverges.

(II) If $0 < p(n, k) < 1$, for all n, $1 \leq k \leq n$ and

$$\Sigma_{j=1}^{\infty} \frac{1}{j} \sum_{k=1}^{n} p(n, k) = \lambda t,$$

for fixed t and some $\lambda > 0$, as $n \to \infty$, then

$$\pi_{k=1}^{n}(1 - p(n, k) = \exp[-\lambda t],$$

for fixed t and some $\lambda > 0$, as $n \to \infty$.

We assume that the coherent systems components are subject to failures according to a well known nonhomogeneous Poisson processes (NHPP). This property characterizes a minimal repair process which means that, at each failure in the set $\{T = T_{(k)}\}$ the system is repaired and continues to work with the same failure rate as it had immediately before failure. In these context we characterize the coherent system through its compensator process

$$A_t^n = \sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} A_t((k))$$

It is well known that, see Arjas and Yashin (1988), in the absolutely continuous case, $A_t((k)) = -\ln(1 - F_{(k)}(t|\Im_{t^-}))$. Therefore $A_t((k)) = \sum_{j=1}^{\infty} \frac{1}{j} P(T_{(k)} \leq t|\Im_{t^-})^j$ and

$$A_t^n = \sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{1}{j} P(T_{(k)} \leq t|\Im_{t^-})^j.$$

Therefore we can state conditions under which we apply the Cesaro Summability Condition to coherent system, by example:

If $F_{(k)}(t|\Im_{t^-})$ are absolutely continuous and

$$A_t^n = \sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{1}{j} P(T_{(k)} \leq t|\Im_{t^-})^j \to \lambda t,$$

for fixed t and some $\lambda > 0$, as $n \to \infty$, then the coherent system converges to a Poisson process.
Under the above hypothesis and the second part of Cesaro Summability Condition we conclude that the asymptotic reliability is equal to

$$\pi_{k=1}^{\infty} P(T_{(k)} > t|\Im_{t^-}) = \exp[-\lambda t].$$

We consider a coherent system of identically distributed component where, for fixed t and some λ, the failure probability of an ordered component depend on the size n of the system and its position, k, and tends to zero with the rate $\frac{\lambda t}{n^{\frac{1}{k}}}$. Follows that

$$P(T_{(k)} \leq t|\Im_{t^-}) = [\frac{\lambda t}{n^{\frac{1}{k}}} + o(\frac{1}{n^{\frac{1}{k}}})]^k.$$

Therefore

$$-\sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{1}{j} [\frac{\lambda t}{n^{\frac{1}{k}}} + o(\frac{1}{n^{\frac{1}{k}}})]^{k^n} \to (\lambda t)^k$$

as $n \to \infty$. Then the coherent system converges to a Weibull process. The above result are the reasons and motivation to asymptotically extend the inaccuracy measure for a double stochastic Poisson processes.

3. Cumulative residual inaccuracy measure

3.1. Cumulative residual inaccuracy measure for coherent systems under double stochastic Poisson processes

As in Bueno (2019) we define cumulative residual inaccuracy measure between coherent system modeled by double stochastic Poisson processes:

Definition 3.1.1
Let $(N_t^T)_{t\geq 0}$, defined by the sequence $(T_n)_{n\geq 1}$, be a double stochastic Poisson processes modeling a coherent system with lifetime T and $\Im_t$-compensator processes $(A_t^T)_{t\geq 0}$ and $(N_t^S)_{t\geq 0}$, defined by the sequence $(S_n)_{n\geq 1}$, be a double stochastic Poisson processes modeling a coherent system lifetime with S and $\Im_t$-compensator processes $(B_t^S)_{t\geq 0}$. The cumulative residual inaccuracy measure between N_t^T and N_t^S is

$$CRI_{S,T} = E[\int_0^T B_s^S ds] + E[\int_0^S A_s^T ds].$$

Theorem 3.1.2
Let $T_1, T_2, ..., T_n, ...$, be the components lifetime of a coherent system with lifetime T and let S be the lifetime of a coherent system with component lifetimes $S_1, S_2, ..., S_m, ...$ under the above conditions and notations. Then the cumulative residual inaccuracy measure of N_t^T and N_t^S on the component level, that is, observing $T_{(i)}, i \geq 1$ and $S_{(i)}, i \geq 1$ is

$$CRI_{S,T} = E[\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(k)},T=T_{(j)}\}}|S_{(k)} - T_{(j)}|].$$

Proof.
As in Bueno (2019),

$$CRI_{S,T} = E[\int_0^T B_s^S ds] + E[\int_0^S A_s^T ds].$$

Now, using results of Section 2 we have

$$E[\int_0^T B_s^S ds] = E[\int_0^T (\int_0^s dB_t^S) ds] = E[[\int_0^T (\int_t^T ds) dB_t^S] = E[\int_0^T (T-t) dB_t^S].$$

As $B_t^S = \sum_{k=1}^{\infty} 1_{\{S=S_{(k)}\}} dB_t((k))$ we have

$$E[\int_0^T B_s^S ds] = E[\int_0^T (T-t) \sum_{k=1}^{\infty} 1_{\{S=S_{(k)}\}} dB_t((k))] =$$

$$E[\sum_{k=1}^{\infty} \int_0^T (T-t) 1_{\{S=S_{(k)}\}} dB_t((k))] = E[\sum_{k=1}^{\infty} \int_0^T (T-t) 1_{\{S=S_{(k)}\}} dN_t^S((k))] =$$

$$E[\sum_{k=1}^{\infty} 1_{\{S=S_{(k)}\}} (T - S_{(k)}) 1_{\{S_{(k)} \leq T\}}].$$

Using the same argument

$$E[\int_0^S A_s^T ds] = E[\sum_{k=1}^{\infty} 1_{\{T=T_{(k)}\}} (S - T_{(k)}) 1_{\{T_{(k)} \leq S\}}].$$

Therefore

$$CRI_{S,T} = E[\sum_{k=1}^{\infty} 1_{\{S=S_{(k)}\}}(T - S_{(k)})1_{\{S_{(k)}\leq T\}} + \sum_{k=1}^{\infty} 1_{\{T=T_{(k)}\}}(S - T_{(k)})1_{\{T_{(k)}\leq S\}}] =$$

$$E[\sum_{k=1}^{\infty} 1_{\{S=S_{(k)}\}}(\sum_{j=1}^{\infty} 1_{\{T=T_{(j)}\}})(T - S_{(k)})1_{\{S_{(k)}\leq T\}}+$$

$$\sum_{k=1}^{\infty} 1_{\{T=T_{(k)}\}}(\sum_{j=1}^{\infty} 1_{\{S=S_{(j)}\}})(S - T_{(k)})1_{\{T_{(k)}\leq S\}}] =$$

$$E[\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(k)},T=T_{(j)}\}}|(T_{(j)} - S_{(k)}|1_{\{S_{(k)}\leq T_{(j)}\}}+$$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{T=T_{(k)},S=S_{(j)}\}}|S_{(j)} - T_{(k)}|1_{\{T_{(k)}\leq S_{(j)}\}}] =$$

$$E[\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(k)},T=T_{(j)}\}}|S_{(k)} - T_{(j)}|].$$

Remark 3.1.3
The interpretation of the cumulative residual inaccuracy measure betweens double stochastic Poisson process is retained. We note that

$$CRI_{S,T} = E[\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(k)},T=T_{(j)}\}}|S_{(k)} - T_{(j)}|] =$$

$$E[\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(k)},T=T_{(j)}\}}|S - T|] =$$

$$E[|S - T|\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(k)},T=T_{(j)}\}}] = E[|S - T|].$$

Therefore, $CRI_{T,S}$ can be seen as a dispersion measure when using a coherent system lifetime S asserted by the experimenter information of the true coherent system lifetime T.

Remark 3.1.4
A nonhomogeneous Poisson process generated by record values: Let $T_1, T_2, ..., T_n$, the component lifetimes of a coherent system subjet to minimal repairs. This process is modulated by a non homogeneous Poisson process (NHPP), $(N_t)_{t\geq 0}$, with deterministic compensator $A(t)$. If $T_{(k)}$ is the k-th occurrence time of the NHPP its survival function is given by

$$\overline{G}_k(t) = P(T_k > t) = P(N(t) < k) = \sum_{j=0}^{k-1} \frac{(A(t))^j}{j!} e^{-A(t)}, k = 0, 1, 2, ...$$

where $A(t) = E[N_t] = -\ln \overline{G}(t)$ and $\overline{G}(t)$ is the reliability function of the first occurrence time.
By its turn, a non homogeneous Poisson process (NHPP) is generated by record values of random variables independent and identically distributed.
The reliability function $\overline{G}_k(t)$ arises naturally as the reliability function of upper record values in a sequence of independent non-negative random variables $T_1, T_2, ...$ generated from $G(t)$, the time distribution function of the first occurrence time. The observation T_j is an upper record value if it exceeds all previous observations, see Arnold et al. (1992).
In other words, an NHPP is essentially a record non-explosive counting process subject that its mean value function $A(t)$ is continuous and goes to ∞ as $t \to \infty$. Therefore the sequence of occurrence times in a non-homogeneous Poisson process can be considered as the record values of a sequence of independent and identically distributed random variables each having distribution function G, in which case the events $\{T = T_{(k)}\}, \ \ 1 \leq k \leq n$ and the occurrence times are independents, see Randles and Wolfe (1979).

Corollary 3.1.5
Let $(N_t^T)_{t\geq 0}$, defined by the sequence $(T_n)_{n\geq 1}$, be a deterministic non-homogeneous Poisson process modeling a coherent system with lifetime T subject to minimal repairs and let $(N_t^S)_{t\geq 0}$, defined by the sequence $(S_n)_{n\geq 1}$, be a deterministic non-homogeneous Poisson process modeling a coherent system with lifetime S subject to minimal repairs, independent of T. Then the cumulative residual inaccuracy measure of N_t^T and N_t^S on the component level is

$$CRI_{S,T} = \sum_{k=1}^{\infty} \int_0^{\infty} s_k^S P(S_{(k)} > t)dt + \sum_{j=1}^{\infty} \int_0^{\infty} s_j^T P(T_{(j)} > t)dt - 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \int_0^{\infty} p_{kj} P(S_{(k)} > t, T_{(j)} > t)dt.$$

Proof.
As in Theorem 3.2 we have

$$CRI_{S,T} = E[\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} 1_{\{S=S_{(k)},T=T_{(j)}\}} |S_{(k)} - T_{(j)}|] =$$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} E\{E[1_{\{S=S_{(k)},T=T_{(j)}\}} |S_{(k)} - T_{(j)}| | S = S_{(k)}, T = T_{(j)}]\} =$$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} E\{1_{\{S=S_{(k)},T=T_{(j)}\}} E[|S_{(k)} - T_{(j)}| | S = S_{(k)}, T = T_{(j)}]\} =$$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} P(S = S_{(k)}, T = T_{(j)}) E[|S_{(k)} - T_{(j)}|] =$$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} p_{k,j} E[(S_{(k)} \vee T_{(j)}) - (S_{(k)} \wedge T_{(j)})] =$$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} p_{k,j} \int_0^{\infty} [P((S_{(k)} \vee T_{(j)}) > t) - P((S_{(k)} \wedge T_{(j)}) > t)]t =$$

$$= \sum_{k=1}^{\infty} \int_0^{\infty} s_k^S P(S_{(k)} > t)dt + \sum_{j=1}^{\infty} \int_0^{\infty} s_j^T P(T_{(j)} > t)dt - 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \int_0^{\infty} p_{kj} P(S_{(k)} > t, T_{(j)} > t)dt.$$

Example 3.1.6
Let $T_1, T_2, ..., T_n, ...$ be the component's lifetimes of a coherent system with lifetime T which are subject to failures according to a Weibull process with parameters $\beta = 2$ and θ_1. Let $S_1, S_2, ..., S_m, ...$ be the component's lifetimes of a coherent system with lifetime S, asserted by the experimenter, which are subject to failures according to a Weibull process with parameters $\beta = 2$ and θ_2. $S \wedge T$ follows a Weibull process with parameters $\beta = 2$ and $\frac{\theta_1^2\theta_2^2}{\theta_1‘2\theta_2^2}$ .In practical we consider the ordered lifetimes $T_1, T_2, ..., T_n, ...$ with a conditional reliability function given by

$$\overline{F_i}(t_i|t_1, t_2, ..., t_{i-1}) = \exp[-(\frac{t_i}{\theta})^\beta + (\frac{t_{i-1}}{\theta})^\beta]$$

for $0 \leq t_{i-1} < t_i$ where t_i are the ordered observations.
Considering $T_1, T_2, ..., T_n, ...$, $S_1, S_2, ..., S_m$ as record values of independent and identically distributed random variables, we can apply Corollary 3.1.5:

$$CRI_{S,T} = \int_0^{\infty} [\sum_{k=1}^{\infty} s_k^T \exp[-(\frac{t}{\theta_1})^\beta + (\frac{t_{k-1}}{\theta_1})^\beta] +$$

$$\sum_{j=1}^{\infty} s_j^S \exp[-(\frac{t}{\theta_2})^\beta + (\frac{s_{j-1}}{\theta_2})^\beta] -$$

$$2\sum_{i=1}^{\infty} s_i^{S\wedge T} \exp[-(\frac{t}{\frac{\theta_1^2\theta_2^2}{\theta_1^2+\theta_2^2}})^\beta + (\frac{u_{i-1}}{\frac{\theta_1^2\theta_2^2}{\theta_1^2+\theta_2^2}})^\beta]]dt$$

where s_k^T,s_j^S and $s_i^{S\wedge T}$ are the components vector of s^T, s^S and $s^{S\wedge T}$ of the ("infinity order") coherent systems signatures with lifetimes T, S and $S \wedge T$, respectively.

3.2. Dynamic cumulative residual inaccuracy measure for coherent system under double stochastic Poisson processes

We can extend the concept to a timing varying form corresponding to residual processes after a fixed time t. Note that there exists a k such that $S_{k-1} < t \leq S_k$ and

$$E[\int_t^T B_s^S ds] = E[\int_t^T (\int_0^s dB_u^S) ds] = E[\int_0^t (\int_t^T ds) dB_u^S + \int_t^T (\int_u^T ds) dB_u^S] =$$

$$E[\int_0^t (T-t) dB_u^S + \int_t^T (T-u) dB_u^S].$$

However

$$dB_u^S = \sum_{k=1}^{\infty} 1_{\{S_{k-1}<u\leq S_k\}} 1_{\{S=S_k\}} dB_u(k).$$

and therefore

$$E[\int_t^T B_s^S ds] = E[\sum_{k=1}^{\infty}\{(T-t)1_{\{S=S_k\}} \int_{S_{k-1}}^{t\wedge S_k} dB_u(k) + 1_{\{S=S_k\}} \int_{t\vee S_{k-1}}^{T\wedge S_k} (T-u)dB_u(k)\} =$$

$$E[\sum_{k=1}^{\infty} 1_{\{S=S_k\}}\{(T-t)1_{\{S_k\leq t\}} + (T-S_k)1_{\{t<S_k\leq T\}}\} =$$

$$E[\sum_{k=1}^{\infty} 1_{\{S=S_k\}}|T-S_k|1_{\{t<S_k\leq T\}}.$$

As $\sum_{n=1}^{\infty} 1_{\{T=T_n\}} = 1$ we have

$$E[\int_t^T B_s^S ds] = E[\sum_{k=1}^{\infty} 1_{\{S=S_k\}}(\sum_{n=1}^{\infty} 1_{\{T=T_n\}})|T-S_k|1_{\{t<S_k\leq T\}} =$$

$$E[\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} 1_{\{S=S_k,T=T_n\}}|T-S_k|1_{\{t<S_k\leq T\}} =$$

$$E[\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} 1_{\{S=S_k,T=T_n\}}|T_n-S_k|1_{\{t<S_k\leq T_n\}}.$$

With the same argument we have

$$E[\int_0^S A_s^T ds] = E[\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} 1_{\{S=S_k,T=T_n\}}|T_n-S_k|1_{\{t<T_n\leq S_k\}}.$$

and therefore

$$E[\int_t^T B_s^S ds + \int_t^S A_s^T ds] = E[\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} 1_{\{S=S_k,T=T_n\}}|T_n-S_k|1_{\{t<T_n\wedge S_k\}}.$$

We can define

Definition 3.2.1

Let $(N_t^T)_{t\geq 0}$, defined by the sequence $(T_n)_{n\geq 1}$, be a double stochastic Poisson processes with $\Im_t$-compensator processes $(A_t^T)_{t\geq 0}$ and $(N_t^S)_{t\geq 0}$, defined by the sequence $(S_n)_{n\geq 1}$, be a double stochastic Poisson processes with $\Im_t$-compensator processes $(B_t^S)_{t\geq 0}$. The dynamic cumulative residual inaccuracy measure between N_t^T and N_t^S is

$$DCRI_{S,T}^t = E[\int_t^T B_s^S ds] + E[\int_t^S A_s^T ds] =$$

$$E[\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} 1_{\{S=S_k,T=T_n\}}|T_n - S_k| 1_{\{t<T_n\wedge S_k\}}.$$

Remark 3.2.2
Let $(N_t^T)_{t\geq 0}$ and $(N_t^S)_{t\geq 0}$ be double stochastic Poisson processes with $\Im_t$-compensator processes $(A_t^T)_{t\geq 0}$ and $(B_t^S)_{t\geq 0}$ respectively. We say that N_t^T and N_t^S satisfies the proportional risk hazard process if $B_t^S = \alpha A_t^T, \quad \forall t \geq 0$ for some $\alpha, \quad 0 < \alpha < 1$.

Theorem 3.2.2 The characterization Problem
If N_t^T and N_t^S satisfies the proportional risk hazard process, then the dynamic cumulative residual inaccuracy measure $DRCI_{S,T}^t < \infty$ uniquely determines the double stochastic Poisson process.
Proof.
We let $(N_t^{T^1})_{t\geq 0}$ and $(N_t^{S^1})_{t\geq 0}$ be two double stochastic Poisson processes with $\Im_t$-compensator processes $(A_t^{T^1})_{t\geq 0}$ and $(B_t^{S^1})_{t\geq 0}$ respectively, with $B_t^{S^1} = \alpha^1 A_t^{T^1}, \quad \forall t \geq 0$ for some $\alpha^1, \quad 0 < \alpha^1 < 1$. Also, let $(N_t^{T^2})_{t\geq 0}$ and $(N_t^{S^2})_{t\geq 0}$ be two double stochastic Poisson processes with $\Im_t$-compensator processes $(A_t^{T^2})_{t\geq 0}$ and $(B_t^{S^2})_{t\geq 0}$ respectively, with $B_t^{S^2} = \alpha^2 A_t^{T^1}, \quad \forall t \geq 0$ for some $\alpha^2 \quad 0 < \alpha^2 < 1$. Then we have :

$$DRCI_{S^1,T^1}^t = DRCI_{S^2,T^2}^t \leftrightarrow$$

$$E[\int_t^{S^1} A_t^{T^1} dt + \int_t^{T^1} \alpha^1 A_t^{T^1} dt] = E[\int_t^{S^2} A_t^{T^2} dt + \int_t^{T^2} \alpha^2 A_t^{T^2} dt].$$

However, for $i = 1,2$ we have

$$E[\alpha^i \int_t^{T^i} A_s^{T^i} ds] = \alpha^i E[\int_0^t (T^i - t) dA_s^{T^i}] + \alpha^i E[\int_t^{T^i} (T^i - s) dA_s^{T^i}] = 0.$$

Without loss of generality, using the Optimal Sampling Theorem , for any $\Im_t$-stopping time S we have,

$$DRCI_{S^1,T^1}^S = DRCI_{S^2,T^2}^S \leftrightarrow E[\int_0^S A_t^{T^1} dt] = E[\int_0^S A_t^{T^2} dt] \leftrightarrow$$

$$E[\int_0^\infty 1_{\{t<S\}} A_t^{T^1} 1_{\{A_t^{T^1} > A_t^{T^2}\}} dt] + E[\int_0^\infty 1_{\{t<S\}} A_t^{T^1} 1_{\{A_t^{T^1} \leq A_t^{T^2}\}} dt] =$$

$$E[\int_0^\infty 1_{\{t<S\}} A_t^{T^2} 1_{\{A_t^{T^1} > A_t^{T^2}\}} dt] + E[\int_0^\infty 1_{\{t<S\}} A_t^{T^2} 1_{\{A_t^{T^1} \leq A_t^{T^2}\}} dt] \leftrightarrow$$

$$E[\int_0^\infty 1_{\{t<S\}} (A_t^{T^1} - A_t^{T^2}) 1_{\{A_t^{T^1} > A_t^{T^2}\}} dt] = E[\int_0^\infty 1_{\{t<S\}} (A_t^{T^2} - A_t^{T^1}) 1_{\{A_t^{T^1} \leq A_t^{T^2}\}} dt] \leftrightarrow$$

$$\int_0^\infty E[1_{\{t<S\}} |A_t^{T^1} - A_t^2| 1_{\{A_t^{T^1} > A_t^{T^2}\}}] dt = \int_0^\infty E[1_{\{t<S\}} |A_t^{T^1} - A_t^{T^2}| 1_{\{A_t^{T^1} \leq A_t^{T^2}\}}] dt \leftrightarrow$$

$$\int_0^\infty E[1_{\{t<S\}} |A_t^{T^1} - A_t^{T^2}| (1_{\{A_t^{T^1} > A_t^{T^2}\}} - 1_{\{A_t^{T^1} \leq A_t^{T^2}\}})] dt = 0.$$

As $\{A_t^{T^1} > A_t^{T^2}\} \cap \{A_t^{T^1} \leq A_t^{T^2}\} = \emptyset$ and the integrand is positive we have $A_t^{T^1} = A_t^{T^2}$. As the compensator is unique we have $N_t^{T^1} = N_t^{T^2}$

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