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ORIGINAL ARTICLE

SLE: differential invariants study

Vladimir Grebenev[1] · **Alexandre Grishkov**[2]

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Abstract

Random curves (produced by the Schramm–Loewner evolution SLE_κ) of the fractal dimension $d_\kappa = 1 + \kappa/8$, $\kappa < 8$ given on a surface $D \subset \mathbb{R}^3$ and the conformal group (CG) that acts on D are considered. We study the action integral $L[\mathbf{X}(t)]$, $\mathbf{X}(t) \in SLE_\kappa$ (the fractal variation of length of a random curve (Kennedy in J Stat Phys 128(6):1263–1277, 2006)) together with the conformal group extension CGE of CG which invariant transforms SLE_κ and $L[\mathbf{X}(t)]$. We calculate the second-order universal differential invariant $\mathbf{J}_2$ (or the multiscale representation of invariants) of the GCE and show that $L[\mathbf{X}(t)]$ generates all second-order differential invariants of CGE by the operators of invariant differentiation. The differential invariants look like invariant quantities of different scales wherein $L[\mathbf{X}(t)]$ plays a role of "the fractal length scale". The method of calculations of differential invariants is a kind of modern multiscale analysis (Olver and Pohjanpelto in Adv Math 222(5):1746–1792, 2009) based on the theory of symmetry group. This investigation is also motivated by Cartan's point of view (in: Cartan, La Théoric des Groupes Finis et Continus et la Géometrie Differentielle traittée par le Méthode du Repére Mobile, Gauthier-Villars, Paris, 1937) that the local geometry properties are entirely governed by differential invariants of the group admitted.

Keywords SLE · Conformal group extension · Fractal length scale · Differential invariants

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✉ Vladimir Grebenev
vngrebenev@gmail.com

1 Federal Research Center for Information and Computational Technologies, Russian Academy of Sciences, Novosibirsk, Russia

2 Institute of Mathematics and Statistics, University of Sao Paulo, Sao Paulo, Brazil

1 Introduction

One-parametric family SLE_κ (Schram–Löwner evolution) presents random process that produce random curves. This is a conformally invariant set arising as scaling limits of various critical models from statistical physics in the plane, see [20, 22] and the bibliography therein. In short, we mean the SLE_κ trace random pathes in $D \subset \mathbb{R}^2$ or $\mathbb{C}$ driven by the standard Brownian motion with the diffusion coefficient κ. We assume that a domain D is endowed with the conformal structure of $\mathbb{C}$. If an embedding of D into $\mathbb{R}^2$ is compact of the genus $g \leq 1$, then D is conformal equivalent to a (part of) Riemannian sphere or a (part of) torus for the oriented case and a (part of) Klein bottle for the non-oriented case. For a domain D of the genus ≥ 2 we have that the conformal group given on D is trivial [19]. Therefore, the substantial case consists in the consideration of D with $g \leq 1$. For non-compact surfaces we have the following surfaces: an open disk or all plane, open annulus or punctured plane (oriented) or an Möbius strip (non-oriented). Speaking about metric on a surface D, we always have in mind a Riemannian metric compatible with the conformal structure. We can get a compact surface from a noncompact one by the so-called Schottki construction or Schottki double, see [10]. If D is a noncompact surface of finite topological type, then there exists a unique oriented compact surface D^{double} with an orientation-reversing conformal involution ζ and an embedding $\imath : D \mapsto D^{double}$ such that $\zeta[\imath(D)] \cap \imath(D) = \emptyset$, the complement $D^{double} \setminus (\zeta[\imath(D)] \cup \imath(D))$ is a disjoint union of finitely many isolated points and closed loops. For a noncompact D we deal with Schottki double D^{double}. With this, D is endowed with a Riemannian metric of the conformal form [3] and the action integral $L[\mathbf{X}(t)]$, $\mathbf{X}(t) \in SLE_\kappa$ can be defined. The conformal transformations of D give rise to CG invariance of $SLE_\kappa \mapsto SLE_\kappa$, [22].

The differential invariants associated with a transformation group acting on a set are the fundamental quantities for understanding the geometry of this set [2]. Exemplarily, the curvature and torsion of a curve defines the shape of the curve. These quantities are differential invariants of the Euclidean group $SE(3)$.

Another topic that has much attention from the geometric multi-scale analysis is the multiscale representation of invariants, see i.e. [18]. The obtained representations allows to compute invariant quantities at different scales. In practise, the length preserving is one of the typical viewing invariants in computer vision. How many invariants can we use for a visualization of the object, in particular, in geometric multi-resolution representations? This question can be identified with generating the minimal set of invariants under transformations.

The basic theory of differential invariants goes to Lie [14] and Tresse [21]. Lie states that all differential invariants can be generated from a finite number of low-order invariants by repeated invariant differentiation. Lie's results were generalized by Tresse and, much later and significantly, Ovsiannikov [17]. A well-known example is that of the differential invariants of a space curve under the action of the Euclidean group $SE(3)$ are generated by two differential invariants, namely its curvature and torsion. More examples can be founded in [7]. Complete classification of differential invariants for many of the fundamental transformation groups of physical and geometrical importance remains undeveloped. The main difficulty in applying Lie's method to complicated examples is that it requires the integration of linear partial differential equations, which can prove to be rather complicated. A rigorous version of the Lie–Tresse Theorem, based on the machinery of Spencer cohomology, was established by Kumpera [12]; see also [11] for a generalization to pseudo-group actions on differential equations (submanifolds of jet space), and [16] for an approach based on Weil algebras. None of these references provide constructive algorithms for solving a system of generating differential invariants, nor methods for classifying the recurrence and commutator formulae, nor do they investigate the finiteness of the generating differential syzygies. Nevertheless, we are in able to derive the minimal system of differential invariants to generate the algebra of differential invariants of a finite-order.

As it was mentioned above, our first aim is to extend *CG* up to the conformal invariant transformations of the functional $L[\mathbf{X}(t)]$, $\mathbf{X}(t) \in SLE_\kappa$ (called the conformal group extension *CGE*). Then we calculate differential invariants of *CGE* up to the second-order. With this we derive the so-called second-order universal differential invariant $\mathbf{J}_2 = \{J_1, \dots, J_{11}\}$. Finally, we demonstrate that $L[\mathbf{X}(t)]$ (equals J_2) is a differential invariant of *CGE* and generates all second-order invariants J_i for $2 < i \leq 11$, $J_1 = t$.

In Sect. 2, the action integral $L[\mathbf{X}(t)]$, $\mathbf{X}(t) \in SLE_\kappa$ is defined and the fractal variation of length, which is finite [9], will be considered. With the usual definition of length, the length of $\mathbf{X}(t)$ is infinite. We construct the conformal group extension *CGE* of *CG* which invariant transforms SLE_κ conserving the fractal variation of length $L[\mathbf{X}(t)]$. In Sect. 3, we calculate the second-order universal differential invariant $\mathbf{J}_2$ of the group *CGE*. We present the minimal set of generating differential invariants of $\mathbf{J}_2$ which consists of $\{J_1, J_2\}$ where $J_1 = t$ and $J_2 = L[\mathbf{X}(t)]$.

2 The length of a random curve from SLE_κ and symmetry transformations

SLE_κ is a one parameter family of random processes that produce random curves. It was proven in [1] that the Hausdorff (fractal) dimension d_H of the SLE_κ curve equals $d_\kappa = 1 + \kappa/8$, $\kappa \leq 8$. For $d_H = 2$, this is the so-called quadratic variation studied by Lévy for Brownian motion [13]. This is a non-random process and the length is proportional to t under suitable conditions on the convergence of the sequence of partitions. With respect to the usual definition of the length of a curve, the length of SLE_κ curves for $\kappa < 8$ is infinite. Namely, the usual definition of the length or total variation of the random curve $\mathbf{X}(t)$ given on $\mathbb{R}^2$ over the time interval [0, t] for each fixed time t would be the supremum over all partitions $0 < t_1 < t_2 < \cdots t_n = t$ of

$$\sum_{j=1}^{n} \left| X(t_j) - X(t_{j-1}) \right|. \tag{1}$$

$|\boldsymbol{X}(t_j) - \boldsymbol{X}(t_{j-1})|$ (or infinitesimally $|d\boldsymbol{X}|$) is of order $(\Delta t_j)^{1/d_H}$, where d_H is the Hausdorff dimension of a curve and $d_H > 1$. With this, the total variation of $\boldsymbol{X}(t)$ will be infinite. Since the length of a segment is of order $(\Delta t_j)^{1/d_H}$, this suggests to consider the quantity [9]

$$fvar(\boldsymbol{X}(t), P) = \sum_{j=1}^{n} |\boldsymbol{X}(t_j) - \boldsymbol{X}(t_{j-1})|^{d_H}, \tag{2}$$

where P denotes the partition, $(t_0, t_1, \dots, t_n)$. The mesh of a partition is the length of the largest subinterval. Then taking a sequence of partitions P_n whose mesh converges to zero, we consider the limit

$$\lim_{n\to\infty} fvar(\boldsymbol{X}(t), P_n) = l_f. \tag{3}$$

This limit exists i.e. $l_f < \infty$ and l_f defines *the fractal variation* of a random curve [9]. This is a non-random constant. Sometimes it is called the p-variation in the stochastic processes. The formula (3) can be written in the integral form

$$l_f = fvar(\boldsymbol{X}(t)) = \int |d\boldsymbol{X}|^{d_H}. \tag{4}$$

For $\boldsymbol{X}(t) \in SLE_\kappa$ given on D supplied with a Riemannian metric dl^2, the integral in (4) reads

$$l_f = fvar(\boldsymbol{X}(t)) = \int (\boldsymbol{X}_t, \boldsymbol{X}_t)_{dl^2}^{d_H/2} (dt)^{d_H}, \tag{5}$$

where $(\cdot,\cdot)_{dl^2}$ is the scalar product generated by the metric dl^2 and $(dt)^{d_H}$ infinitesimally closes to $(\Delta t_n)^{d_H}$ or $(dt)^{d_H} \sim (\Delta t_n)^{d_H}$ in the limit $(\Delta t_n)^{d_H} \to 0$ as $n \to \infty$. With this, we define the length of $\boldsymbol{X}(t)$ or the action integral:

$$L[\boldsymbol{X}(t)] = \int (\boldsymbol{X}_t, \boldsymbol{X}_t)_{dl^2}^{d_H/2} (dt)^{d_H}. \tag{6}$$

First of all, we show that this functional is a scalar invariant of *CGE*. Refering to CGE, we will mean the infinitesimal operator I_{CGE} defined by the Formula (10).

The geometry of D is linked with a group of transformations acting on D i.e. here with *CG*. Indeed, let us assume that D is a compact surface of the genus ≥ 2. Consider the infinitesimal operator of the conformal group *CG*

$$I_{CG} = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y}, \tag{7}$$

where the functions ξ^1 and ξ^2 satisfy the Cauchy-Riemann conditions, $\xi^1_x = \xi^2_y$ and $\xi^1_y = -\xi^2_x$. *CG* acts on D as a tangent transformation i.e. there exists no any transformations along the normal vector. Therefore, *CG* looks like sliding transformations of D and the following result holds [19]: *Let an embedded compact surface in* $\mathbb{R}^3$ *of a topology genus g admit a sliding transformation. Then* $g \leq 1$*and the surface is conformal equivalent to a (part of) sphere or a (part of) torus for the oriented case* *and a (part of) Klein bootle (non-oriented). For $g \geq 2$the group of transformations is trivial.*

Notice that any metric on D can be written in the conformal form [3]

$$dl^2 = \lambda^2(x, y)\left(dx^2 + dy^2\right). \tag{8}$$

Further, $\lambda^2(x, y)$ is transformed under the conformal transformations $F(z) : D \mapsto D$, $z = x + iy$ as follows (see [19])

$$\lambda^{2^*} = \lambda^2(x, y)|F_z|^{-2}. \tag{9}$$

With this, the infinitesimal operator

$$I_{CGE} = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} - 2\lambda^2 \xi^1_x \frac{\partial}{\partial \lambda^2} \tag{10}$$

invariant transforms the metric (8). To prove it, we use the ansatz for ξ^1 and ξ^2 suggested in [5]:

$$\xi^1 = c^{11}(\boldsymbol{x})x + c^{12}(\boldsymbol{x})y + d^1(\boldsymbol{x}), \tag{11}$$

$$\xi^2 = c^{21}(\boldsymbol{x})x + c^{22}(\boldsymbol{x})y + d^2(\boldsymbol{x}). \tag{12}$$

Here

$$d^1_x(\boldsymbol{x}) = 2c^{11}(\boldsymbol{x}) - c^{11}_x(\boldsymbol{x})x - c^{12}_x(\boldsymbol{x})y, \tag{13}$$

$$d^1_y(\boldsymbol{x}) = - c^{11}_y(\boldsymbol{x})x - c^{12}_y(\boldsymbol{x})y, \tag{14}$$

$$d^2_x(\boldsymbol{x}) = c^{12}_x(\boldsymbol{x})x - c^{11}_x(\boldsymbol{x})y, \tag{15}$$

$$d^2_y(\boldsymbol{x}) = 2c^{11}(\boldsymbol{x}) + c^{12}_y(\boldsymbol{x})x - c^{22}_y(\boldsymbol{x})y, \tag{16}$$

and c^{11}, c^{12} are the harmonic functions of $\boldsymbol{x}$, that follow from the compatibility conditions $d^1_{yx} = d^1_{xy}$, $d^2_{yx} = d^2_{xy}$, see for details [5]. Further, we get $c^{11} = c^{22}$, $c^{12} = -c^{21}$ and

$$\xi^1_y = -\xi^2_x, \quad \xi^1_x = \xi^2_y. \tag{17}$$

Moreover, it follows from (11), (12) and (13–16) that

$$\xi^1_x = 3c^{11}, \quad \xi^1_y = c^{12}. \tag{18}$$

Therefore, we see that the ansatz for ξ^1 and ξ^2 indeed generates the conformal transformations. Thus, *CGE* group action reads

$$x^* = U(x, y, a), \quad y^* = V(x, y, a), \tag{19}$$

$$\lambda^{2^*} = \left(U_x^2 + V_x^2\right)^{-1}\lambda^2. \tag{20}$$

Here $U(x, y, a)$, $V(x, y, a)$ are the global forms of the infinitesimals (11), (12), which also depend on an arbitrary group parameter a. Moreover, $U(x, y, a)$ and $V(x, y, a)$ are conjugate harmonic functions, for details see [5]. Exemplarily, we prove the formula (20). The derivative of equation (20) with respect to the parameter a gives

$$\lambda_a^*\big|_{a=0} = -2\left(U_x^2 + V_x^2\right)^{-2}\Big|_{a=0}\left(U_x U_{xa}\big|_{a=0} + V_x V_{xa}\big|_{a=0}\right)\lambda^2, \tag{21}$$

where U_x, U_{xa} (V_x, V_{xa}) denote the partial derivatives of U (or V) with respect to x and a variables. Then, calculating the derivatives of the right hand side of (21), we get

$$U_x\big|_{a=0} = x_x = 1, \quad U_{xa}\big|_{a=0} = \xi_x^1, \tag{22}$$

$$V_x\big|_{a=0} = y_x = 0, \quad V_{xa}\big|_{a=0} = \xi_x^2. \tag{23}$$

Hence equation (21) gives exactly the coordinate $-2\xi_x^1\lambda^2$ of the infinitesimal operator (10).

In the complex variable frame, the infinitesimal operator I_{CGE} has the form

$$I_{CGE} = \psi\frac{\mathrm{d}}{\mathrm{d}z} + \bar{\psi}\frac{\mathrm{d}}{\mathrm{d}\bar{z}} - \lambda^2\left(\psi_z + \bar{\psi}_{\bar{z}}\right)\frac{\partial}{\partial\lambda^2} \tag{24}$$

where $\psi = \xi^1 + i\xi^2$, $\bar{\psi} = \xi^1 - i\xi^2$. Using the notation $F = U + iV$, the formulas (19, (20) in the complex variables frame are written as follows [6]:

$$z^* = F(z, a), \tag{25}$$

$$\lambda^{2^*} = |F_z|^{-2}\lambda^2, \tag{26}$$

where $z^* = x^* + iy^*$. The symbol $|F_z|$ denotes the modulo of F_z i.e.

$$|F_z|^2 = \left(U_x^2 + V_x^2\right) = [1 + 6c^{11}a + O(a^2)]. \tag{27}$$

The last equality follows from the formula (18). The metric (8) in the complex variables frame reads

$$dl^2 = \lambda^2(z, \bar{z})\mathrm{d}z\mathrm{d}\bar{z}, \tag{28}$$

wherein the differential complex $\mathrm{d}z\mathrm{d}\bar{z}$ is transformed as

$$(\mathrm{d}z\mathrm{d}\bar{z})^* = F_z\bar{F}_z\mathrm{d}z\mathrm{d}\bar{z} = |F_z|^2\mathrm{d}z\mathrm{d}\bar{z} = [1 + 6c^{11}a + O(a^2)]\mathrm{d}z\mathrm{d}\bar{z}, \tag{29}$$

where $\bar{F}_z = \overline{dF/dz}$. Collecting the terms transformed in (28), we get that

$$dl^{2^*} = \lambda^2(z, \bar{z})|F_z|^{-2}\mathrm{d}z\mathrm{d}\bar{z}|F_z|^2 = dl^2 \tag{30}$$

that proves the invariance of dl^2.

To show the invariance of the functional (6), we write this functional in the coordinate frame

$$L[\boldsymbol{X}(t)] = \int \left(\lambda^2(X,Y)(X_t^2 + Y_t^2)\right)^{d_H/2} (dt)^{d_H}, \quad \boldsymbol{X} = (X,Y). \tag{31}$$

In the transformed form and neglecting terms of order $O(a^2)$ the functional reads

$$\begin{aligned} L^*[\boldsymbol{X}^*(t)] =& \int \left(\lambda^{2^*}(X^*,Y^*)({X^*}_t^2 + {Y^*}_t^2)\right)^{d_H/2} (dt)^{d_H} \\ =& \int \left(\lambda^2(1 - 6c^{11}a + O(a^2))(1 + 6c^{11}a + O(a^2))(X_t^2 + Y_t^2)\right)^{d_H/2} (dt)^{d_H} \\ =& L[\boldsymbol{X}(t)]. \end{aligned} \tag{32}$$

Cancelling terms of the order $O(a^2)$, the last equality follows. Therefore, $L[\boldsymbol{X}(t)]$ is *CGE* invariant functional. Other words, the fractal length of $\boldsymbol{X}(t)$ is conserved under the transformations (25), (26).

We prove that *CGE* is more widest group of symmetry transformations of $L[\boldsymbol{X}(t)]$. We take $l = l_f$ as a parametrization (the fractal arch-length parametrization) of $\boldsymbol{X}(t)$ i.e. $l = l_f$ is the fractal length of the curve $\boldsymbol{X}(t)$, that is,

$$\lambda^2(X_l^2 + Y_l^2) = 1. \tag{33}$$

Hence, we can write

$$L[\boldsymbol{X}(l)] = \int 1 \cdot (dl)^{d_H} = l_f. \tag{34}$$

We consider the co-vector (l_X, l_Y) instead of the tangent vector $\boldsymbol{X}_l = (X_l, Y_l)$. Here (l_X, l_Y) is defined by the equations

$$X_l = \frac{l_X}{\lambda^2}, \quad Y_l = \frac{l_Y}{\lambda^2}, \tag{35}$$

where l_X, l_Y denote the derivative of l with respect to the variable X, Y. Then (33) is transformed as

$$l_X^2 + l_Y^2 = \lambda^2. \tag{36}$$

To find symmetries of the functional $L[\boldsymbol{X}(l)]$, we can consider symmetries of (36) such that l (the fractal length of the curve $\boldsymbol{X}(l)$) is not transformed. This is an extension of the notion of variational symmetry [8]. To proceed, we perform the corresponding symmetry analysis for the functional $L[\boldsymbol{X}(l)]$. We do it by the equivalence transformations i.e. with such transformations of differential equations which map the equation to another equation of the same form [8]. Equivalence transformation admitted by (36) is a Lie point transformation of (X, Y, l, λ^2) variables. We look for an infinitesimal operator in the following form

$$\begin{aligned}\mathcal{Q} &= \alpha^1(X, X, l, \lambda^2)\frac{\partial}{\partial X} + \alpha^2(X, Y, l, \lambda^2)\frac{\partial}{\partial Y} \\ &+ \gamma^1(X, Y, l, \lambda^2)\frac{\partial}{\partial l} + \gamma^2(X, Y, l, \lambda^2)\frac{\partial}{\partial \lambda^2}.\end{aligned} \tag{37}$$

The functions α^1, α^2 and γ^i, $i = 1, 2$ are defined due to the equation

$$\mathcal{Q}_1\big|_{(36)} = 0, \tag{38}$$

where $\mathcal{Q}_1$ is the first prolongation of $\mathcal{Q}$. The infinitesimal operator $\mathcal{Q}$ reads (the calculations are performed in the Appendix):

$$\begin{aligned}\mathcal{Q} &= \xi^1(X, Y)\frac{\partial}{\partial X} + \xi^2(X, Y)\frac{\partial}{\partial X} + \Theta(l)\frac{\partial}{\partial l} \\ &+ 2\left(\frac{d\Theta}{dl} - \xi^1_X(X, Y)\right)\lambda^2\frac{\partial}{\partial \lambda^2}.\end{aligned} \tag{39}$$

Here $\xi^1(X, Y)$ and $\xi^2(X, Y)$ are arbitrary conjugate harmonic functions and $\Theta(l)$ is an arbitrary function of the fractal parametrization l. With this, we can take $\Theta(l) \equiv 0$ and the following infinite dimensional Lie subalgebra appears

$$\mathcal{P} = \xi^1(X, Y)\frac{\partial}{\partial X} + \xi^2(X, Y)\frac{\partial}{\partial Y} - 2\xi^1_X(X, Y)\lambda^2\frac{\partial}{\partial \lambda^2}, \tag{40}$$

which does not transform l or $L(\boldsymbol{X}(l))$ is an (scalar) invariant of $\mathcal{P}$. Comparing now $\mathcal{P}$ and I_{CGE}, we see that these infinitesimal operators coincide. Therefore, CGE is the maximal group of symmetry transformations of $L(\boldsymbol{X}(l))$. The following quantities

$$J_2 = l(\boldsymbol{X}), \quad J_4 = \lambda^{-2}(l_X^2 + l_Y^2) \tag{41}$$

are invariant of the group CGE. Here J_2 is a scalar invariant and J_4 is the first-order differential invariant. These invariants are solutions of the equation

$$\mathcal{P}_1 J = 0, \tag{42}$$

where $\mathcal{P}_1$ is the first prolongation of $\mathcal{P}$, that is,

$$\begin{aligned}\mathcal{P}_1 &= \xi^1\frac{\partial}{\partial X} + \xi^2\frac{\partial}{\partial Y} \\ &- 2\xi^1_X\left(\lambda^2\frac{\partial}{\partial \lambda^2} + l_X\frac{\partial}{\partial l_x} + l_Y\frac{\partial}{\partial l_Y} + 3\lambda^2_X\frac{\partial}{\partial \lambda^2_X} + 3\lambda^2_Y\frac{\partial}{\partial \lambda^2_Y}\right) \\ &- \xi^2_X\left(2l_Y\frac{\partial}{\partial Y} - l_X\frac{\partial}{\partial l_Y} + \lambda^2_Y\frac{\partial}{\partial \lambda^2_X} - \lambda^2_X\frac{\partial}{\partial \lambda^2_Y}\right) - 2\xi^1_{XX}\lambda^2\frac{\partial}{\partial \lambda^2_X} - 2\xi^1_{XY}\lambda^2\frac{\partial}{\partial \lambda^2_Y}.\end{aligned} \tag{43}$$

3 Differential invariants of *CGE*

The differential invariant of an order n with respect to CGE, which is admitted by SLE_κ, is a real-valued function J defined on the curves $\boldsymbol{X} \in SLE_\kappa$ which is invariant under the prolonged group action CGE on the variables $(X, Y, l, \lambda^2, \dots, l_t, l_X, l_Y, \lambda^2_X, \lambda^2_Y, \dots)$ up to the n-th order of partial derivatives of the dependent variables (X, Y, l, λ^2).

The first-order differential invariants of CGE are calculated directly by solving the equation

$$\begin{aligned} \mathcal{P}_1 J = \Bigg[& \xi^1 \frac{\partial}{\partial X} + \xi^2 \frac{\partial}{\partial Y} \\ & - 2\xi^1_X \left(\lambda^2 \frac{\partial}{\partial \lambda^2} + l_X \frac{\partial}{\partial l_x} + l_Y \frac{\partial}{\partial l_Y} + 3\lambda^2_X \frac{\partial}{\partial \lambda^2_X} + 3\lambda^2_Y \frac{\partial}{\partial \lambda^2_Y} \right) \\ & - \xi^2_X \left(2 l_Y \frac{\partial}{\partial Y} - l_X \frac{\partial}{\partial l_Y} + \lambda^2_Y \frac{\partial}{\partial \lambda^2_X} - \lambda^2_X \frac{\partial}{\partial \lambda^2_Y} \right) - 2\xi^1_{XX} \lambda^2 \frac{\partial}{\partial \lambda^2_X} - 2\xi^1_{XY} \lambda^2 \frac{\partial}{\partial \lambda^2_Y} \Bigg] J = 0 \end{aligned} \tag{44}$$

Equation (44) is solved by splitting with respect to the variables

$$(X, Y, l, \lambda^2, l_X, l_Y, \lambda^2_X, \lambda^2_Y).$$

As a result, we get

$$\frac{\partial J}{\partial X} = 0, \quad \frac{\partial J}{\partial Y} = 0, \quad \frac{\partial J}{\partial \lambda^2_X} = 0, \quad \frac{\partial J}{\partial \lambda^2_Y} = 0, \tag{45}$$

$$2\lambda^2 \frac{\partial J}{\partial \lambda^2} + l_X \frac{\partial J}{\partial l_X} + l_Y \frac{\partial J}{\partial l_Y} = 0, \tag{46}$$

$$- l_X \frac{\partial J}{\partial l_Y} + l_Y \frac{\partial J}{\partial l_X} = 0. \tag{47}$$

This system has the following third independent solutions

$$J^1 = t, \quad J^2 = l, \quad J^3 = l_t, \quad J^4 = \frac{l_X^2 + l_Y^2}{\lambda^2}, \tag{48}$$

and the set

$$\mathbf{J}_1 = \{J^1, J^2, J^3, J^4\}. \tag{49}$$

denotes the first-order universal invariant of the group $G_{\mathcal{Y}}$.

The second-order invariants are calculated similar by solving the determining equation

$$\mathcal{P}_2 J = 0, \tag{50}$$

where $\mathcal{P}_2$ denotes the second prolongation of $\mathcal{P}$, that is

$$
\begin{aligned}
\mathcal{P}_2 = {} & \xi^1 \frac{\partial}{\partial X} + \xi^2 \frac{\partial}{\partial Y} \\
& - \xi^1_X \left(2\lambda^2 \frac{\partial}{\partial \lambda^2} + l_X \frac{\partial}{\partial l_X} + l_Y \frac{\partial}{\partial l_Y} + 3\lambda^2_X \frac{\partial}{\partial \lambda^2_X} + 3\lambda^2_Y \frac{\partial}{\partial \lambda^2_Y} \right. \\
& \left. + 2l_{XX} \frac{\partial}{\partial l_{XX}} + 2l_{XY} \frac{\partial}{\partial l_{XY}} + 2l_{YY} \frac{\partial}{\partial l_{YY}} + 4\lambda^2_{XX} \frac{\partial}{\partial \lambda^2_{XX}} + 4\lambda^2_{YY} \frac{\partial}{\partial \lambda^2_{YY}} \right) \\
& - \xi^2_X \left(l_Y \frac{\partial}{\partial l_Y} - l_X \frac{\partial}{\partial l_Y} + \lambda^2_Y \frac{\partial}{\partial \lambda^2_X} - \lambda^2_X \frac{\partial}{\partial \lambda^2_Y} + \lambda^2_{XY} \frac{\partial}{\partial \lambda^2_{XX}} \right. \\
& \left. -(l_{XX} - l_{YY}) \frac{\partial}{\partial \lambda^2_{XY}} - l_{XY} \frac{\partial}{\partial \lambda^2_{YY}} + +2\lambda^2_{XY} \frac{\partial}{\partial \lambda^2_{XX}} + 2\lambda^2_{XY} \frac{\partial}{\partial \lambda^2_{YY}} \right) \\
& - \xi^1_{XX} \left(2\lambda^2 \frac{\partial}{\partial \lambda^2_X} + l_X \frac{\partial}{\partial l_{XX}} + l_Y \frac{\partial}{\partial l_{XY}} - l_X \frac{\partial}{\partial l_{YY}} + 5\lambda^2_X \frac{\partial}{\partial \lambda^2_{XX}} - \lambda^2_X \frac{\partial}{\partial \lambda^2_{YY}} \right) \\
& - \xi^1_{XY} \left(2\lambda^2 \frac{\partial}{\partial \lambda^2_Y} - l_Y \frac{\partial}{\partial l_{XX}} + l_X \frac{\partial}{\partial l_{XY}} + l_Y \frac{\partial}{\partial l_{YY}} - \lambda^2 \frac{\partial}{\partial \lambda^2_{XX}} + 5\lambda^2_Y \frac{\partial}{\partial \lambda^2_{YY}} \right) \\
& - 2\xi^1_{XXX} \lambda^2 \left(\frac{\partial}{\partial \lambda^2_{XX}} - \frac{\partial}{\partial \lambda^2_{YY}} \right) - 4\xi^1_{XXY} \lambda^2 \frac{\partial}{\partial \lambda^2_{XY}}.
\end{aligned}
\tag{51}
$$

For solving Eq. (50), we apply the operators of invariant differentiation [17]. Consider the algebra of differential invariants **A**. The invariant differential operator $\mathcal{D}^i$ is defined as a differential operator which maps $\mathbf{A} \mapsto \mathbf{A}$ such that every differential invariant from **A** is locally presented as a function of the generating invariants and their invariant derivatives. More exactly, the operators of invariant differentiation are determined by the formula [17]

$$
\mathcal{D}^i = \mathbf{a}^i \cdot \left(D_x, D_y, D_t \right), \quad \mathbf{D} = (D_X, D_Y, D_t), \quad i = 1, 2, 3. \tag{52}
$$

First of all, we find functional independent solutions I_j, $j = 1, 2, 3$ of the equation

$$
\left[\mathcal{P}_1 + (\mathbf{a} \cdot \mathbf{D}) \mathbf{b} \cdot \partial_\theta \right] I = 0, \tag{53}
$$

where $\mathbf{b} = (b_1, b_2, b_3) = (\xi^1, \xi^2, 0)$ and

$$
\partial_\theta = a_1 \partial / \partial a_1 + a_2 \partial / \partial a_2 + a_3 \partial / \partial a_3. \tag{54}
$$

Equation (53) has the following functionally independent solutions

$$
I_1 = a_3, \quad I_2 = \lambda^2 (a_1^2 + a_2^2), \quad I_3 = a_1 l_Y - a_2 l_X
$$

With this, the vector $\mathbf{a}^i$ is calculated from the equations

$$
I_i(\mathbf{a}^i) = \mathbf{c}^i, \quad \mathbf{c}^i \in \mathbb{R}^3. \tag{55}
$$

Here the vectors $\mathbf{c}^i$ are chosen equal $\mathbf{c}^1 = (0,0,1)$, $\mathbf{c}^2 = (0,1,0)$, $\mathbf{c}^3 = (0,1,1)$. Then the vectors $\mathbf{a}^i$ are of the form

$$\mathbf{a}^1 = (0,0,1), \quad \mathbf{a}^2 = \left(\frac{l_X}{\lambda^2}, \frac{l_Y}{\lambda^2}, 0\right), \quad \mathbf{a}^3 = \left(\frac{l_Y}{\lambda^2}, -\frac{l_X}{\lambda^2}, 0\right). \tag{56}$$

For the verification of these formulas, we consider nontrivial solutions of the equation

$$I_2(\mathbf{a}^2) \equiv \lambda^2(a_1^{2^2} + a_2^{2^2}) = 1. \tag{57}$$

Then substituting the vector $\mathbf{a}^2$ into (57), we get

$$l_X^2 + l_Y^2 = \lambda^2. \tag{58}$$

Hence, Eq. (57) is fulfilled identically. Let us consider I_3 and the vector $\mathbf{a}^3$ then we get again the equation of the this form.

With this, the operators of invariant differentiations read

$$\mathcal{D}^1 = D_t, \quad \mathcal{D}^2 = \frac{1}{\lambda^2}\big(l_X D_X + l_Y D_Y\big), \quad \mathcal{D}^3 = \frac{1}{\lambda^2}\big(l_Y D_X - l_X D_Y\big). \tag{59}$$

By applying the operators $\mathcal{D}^i$, $i = 1,2,3$ to the invariants J^j, $j = 1,\dots,4$, we get the following functionally independent differential invariants of the second order

$$J^5 = \frac{\Delta l}{\lambda^2}, \tag{60}$$

$$J^6 = l_{tt}, \tag{61}$$

$$J^7 = \frac{2[l_X^2 l_{XX} + l_X^2 l_{YY}^1 + 2 l_X l_Y l_{XY}] - (l_X\lambda_X^2 + l_Y\lambda_Y^2)(l_X^2 + l_Y^2)(\lambda^2)^{-1}}{\lambda^2}, \tag{62}$$

$$J^8 = \frac{2[l_X l_Y(l_{XX} - l_{YY}) + (l_Y^2 - l_X^2) l_{XY}] + (l_X\lambda_Y^2 - l_Y\lambda_X^2)(l_X^2 + l_Y^2)(\lambda^2)^{-1}}{\lambda^2}, \tag{63}$$

$$J^9 = -\frac{1}{2}\left[\frac{\Delta\lambda^2}{\lambda^2} - \frac{\lambda_X^2 + \lambda_Y^2}{\lambda^{2^3}}\right]. \tag{64}$$

Here Δ is the Laplace operator. The set $\mathbf{J}_2 = \{J^1,\dots,J^9\}$ is the complete set of differential invariants of order ≤ 2 wherein $\mathbf{J}_2$ denotes the second-order universal differential invariant. It is easily to verify that the set $\{J^1,\dots,J^9\}$ consists of all second-order differential invariants. The number of functionally independent differential invariants of order ≤ 2 equals $m_2 = \dim \mathcal{J}^2 - r_2$. Here $\mathcal{J}^2$ denotes the jet bundle of order 2 and r_2 is the maximal prolonged orbit dimension of the prolonged group orbit of *CGE*. Simple calculations give that $\dim \mathcal{J}^2 = 18$ and $r_2 = 9$. Therefore, we have $m_2 = 9$ and $\mathbf{J}_2$ contains 9 scalar functional independent invariants. Hence, $\mathbf{J}_2$ forms the complete set of differential invariants of the order $n \leq 2$. The table for commutators of the operators $\mathcal{D}^i$ reads

$$(\mathcal{D}^1\mathcal{D}^2 - \mathcal{D}^2\mathcal{D}^1) = 0, \tag{65}$$

$$(\mathcal{D}^1\mathcal{D}^3 - \mathcal{D}^3\mathcal{D}^1) = 0, \tag{66}$$

$$(\mathcal{D}^3\mathcal{D}^2 - \mathcal{D}^2\mathcal{D}^3) = \frac{J^8}{J^4}\mathcal{D}^2 - \frac{J^7}{J^4}\mathcal{D}^3 + J^5J^4\mathcal{D}^2. \tag{67}$$

We show how to obtain the differential invariants J^k with $3 \leq k \leq 9$ by applying the operators $\mathcal{D}^i$ to the single differential invariant J^2. The first series of the representations reads

$$J^3 = \mathcal{D}^1J^2, \tag{68}$$

$$J^4 = \mathcal{D}^2J^2, \tag{69}$$

$$J^6 = \mathcal{D}^1J^3, \tag{70}$$

$$J^7 = \mathcal{D}^2J^4, \tag{71}$$

$$J^8 = \mathcal{D}^3J^4. \tag{72}$$

The formulas above are obtained by direct calculations using the explicit forms of the operators $\mathcal{D}^i$. Then, we get that

$$J^5 = \frac{1}{J^8}(\mathcal{D}^3J^7 - \mathcal{D}^2J^8), \tag{73}$$

$$\begin{aligned} J^9 = \frac{1}{2(J^4)^2}\Big(&\mathcal{D}^2J^7 + \mathcal{D}^3J^8 \\ &- 2J^4\mathcal{D}^2J^5 + 3J^5J^7 - 2J^4(J^5)^2 - 2\frac{(J^7)^2 + (J^8)^2}{J^4}\Big). \end{aligned} \tag{74}$$

Exemplarily, the formulas (73), (74) follow by applying (67) to the invariants J^4. To prove the formula (74), we use the following syzygies

$$(\mathcal{D}^3\mathcal{D}^2 + \mathcal{D}^2\mathcal{D}^3) = J^4\frac{\Delta}{\lambda^2} + \frac{J^7}{J^4}\mathcal{D}^2 + \frac{J^8}{J^4}\mathcal{D}^3 - J^5\mathcal{D}^2 \tag{75}$$

which is verified by the direct substitution of the quantities into (75). Hence, the formula (74) follows by the operator (75) which is applied to J^4.

Notice that the invariant J^9 coincides with the curvature of the metric due to the identity

$$J^9 = -\frac{1}{2}\left[\frac{\Delta\lambda^2}{\lambda^{2^2}} - \frac{\lambda_X^{2^2} + \lambda_Y^{2^2}}{\lambda^{2^3}}\right] \equiv -\frac{1}{2}\frac{\Delta \ln \lambda^2}{\lambda^2}, \tag{76}$$

where the last term is the curvature of metric K.

To give more calculations, we consider the Beltrami parameters [4]

$$B_1 = \frac{K_X^2 + K_Y^2}{\lambda^2}, \tag{77}$$

$$B_2 = \frac{K_{XX} + K_{YY}}{\lambda^2}, \tag{78}$$

which are also differential invariants but of the third- and fourth-orders respectively of the group CGE. Since K is a differential invariant of CGE, the invariance of S_2 is obtained by the formula

$$\frac{\Delta K}{u^2} = \left(\frac{\mathcal{D}^3\mathcal{D}^2 + \mathcal{D}^2\mathcal{D}^3}{J^4} - \frac{J^7}{J^{4^2}}\mathcal{D}^2 - \frac{J^8}{J^{4^2}}\mathcal{D}^3 - \frac{J^5}{J^4}\mathcal{D}^2\right)K \tag{79}$$

due to the formula (75). The invariance of B_1 follows from the action of CGE on the variables X, Y, K, λ^2 that leads to the invariant (substituting K into J^4 instead of l in the formula (48))

$$B_1 = \frac{K_X^2 + K_Y^2}{\lambda^2}. \tag{80}$$

Consider the relationships (68)–(72) and (73), (74). Together with the commutators (65)–(67) they completely describe the second-order universal differential invariant $\mathbf{J}_2$ or the algebra of differential invariants of orders ≤ 2 with the identity

$$\left(\mathcal{D}^3\mathcal{D}^2 + \mathcal{D}^2\mathcal{D}^3\right)J = \left(J^4\frac{\Delta}{\lambda^2} + \frac{J^7}{J^4}\mathcal{D}^2 + \frac{J^8}{J^4}\mathcal{D}^3 - J^5\mathcal{D}^2\right)J \tag{81}$$

for arbitrary $J \in \mathbf{J}_2$. Moreover, we established that the invariant J^2 generates the invariants J^n with $n \geq 3$ from $\mathbf{J}_2$ by the operators $\mathcal{D}^i$, $i = 1, 2, 3$ and algebraic manipulations. Thus $\{J^1, J^2\}$ is the basis of generating differential invariants of $\mathbf{J}_2$.

4 Conclusions and outlook

The invariants of admitted groups of invariant transformations is an important part of physics. In particular, they are presented in practically all acts of measurement procedure for random vectors in thermodynamics [15]. The main geometric image that stands behind any measurement procedure is the Lagrangian manifold with Lagrangians in the corresponding variation problems with an action integral and the group of affine invariant transformations has a sense in thermodynamics. Therefore, the measurement quantities have much more and very special invariants. Central moments or

kth degree differential forms on the Lagrangian manifold define invariants which are specific i.e. look like differential invariants, see for details [15]. More formally, the measurement process of velocity requires to use the images of probability measures as measures on the vector space and with this, an admissible Legendrian manifold has to be considered. If we consider now the parallels between the above picture of measurement velocity vector with the measurement process of length of a random curve from SLE_κ we have to consider the *SLE*–measures which describe probability distributions on phase-separating curves. Then the Malliavin measures can be used which are conformally covariant too [10]. The fine properties discovered for SLE_κ process, that is, the Hausdorff dimension of the *SLE* equals $d_\kappa = 1 + \kappa/8$, $\kappa < 8$ leads to the existence of fractal variation along the curves of SLE_κ, $k < 8$ that was supported also in LERW, SAW, Ising and percolation lattice models, see [9]. With this, we define the action integral $L[\boldsymbol{X}(t)]$ using the notion of the fractal variation of random curves $\boldsymbol{X}(t) \in SLE_\kappa$. In fact, $L[\boldsymbol{X}(t)]$ looks like the (fractal) length scale defined on SLE_κ curves. Further, having it in mind and calculating the differential invariants up to the second-order, we gave the multiscale representation of invariants. Namely, for the second-order differential invariants, we proved that the differential invariants of orders ≤ 2 are presented by the universal invariant $\mathbf{J}_2$ (the multiscale representation of invariants). We used the machinery of operators of invariant differentiation $\mathcal{D}^i$, $i = 1, 2, 3$ to calculate all invariants J^n, $2 < n \leq 11$ from $\mathbf{J}_2$ by applying $\mathcal{D}^i$ to the generating invariant J^2. Further we can prove that $\{J^1, J^2\}$ is also the minimal set of generating invariants for arbitrary universal differential invariant $\mathbf{J}_k$ of order $k > 2$. It is based on a recurrence relation which expresses invariantly differential invariants of the algebra $\mathbf{A}$ in terms of a function of the differential invariants and their invariant derivatives by the operators $\mathcal{D}^i$ of the universal differential invariant $\mathbf{J}_{k-1}$ of a low order. The proof of this general result will be given in another paper.

Appendix: Lie group analysis of Eq. (36)

For convenience of the calculations, we denote $x^1 = X$, $x^2 = Y$ and $u^1 = l$ and $u^2 = \lambda^2$. The infinitesimal operator (37) we write in the form

$$\mathcal{Q} = \alpha^i(x^1, x^2, u^1, u^2)\frac{\partial}{\partial x^i} + \gamma^k(x^1, x^2, u^1, u^2)\frac{\partial}{\partial u^k}, \tag{82}$$

where $i, k = 1, 2$. The operator of the first prolongation of $\mathcal{Q}$ has the form

$$\mathcal{Q}_1 = \mathcal{Q} + \zeta_i^k \frac{\partial}{\partial u_i^k}. \tag{83}$$

Here

$$\zeta_i^k = \frac{\partial \gamma^k}{\partial x^i} + u_i^l \frac{\partial \gamma^k}{\partial u^l} - u_j^k \frac{\partial \alpha^j}{\partial x^i} - u_i^l u_j^k \frac{\partial \alpha^j}{\partial u^l}, \tag{84}$$

where $l = k = 1, 2$ and u^l_i (u^k_j) denotes the partial derivative of u^l (u^k) with respect to the variable x^i (x^j). The invariance of Eq. (36) under the action of $\mathcal{Q}$ means

$$\mathcal{Q}_1\big|_{(36)} \equiv u^1_i\left\{ \frac{\partial \gamma^1}{\partial x^i} + u^1_i \frac{\partial \gamma^1}{\partial u^2} - u^1_j\left(\frac{\partial \alpha^j}{\partial x^i} + u^1_i \frac{\partial \alpha^j}{\partial u^1} + u^2_i \frac{\partial \alpha^j}{\partial u^2} \right) \right\} \\ - \gamma^2 = 0. \tag{85}$$

Splitting with respect to the monomials $u^1_i u^2_i$ and $u^1_i u^2_i u^1_j$ gives

$$\frac{\partial \gamma^1}{\partial u^2} = 0, \quad \frac{\partial \alpha^j}{\partial u^2} = 0, \tag{86}$$

i.e. $\gamma^1 = \gamma^1(x^1, x^2, u^1)$, $\alpha^j = \alpha(x^1, x^2, u^1)$. Grouping the terms and doing simple algebraic manipulations in (85), we obtain

$$\frac{\partial \alpha^1}{\partial x^1} - \frac{\partial \alpha^2}{\partial x^2} = 0, \tag{87}$$

$$\frac{\partial \alpha^1}{\partial x^2} + \frac{\partial \alpha^2}{\partial x^1} = 0, \tag{88}$$

$$\frac{\partial \gamma^1}{\partial x^i} - u^2 \frac{\partial \alpha^i}{\partial u^1} = 0 \quad i = 1, 2, \tag{89}$$

$$2u^2\left(\frac{\partial \gamma^1}{\partial u^1} - \frac{\partial \alpha^1}{\partial x^1} \right) - \gamma^2 = 0. \tag{90}$$

Splitting (89) with respect to u^2 we obtain due to (89) the following equalities

$$\frac{\partial \gamma^1}{\partial x^1} = \frac{\partial \gamma^1}{\partial x^2} = 0, \tag{91}$$

i.e. $\gamma^1 = \gamma^1(u^1)$ and

$$\frac{\partial \alpha^1}{\partial u^1} = \frac{\partial \alpha^2}{\partial u^1} = 0. \tag{92}$$

It means that $\alpha^i = \alpha^i(x^1, x^2)$, $i = 1, 2$. Thus, we have $\alpha^j = \alpha^j(x^1, x^2)$ for $j = 1, 2$ and $\gamma^2 = \gamma^2(x^1, x^2, u^2)$. It follows from (90) and (92) that

$$\gamma^1 = \Theta(u^1), \quad \gamma^2 = 2\left(\frac{d\Theta}{du^1} - \frac{\partial \Phi}{\partial x^1} \right) u^2, \tag{93}$$

where Θ is an arbitrary function. Equations (87), (88) are the Cauchy-Riemann conditions that leads to $\alpha^1 = \xi^1(x^1, x^2)$ and $\alpha^2 = \xi^2(x^1, x^2)$ where $\xi^1(x^1, x^2)$ and $\xi^2(x^1, x^2)$ are arbitrary conjugate harmonic functions. Therefore, we get

$$\mathcal{Q} = \xi^1(X,Y)\frac{\partial}{\partial X} + \xi^2(X,Y)\frac{\partial}{\partial Y} + \Theta(l)\frac{\partial}{\partial l} + 2\left(\frac{d\Theta}{dl} - \xi^1_X\right)\lambda^2\frac{\partial}{\partial \lambda^2}. \tag{94}$$

A Lie algebra generated by the infinitesimal operator $\mathcal{Q}$ contains an infinite dimensional Lie subalgebra with the infinitesimal operator

$$\mathcal{P} = \xi^1(X,Y)\frac{\partial}{\partial X} + \xi^2(X,Y)\frac{\partial}{\partial Y} - 2\xi^1_X\lambda^2\frac{\partial}{\partial \lambda^2}. \tag{95}$$

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Declarations

Conflict of interest The authors declare there is no conflict of interest.

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