Stability of relativistic quantum electrodynamics in the Coulomb gauge

Christian D. Jäkel, and Walter F. Wreszinski

Stability of relativistic quantum electrodynamics in the Coulomb gauge

Christian D. Jäkel[1,a)] and Walter F. Wreszinski[2,b)]

[1]*Departamento de Matemática Aplicada, Universidade de São Paulo (USP), São Paulo, Brazil*
[2]*Instituto de Fisica, Universidade de São Paulo (USP), São Paulo, Brazil*

We show that relativistic quantum electrodynamics in the Coulomb gauge satisfies the following bound, which establishes stability: let $H(\Lambda, V)$ denote the Hamiltonian of QED_{1+3} on the three-dimensional torus of volume V and with ultraviolet cutoff Λ. Then there exists a constant $0 < \mu(\Lambda, V) < \infty$ (the vacuum energy renormalization) such that the renormalized Hamiltonian is positive: $H_{ren}(\Lambda, V) \equiv H_{\Lambda,V} + \mu_{\Lambda,V} \cdot \mathbb{1} \geq 0$. *Published by AIP Publishing.* https://doi.org/10.1063/1.5011031

I. INTRODUCTION

The proof of stability of non-relativistic matter interacting with a classical electromagnetic field is one of the crown jewels of mathematical physics, both from the point of view of physics and mathematics (see Ref. 14, Chapters 1–7 and 9 for a comprehensive exposition and references). It accounts for a wide, enormously rich class of phenomena in quantum mechanics, which are of crucial importance to the macroscopic world and even to everyday life.

There are, however, various phenomena which require the quantization of the electromagnetic field, such as spontaneous emission[16] and the black-body radiation, with its astoundingly perfect fit to the spectrum of the cosmic radiation background.[19] In addition, there is the well-known conceptual necessity to quantize the electromagnetic field,[3] an argument which does not extend to the gravitational field (as Dyson recently remarked[6]).

Quantization of the electromagnetic field is, however, a well-known source of trouble. Its coupling to non-relativistic matter has been studied extensively; see Ref. 14, Chap. 11, and the references given therein and Spohn's treatise,[17] Part II, Chapters 19 and 20, and also the references he cites. One particular important step was taken by Lieb and Loss[5] (see also Ref. 17, pp. 314–315), who established an upper bound $O(\Lambda^{12/7})$ to the ground state energy. As the latter disagrees with the result $O(\Lambda^2)$ suggested by perturbation theory, the bound by Lieb and Loss implies that perturbation theory cannot converge: it simply would yield a wrong picture of the electron cloud (see also Ref. 10 for an illuminating discussion).

For non-relativistic matter interacting with the radiation field, the term $\vec{p} \cdot \vec{A}_\Lambda$ in the Hamiltonian, where $\vec{p}$ stands for the momentum operator of a particle and $\vec{A}_\Lambda$ stands for the quantized electromagnetic vector potential field with ultraviolet cutoff Λ, *seems to indicate* a lower bound to the Hamiltonian only of type $-c\Lambda$, with $c > 0$ proportional to the number of static nuclei. Fröhlich[7] has remarked that, while such a bound proves stability of matter if an ultraviolet cutoff is imposed on the theory, the linear dependence on Λ is disastrous, physically speaking. He raised the question whether such a catastrophe does indeed prevail, relating it to Landau's conjectures (the so-called Landau pole; see Ref. 15). He also remarked, at the time (and in this respect the situation has not changed since) that *it was not known* (provided a mass renormalization M_Λ and a chemical potential renormalization μ_Λ are chosen appropriately) whether a lower bound on $H_\Lambda(M_\Lambda, \mu_\Lambda)$ can be found which is *uniform* in Λ.

---

a)jaekel@ime.usp.br
b)wreszins@gmail.com

Similar problems are expected in the case of relativistic quantum electrodynamics in the Coulomb gauge, where the term $\vec{j} \cdot \vec{A}_\Lambda$ plays a role similar to the above mentioned term $\vec{p} \cdot \vec{A}_\Lambda$, where now $\vec{j}$ denotes a (regularized) electron-positron current, but where, in addition, a charge renormalization is expected from perturbation theory. In this paper, we show that for the Hamiltonian of relativistic quantum electrodynamics in the Coulomb gauge, a suitable vacuum energy renormalization can be found such that the renormalized Hamiltonian (with no mass and no charge renormalization) is positive. In doing so, we provide a more complete picture of the electron cloud. It affects both the effective interaction between the electrons and their kinetic energy, yielding a picture of "dressed" electrons and positrons.

Our results are valid for any strictly positive and finite values of the (modulus of) bare electron charge and electron mass, but further finite renormalizations may be necessary to arrive at the corresponding physical values; see Sec. V.

In spite of the open problems regarding the form of the vacuum energy renormalization $-\mu_{\Lambda,V}$ (see Sec. VI), it is a c-number, and Theorem 4.5 proves that, after subtracting it, a positive renormalized Hamiltonian results. Thus, in the sense of the preceding discussion above, we have shown the non-perturbative existence of relativistic quantum electrodynamics.

II. QED$_{1+3}$ ON THE THREE-TORUS

We will study quantum electrodynamics in the Coulomb gauge, with a cutoff Hamiltonian

$$H(\Lambda, V) \doteq \underbrace{H^\circ_{ferm.}(V) + H^\circ_{bos.}(V)}_{=:H_\circ(V)} + H_{int}(V,\Lambda) + :H_{Coul.}(V,\Lambda):, \tag{1}$$

acting on a Hilbert space $\mathcal{H} \doteq \mathcal{H}_{ferm.} \otimes \mathcal{H}_{bos.}$ that is the tensor product of an antisymmetric Fock space $\mathcal{H}_{ferm.}$ and a symmetric Fock space $\mathcal{H}_{bos.}$. The one-particle space for both Fock spaces is $\ell^2(\Gamma_\kappa)$, with

$$\Gamma_V \doteq \left\{ \frac{2\pi\nu}{V^{1/3}} \mid \nu \in \mathbb{Z}^3 \right\} \quad \text{and} \quad \Gamma_\kappa \doteq \Gamma_V \cap \Lambda.$$

Here Λ is a *finite* set, symmetric under inversion about each coordinate plane containing the origin, and therefore invariant under inversion through the origin. The number of sites in Λ will be denoted by $|\Lambda|$. Since $|\Lambda|$ is finite, there are no antisymmetric *n*-particle functions for $n > |\Lambda|$. Hence, Pauli's principle ensures that $\mathcal{H}_{ferm.}$ is finite-dimensional; however, this argument does not hold for bosons, and $\mathcal{H}_{bos.}$ is, in fact, infinite-dimensional.

The first two operators on the r.h.s. in (1) denote the free massive fermion Hamiltonian

$$H^\circ_{ferm.}(V) = \sum_{p \in \Gamma_\kappa} \sum_{\ell \in \{1,2\}} \underbrace{\sqrt{p^2 + m^2}}_{=:\omega(p)} \left(b^*_\ell(p)b_\ell(p) + d^*_\ell(p)d_\ell(p) \right) \otimes \mathbb{1}, \tag{2}$$

and the free massless boson Hamiltonian

$$H^\circ_{bos.}(V) = \mathbb{1} \otimes \sum_{k \in \Gamma_\kappa} \sum_{J \in \{1,2\}} |k| \, a^*_J(k)a_J(k), \tag{3}$$

respectively. The constant $m > 0$ appearing in (2) is the electron mass. The photon (respectively, electron and positron) annihilation and creation operators $a_J(k), a^*_J(k)$ [respectively, $b_\ell(p)$, $b^*_\ell(p)$, $d_\ell(p)$, $d^*_\ell(p)$] are normalized so that

$$[a_J(k), a^*_{J'}(k')] = \delta_{k,k'} \delta_{J,J'},$$

and

$$\{b_\ell(p), b_{\ell'}(p')\} = \delta_{p,p'}\delta_{\ell,\ell'}, \qquad \{d_\ell(p), d^*_{\ell'}(p')\} = \delta_{p,p'}\delta_{\ell,\ell'},$$

where δ is the Kronecker delta. All other commutators (or anti-commutators) are zero. The operators $H^\circ_{ferm.}(V)$ and $H^\circ_{bos.}(V)$ depend on V (but not on Λ). In the sequel, however, we frequently deal with operators that depend on both V and Λ, and so it will be convenient to set $\kappa = (V, \Lambda)$.

The *photon-fermion interaction*, with periodic boundary conditions, is

$$H_{int}(\kappa) = -\int_V \mathrm{d}^3x \, J_\kappa(x) \cdot eA_\kappa(x). \tag{4}$$

The dot $\cdot$ denotes the scalar product of three-vectors in Euclidean space. The *electric current* $J_\kappa(x) = (J_\kappa^{(1)}(x), J_\kappa^{(2)}(x), J_\kappa^{(3)}(x))$ appearing in (4) is a three-vector-valued operator on $\mathcal{H}_{ferm}$, whose components can be expressed, using the Pauli matrices, in terms of the electron-positron fields $\Psi_\kappa(x)$,

$$J_\kappa^{(i)}(x) \doteq \langle \Psi_\kappa^\dagger(x), \alpha_i \Psi_\kappa(x) \rangle, \qquad \text{where } \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{5}$$

is a 4×4-matrix for each $i = 1, 2, 3$. The symbol $\langle .,. \rangle$ denotes the scalar product in $\mathbb{C}^4$. The star on an operator denotes adjoint and the dagger above in the definition of the current does not change the four vectors, in conformance with the above reference to the scalar product in $\mathbb{C}^4$; see below. The *electron-positron field* itself is given by

$$\Psi_\kappa(x) \doteq \frac{1}{V^{1/2}} \sum_{p \in \Gamma_\kappa} \sum_{\ell \in \{1,2\}} \left(b_\ell(p) \, u_\ell(p) \, e^{ip \cdot x} + d_\ell^*(p) \, v_\ell(p) \, e^{-ip \cdot x} \right), \tag{6}$$

$$\Psi_\kappa^\dagger(x) \doteq \frac{1}{V^{1/2}} \sum_{p \in \Gamma_\kappa} \sum_{\ell \in \{1,2\}} \left(b_\ell^*(p) \, u_\ell(p) \, e^{-ip \cdot x} + d_\ell(p) \, v_\ell(p) \, e^{ip \cdot x} \right), \tag{7}$$

with four-vectors

$$u_1(p) = \frac{1}{\sqrt{2\omega(p)(\omega(p)+m)}} \begin{pmatrix} 1 \\ 0 \\ p_3 \\ p_1 + ip_2 \end{pmatrix}, \qquad u_2(p) = \frac{1}{\sqrt{2\omega(p)(\omega(p)+m)}} \begin{pmatrix} 0 \\ 1 \\ p_1 - ip_2 \\ -p_3 \end{pmatrix},$$

$$v_1(p) = \frac{1}{\sqrt{2\omega(p)(\omega(p)+m)}} \begin{pmatrix} -p_1 + ip_2 \\ p_3 \\ 0 \\ 1 \end{pmatrix}, \qquad v_2(p) = \frac{1}{\sqrt{2\omega(p)(\omega(p)+m)}} \begin{pmatrix} p_3 \\ p_1 + ip_2 \\ 1 \\ 0 \end{pmatrix}.$$

The Fermi field $\Psi_\kappa(x)$ is a *bounded* operator for each x, but this statement does not hold for the vector potential $A_\kappa(x)$ appearing in (4). The latter is given by

$$A_\kappa(x) \doteq \frac{1}{V^{1/2}} \sum_{k \in \Gamma_\kappa \setminus \{0\}} \sum_{J \in \{1,2\}} \frac{\left(a_J(k) e^{ik \cdot x} + a_J^*(k) e^{ik \cdot x} \right)}{\sqrt{2|k|}} \, \epsilon_J(k). \tag{8}$$

The *polarisation vectors* can be chosen (see Ref. 12) such that for $k_i \geq 0$, $i = 1, 2, 3$,

$$\epsilon_1(k) \doteq \frac{1}{\sqrt{k_1^2 + k_2^2}} \begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix}, \qquad \epsilon_2(k) \doteq \frac{k}{|k|} \times \epsilon_1(k), \tag{9}$$

and $\epsilon_1(-k) = \epsilon_2(k)$, as well as $\epsilon_2(-k) = \epsilon_1(k)$. The question of the choice of polarization vectors in qed is a nontrivial matter; see also Ref. 13 and the forthcoming (38). Note that the three vectors $(\epsilon_1(k), \epsilon_2(k), \frac{k}{|k|})$ form an oriented orthonormal basis in $\mathbb{R}^3$. Hence, by (8),

$$\nabla \cdot A_\kappa(x) = 0,$$

which characterises *Coulomb gauge*. The final term in (1) represents the *Coulomb interaction*

$$:H_{Coul.}(\kappa): \doteq e^2 \int_{V \times V} d^3x \, d^3y \, \mathcal{V}_\kappa(x - y) :\Psi_\kappa^*(x) \Psi_\kappa(x) \Psi_\kappa^*(y) \Psi_\kappa(y): \otimes \mathbb{1}. \tag{10}$$

We note that, with the above notation, $H_{Coul.}(\kappa)$ denotes the instantaneous Coulomb interaction *without* normal ordering, i.e., without the Wick dots. Here $\mathcal{V}_\kappa(x)$ denotes a regularized Coulomb potential on the torus,

$$\mathcal{V}_\kappa(x) = \frac{1}{V} \sum_{k \in \Gamma_\kappa \setminus \{0\}} \frac{e^{ik \cdot x}}{|k|^2}, \qquad \kappa = (\Lambda, V). \tag{11}$$

In the limit $\lim_{V \to \infty} \lim_{\Lambda \to \infty} \mathcal{V}_{(\Lambda, V)}$ tends to the Coulomb potential in the distributional sense. Note that due to the exclusion of $k = 0$ in (8): $H_{int}(\kappa): = H_{int}(\kappa)$.

It was proven in Ref. 9 that there exists a dense set of vectors $\mathcal{D}$ in $\mathcal{H}$ on which $H(\Lambda, V)$ is essentially self-adjoint. The closure of $H(\Lambda, V)$ has a purely discrete spectrum with finite multiplicity, it is bounded from below, and the eigenfunctions of $H(\Lambda, V)$ lie in $\mathcal{D}$.

III. A FINITE-DIMENSIONAL GRASSMANN ALGEBRA

In the sequel, $c_\ell^*(p) = b_\ell^*(p)$, $d_\ell^*(p)$ will denote either an electron or a positron creation operator. To every vector $\Psi \in \mathcal{H}_{ferm.}$, written in the form

$$\Psi = \sum_{n=0}^\infty \sum_{p_i \in \Gamma_\kappa} \sum_{\ell_i \in \{1,2\}} K^{(n)}_{\ell_1,\ldots,\ell_n}(p_1,\ldots,p_n)\, c_{\ell_1}^*(p_1) \cdots c_{\ell_n}^*(p_n) \Omega_\circ, \tag{12}$$

we will now associate an element

$$\Psi(\mathbb{c}^*) = \sum_{n=0}^\infty \frac{1}{n!^{1/2}} \sum_{p_i \in \Gamma_\kappa} \sum_{\ell_i \in \{1,2\}} K^{(n)}_{\ell_1,\ldots,\ell_n}(p_1,\ldots,p_n)\, \mathbb{c}_{\ell_1}^*(p_1) \cdots \mathbb{c}_{\ell_n}^*(p_n)$$

of the Grassmann algebra $\mathcal{G}_n$ generated by anti-commuting symbols $\mathbb{c}_\ell(p)$ and $\mathbb{c}_{\ell'}^*(p')$,

$$\{\mathbb{c}_\ell(p), \mathbb{c}_{\ell'}(p')\} = \{\mathbb{c}_\ell(p), \mathbb{c}_{\ell'}^*(p')\} = \{\mathbb{c}_\ell^*(p), \mathbb{c}_{\ell'}^*(p')\} = 0, \tag{13}$$

with $p, p' \in \Gamma_\kappa$ and $\ell, \ell' \in \{1, 2\}$. The algebra $\mathcal{G}_n$ is of dimension $n = 2^{8|\Lambda|}$; see Ref. [1], p. 49. The set of Ψ's corresponding to the state vectors $\Psi \in \mathcal{H}_{ferm.}$ is denoted by $\mathcal{L}$.

To further simplify the notation, we enumerate the momenta in $p_i \in \Gamma_\kappa$, and the set

$$\mathbb{c}_{2i+\ell-1} \equiv \mathbb{c}_\ell(p_i), \qquad i = 1, 2, \ldots, 2^{8|\Lambda|-1}, \quad \ell \in \{1, 2\}.$$

Next, one introduces symbols $d\mathbb{c}_1, \ldots, d\mathbb{c}_n$, subject to the commutation relations

$$\{d\mathbb{c}_i, d\mathbb{c}_k\} = \{\mathbb{c}_k, d\mathbb{c}_i\} = \{d\mathbb{c}_i^*, d\mathbb{c}_k^*\} = \{\mathbb{c}_k^*, d\mathbb{c}_i^*\} = 0, \ \ i, k = 1, \ldots, 2^{8|\Lambda|}, \tag{14}$$

and defines single integrals

$$\int d\mathbb{c}_i = 0, \quad \int d\mathbb{c}_i^* = 0, \quad \int d\mathbb{c}_i\, \mathbb{c}_i = 1, \quad \text{and} \quad \int d\mathbb{c}_i^*\, \mathbb{c}_i^* = 1. \tag{15}$$

Multiple integrals are understood as iterated integrals. Thus, (14) and (15) define the integral

$$\int d\mathbb{c}_n \cdots d\mathbb{c}_1 f(\mathbb{c})$$

for all monomials and one can than extend the integrals to arbitrary elements by linearity.

The Grassmann algebra $\mathcal{G}_n$ has an involution (Ref. [1], p. 66) and, corresponding to the inner product $\langle \Psi_1, \Psi_2 \rangle$ in $\mathcal{H}_{ferm.}$, there is an associated inner product in $\mathcal{L}$,

$$\langle \Psi_1, \Psi_2 \rangle = \int \prod d\mathbb{c}d\mathbb{c}^*\, e^{-\mathbb{c}\mathbb{c}^*}\, \Psi_1 \Psi_2, \tag{16}$$

where $e^{-\mathbb{c}\mathbb{c}^*} \doteq \prod_{i=1}^n e^{-\mathbb{c}_i\mathbb{c}_i^*}$ and $\prod d\mathbb{c}d\mathbb{c}^* \doteq \prod_{i=1}^n d\mathbb{c}_i d\mathbb{c}_i^*$.

Lemma 3.1. The identity (16) converts $\mathcal{L}$ into a Hilbert space, which serves as a realization of the fermionic Fock space $\mathcal{H}_{ferm.}$ (Ref. [1], Theorem 3.1, p. 83).

We next discuss how a bounded operator A acting on $\mathcal{H}_{ferm.}$ is represented by an element $\mathbb{A} \in \mathcal{G}_n$, which naturally acts on $\mathcal{L}$. The connection is particularly simple, if A is given in its *normal form*

$$A = \sum_{m,n} \sum_{p_i, p_j' \in \Gamma_\kappa} \sum_{\ell_i, \ell_j' \in \{1,2\}} K^{(m,n)}_{\ell_1,\ldots,\ell_m,\ell_1',\ldots,\ell_n'}(p_1,\ldots,p_m,p_1',\ldots,p_n')$$
$$\times a_{\ell_1}^*(p_1) \cdots a_{\ell_m}^*(p_m) a_{\ell_1'}(p_1') \cdots a_{\ell_n'}(p_n'), \tag{17}$$

as this allows us to represent A by (see Ref. [1], p. 26),

$$\mathbb{A}(\mathbb{c}^*, \mathbb{c}) \doteq \sum_{m,n} \sum_{p_i, p_j' \in \Gamma_\kappa} \sum_{\ell_i, \ell_j' \in \{1,2\}} K^{(m,n)}_{\ell_1,\ldots,\ell_m,\ell_1',\ldots,\ell_n'}(p_1,\ldots,p_m,p_1',\ldots,p_n') \tag{18}$$
$$\times \mathbb{c}_{\ell_1}^*(p_1) \cdots \mathbb{c}_{\ell_m}^*(p_m) \mathbb{c}_{\ell_1'}(p_1')) \cdots \mathbb{c}_{\ell_n'}(p_n').$$

In fact, if $\Psi = A\,\Phi$, then Ψ corresponds to the element (using a suggestive notation)

$$\Psi(\mathbb{c}^*) = \int \prod d\mathbb{f}^* d\mathbb{f}\, e^{-(\mathbb{f}^* - \mathbb{c}^*)\mathbb{f}}\, \mathbb{A}(\mathbb{c}^*, \mathbb{f})\Psi(\mathbb{f}^*) \in \mathcal{L}\,.$$

See Ref. 1, Eq. (3.68), p. 84.

   Remark 3.1. It is important to notice that the correspondence between (17) and (18) only holds for operators A equal to sums of normal forms, in which A are (at most) linear in each of the $b_\ell(p)$, $d_\ell(p)$, $b_\ell^*(p)$, and $d_\ell^*(p)$, due to the Pauli exclusion principle.

   Returning to the model introduced in Sec. II, we shall consider expressions of the form

$$\langle \Omega, \mathbb{H}(\Lambda, V)\Omega \rangle,$$

where $\Omega \in \mathcal{L} \otimes \mathcal{H}_{bos.}$ is a vector of unit norm of the form

$$\Omega = \sum_{n,m} \frac{1}{\sqrt{n!\,m!}} \sum_{p_i, p_j' \in \Gamma_\kappa} \sum_{\ell_i, J_k \in \{1,2\}} K_{\ell_1,\dots,\ell_m, J_1,\dots,J_n}^{(m,n)}(p_1, \dots, p_m, p_1', \dots, p_n')$$

$$\times \mathbb{c}_{\ell_1}^*(p_1) \cdots \mathbb{c}_{\ell_m}^*(p_m)\, a_{J_1}^*(p_1') \cdots a_{J_n}^*(p_n')\Omega_{bos.}^\circ. \tag{19}$$

Here $\Omega_{bos.}^\circ$ denotes the vacuum (no-particle state) in $\mathcal{H}_{bos.}$, the sums over n and m are finite, and

$$\sum_{m,n} \sum_{p_i, p_j' \in \Gamma_\kappa} \sum_{\ell_i, J_k \in \{1,2\}} |K_{\ell_1,\dots,\ell_m, J_1,\dots,J_n}^{(m,n)}(p_1, \dots, p_m, p_1', \dots, p_n')|^2 = 1.$$

The $a_J^*(k)$'s appearing in (19) are the photon creation operators.

   Proposition 3.2. The set of vectors Ω of the form (19) form a dense set in $C^\infty(H_\circ(V))$, on which $H(\kappa)$ is essentially self-adjoint by Ref. 9, Theorem 3.1.

IV. THE FIRST UNITARY TRANSFORMATION

   In this section, we apply a unitary transformation to the Hamiltonian

$$\mathbb{H}(\kappa) = \mathbb{H}_{ferm.}^\circ(V) + \mathbb{H}_{Coul.}(\kappa) + \mathbb{H}_{int}(\kappa) + H_{bos.}^\circ(V), \tag{20}$$

where both

$$\mathbb{H}_{ferm.}^\circ(V) = \sum_{p \in \Gamma_\kappa} \sum_{\ell \in \{1,2\}} \sqrt{|p|^2 + m^2}\left(\mathbb{b}_\ell^*(p)\mathbb{b}_\ell(p) + \mathbb{d}_\ell^*(p)\mathbb{d}_\ell(p)\right) \otimes \mathbb{1}$$

and

$$\mathbb{H}_{Coul.}(\kappa) = e^2 \int_{V \times V} d^3x\, d^3y\, \mathcal{V}_\kappa(x - y)\Psi_\kappa^*(x)\Psi_\kappa(x)\Psi_\kappa^*(y)\Psi_\kappa(y) \otimes \mathbb{1}$$

act trivially on the bosonic Fock space $\mathcal{H}_{bos.}$.
   Inspecting the parts of $\mathbb{H}(\kappa)$, which involve bosonic creation and annihilation operators, i.e.,

$$H_{bos.}^\circ(\kappa) = \sum_{k \in \Gamma_\kappa} \sum_{J \in \{1,2\}} |k|\, a_J^*(k)a_J(k),$$

$$\mathbb{H}_{int}(\kappa) = -e \int_V d^3x\, \mathbb{J}_\kappa(x) \cdot \underbrace{\sum_{\substack{k \in \Gamma_\kappa \setminus \{0\} \\ J \in \{1,2\}}} \frac{\left(a_J(k)e^{ik \cdot x} + a_J^*(k)e^{ik \cdot x}\right)\epsilon_J(k)}{\sqrt{2V\,|k|}}}_{=A_\kappa(x)},$$

one may try to "complete the square" by adding and subtracting a term of the form

$$\mathbb{H}_{Curr.}(\kappa) = e^2 \sum_{k \in \Gamma_\kappa \setminus \{0\}} \sum_{J \in \{1,2\}} \frac{\left(\widetilde{\mathbb{J}_\kappa}(k) \cdot \epsilon_J(k)\right)\left(\widetilde{\mathbb{J}_\kappa}(-k) \cdot \epsilon_J(k)\right)}{2|k|^2},$$

which is quadratic in $\mathbb{J}_\kappa(x)$. In fact, there is a unitary operator, namely,

$$\mathbb{U}_\kappa \doteq \prod_{\substack{k \in \Gamma_\kappa \backslash \{0\} \\ J \in \{1,2\}}} \exp\left(-\frac{e\left((\widetilde{\mathbb{J}_\kappa}(k) \cdot \epsilon_J(k)) a_J^*(k) - (\widetilde{\mathbb{J}_\kappa}(k) \cdot \epsilon_J(k))^* a_J(k) \right)}{\sqrt{2|k|^3}} \right),$$

which accomplishes this task.

Proposition 4.1. As a quadratic form on $\mathcal{L} \otimes \mathcal{H}_{bos}$,

$$\mathbb{U}_\kappa^* \left(H_{bos.}^\circ(\kappa) + \mathbb{H}_{int}(\kappa) \right) \mathbb{U}_\kappa = \underbrace{\sum_{J \in \{1,2\}} \sum_{k \in \Gamma_\kappa} |k| \, \mathbb{o}_J^*(k) \mathbb{o}_J(k)}_{:= \mathbb{H}_{bos.}^{mod.} \geq 0} - \mathbb{H}_{Curr.}(\kappa), \tag{21}$$

where the positive first term on the r.h.s. is build up from dressed bosonic creation and annihilation operators

$$\mathbb{o}_J^*(k) = \mathbb{1} \otimes a_J^*(k) + \frac{\widetilde{\mathbb{J}_\kappa}(k) \cdot \epsilon_J(k)}{\sqrt{2|k|^3}} \otimes \mathbb{1}$$

and

$$\mathbb{o}_J(k) = \mathbb{1} \otimes a_J(k) + \frac{\widetilde{\mathbb{J}_\kappa}(-k) \cdot \epsilon_J(k)}{\sqrt{2|k|^3}} \otimes \mathbb{1}.$$

Moreover,

$$\mathbb{U}_\kappa^* \mathbb{H}_{Coul.}(\kappa) \mathbb{U}_\kappa = \mathbb{H}_{Coul.}(\kappa). \tag{22}$$

Remark 4.1. Note that the action of $\mathbb{o}_J^*(k)$ and $\mathbb{o}_J(k)$ on $\mathcal{H}_{ferm.}$ is no longer trivial, as U_κ mixes the components in the tensor product $\mathcal{H}_{ferm.} \otimes \mathcal{H}_{bos.}$. The photons are thus "dressed" by the interaction too. This linkage of the electron Hilbert space to the photon Fock space also appears in the relativistic model of Lieb and Loss[11] in which, however, there are no positrons and the fermion subspace is restricted to N electrons. A better understanding of the relation between the two models is desirable.

Proof. The $-e\widetilde{\mathbb{J}_\kappa}(k) \cdot \epsilon_J(k)$ are Grassmann variables, which commute with all the other Grassmann variables. Thus, (21) is a consequence of

$$\mathbb{U}_\kappa^* \, a_J(k) \, \mathbb{U}_\kappa = \mathbb{o}_J(k), \qquad \mathbb{U}_\kappa^* \, a_J^*(k) \, \mathbb{U}_\kappa = \mathbb{o}_J^*(k).$$

The final statement follows from the fact that on $\mathcal{L}$, the Grassmann variables commute. Hence, $[\mathbb{H}_{Coul.}(\kappa), \widetilde{\mathbb{J}_\kappa}(k)] = 0$. $\square$

In the original representation on $\mathcal{H}_{ferm.} \otimes \mathcal{H}_{bos.}$, the current-current interaction $\mathbb{H}_{Curr.}(\kappa)$ is represented [using the correspondence (17) and (18)] by the normal product $:H_{Curr.}(\kappa):$ of

$$H_{Curr.}(\kappa) \doteq \sum_{J \in \{1,2\}} \frac{e^2}{2|k|^2} (\widetilde{J_\kappa}(k) \cdot \epsilon_J(k)) (\widetilde{J_\kappa}(-k) \cdot \epsilon_J(k)) \otimes \mathbb{1}. \tag{23}$$

Recalling (5), we conclude that the difference

$$H_{Curr.}^{trunc.}(\kappa) := H_{Curr.}(\kappa) - :H_{Curr.}(\kappa): \tag{24}$$

consists of a sum of four terms

$$H_{Curr.}^{trunc.}(\kappa) = \sum_{i=1}^4 \underbrace{\left(\sum_{p,k \in \Gamma_\kappa \backslash \{0\}} \sum_{\ell,\ell',\ell'' \in \{1,2\}} c_{\ell,\ell',\ell''}^{(i)}(p,k) E_{\ell,\ell',\ell''}^{(i)}(p,k) \right)}_{=E_i}, \tag{25}$$

obtained by replacing terms of the form $b_\ell(p) b_\ell^*(p)$ and $d_\ell(p) d_\ell^*(p)$ occurring in $H_{Curr.}(\kappa)$ by the identity operator $\mathbb{1}$: let $\mathcal{E}(k)$ denote the 4×4 matrix

$$\mathcal{E}(k) = \frac{1}{2V|k|^2} \sum_{J \in \{1,2\}} \sum_{i=1}^3 \epsilon_J^{(i)}(k) \alpha_i,$$

then

$$E^{(1)}_{\ell,\ell',\ell''}(p,k) = b^*_{\ell'}(p+k)b_{\ell''}(p-k) \otimes \mathbb{1},$$

$$E^{(2)}_{\ell,\ell',\ell''}(p,k) = b^*_{\ell'}(p+k)d^*_{\ell''}(k-p) \otimes \mathbb{1},$$

$$E^{(3)}_{\ell,\ell',\ell''}(p,k) = d_{\ell'}(-p-k)b_{\ell''}(p-k) \otimes \mathbb{1},$$

$$E^{(4)}_{\ell,\ell',\ell''}(p,k) = d^*_{\ell''}(-p+k)d_{\ell'}(-p+k) \otimes \mathbb{1},$$

with coefficients

$$c^{(1)}_{\ell,\ell',\ell''}(p,k) = \langle u_{\ell'}(p+k), \mathcal{E}(k)u_\ell(p)\rangle\langle u_\ell(p), \mathcal{E}(k)u_{\ell''}(p-k)\rangle,$$

$$c^{(2)}_{\ell,\ell',\ell''}(p,k) = \langle u_{\ell'}(p+k), \mathcal{E}(k)u_\ell(p)\rangle\langle u_\ell(p), \mathcal{E}(k)v_{\ell''}(k-p)\rangle,$$

$$c^{(3)}_{\ell,\ell',\ell''}(p,k) = \langle u_\ell(p), \mathcal{E}(k)u_{\ell''}(p-k)\rangle\langle v_{\ell'}(-p-k), \mathcal{E}(k)u_\ell(p)\rangle,$$

$$c^{(4)}_{\ell,\ell',\ell''}(p,k) = \langle v_{\ell'}(-p-k), \mathcal{E}(k)u_\ell(p)\rangle\langle u_\ell(p), \mathcal{E}(k)v_{\ell''}(-p+k)\rangle. \tag{26}$$

Note that $E_1 = E_1^*$, $E_3 = E_2^*$, $E_4 = E_4^*$.

An interesting aspect of the explicit form of $H_{Curr.}(\kappa)$ given in (23) is that is shows a certain similarity to the Coulomb interaction $H_{Coul.}(\kappa)$ introduced in (10). We can explore this fact by establishing an inequality which generalizes the positivity bound for the Coulomb energy.

Proposition 4.2. Let $M(k)$ be the $\mathbb{C}^4 \otimes \mathbb{C}^4$-valued matrix given by

$$M(k) \equiv \mathbb{1} \otimes \mathbb{1} - \frac{1}{2}\alpha\epsilon_1(k) \otimes \alpha\epsilon_1(k) - \frac{1}{2}\alpha\epsilon_2(k) \otimes \alpha\epsilon_2(k) \tag{27}$$

and let $\mathcal{V}$ be (up to a constant $\frac{1}{V}$) the Coulomb kernel introduced in (11). Then

$$\int_{V \times V} d^3x d^3y \, \mathcal{V}(x-y)\big((\Psi(x)^* \otimes \Psi(y)^*), M(k)(\Psi(x) \otimes \Psi(y))\big) \geq 0 \tag{28}$$

in the sense of quadratic forms on $\mathcal{H}_{ferm.}$. Note that the $\mathbb{C}^4$-valued bounded operators $\Psi(x)$ and $\Psi^*(x)$ were defined in (6).

Proof. We may write $M(k)$ in the form

$$4M(k) = \big((\mathbb{1} - \alpha\epsilon_1(k)) \otimes (\mathbb{1} + \alpha\epsilon_1(k)) + (\mathbb{1} + \alpha\epsilon_1(k)) \otimes (\mathbb{1} - \alpha\epsilon_1(k))\big)$$

$$+ \big((\mathbb{1} - \alpha\epsilon_2(k)) \otimes (\mathbb{1} + \alpha\epsilon_2(k)) + (\mathbb{1} + \alpha\epsilon_2(k)) \otimes (\mathbb{1} - \alpha\epsilon_2(k))\big).$$

By a unitary transformation, we can convert each of the four summands in $4M(k)$ to a direct product of diagonal matrices $D_{1,i} \otimes D_{2,i}$, $i = 1, \ldots, 4$. The eigenvalues of each $D_{1,i}, D_{2,i}$, $i = 1, \ldots, 4$ are zero and two, each with multiplicity two, as may be verified by straightforward diagonalisation, independent of k, because they depend only on the Euclidean norms $|\epsilon_1(k)|$ and $|\epsilon_2(k)|$, which are both equal to one. The inequality (28) then follows. □

Corollary 4.3. $H_{CC} \doteq H_{Coul.}(\kappa) - H_{Curr.}(\kappa) \geq 0$.

Proof. Inspecting (27), we see that $H_{Coul.}(\kappa)$ as defined in (10) and $H_{Curr.}(\kappa)$ defined in (23) can be combined to yield the expression on the l.h.s. in (28). □

We will also need a bound on general Fermi operators. As before, the Fermi creation operators c_i^* and annihilation operators c_j will satisfy canonical anti-commutation relations, i.e., $\{c_i, c_k^*\} = \delta_{i,k}$, and the remaining anti-commutators are all equal to zero.

Lemma 4.4 (Bogolubov and Bogolubov[2]). Let F be a self-adjoint fermionic Hamiltonian of the general form

$$F = \sum_{i,j=1}^n A_{i,j}c_i^*c_j + \frac{1}{2}\sum_{i,j=1}^n B_{i,j}c_i^*c_j^* + \frac{1}{2}\sum_{i,j=1}^n B_{i,j}^*c_ic_j, \tag{29}$$

where the matrix $A = (A_{i,j})$ is hermitian and the matrix $B = (B_{i,j})$ is anti-symmetric, i.e.,

$$A = A^*, \qquad B = -B^T \qquad (\textit{T denotes the transpose}).$$

Then there exists a constant μ such that $F + \mu \cdot \mathbb{1} \geq 0$.

  Proof. It has been shown in Ref. 2 that a Hamiltonian of the form (29) can always be written as

$$F = \sum_{\ell=1}^{n} \lambda_\ell \, q_\ell^* q_\ell + \mu \cdot \mathbb{1}, \qquad \lambda_\ell \geq 0, \tag{30}$$

with new Fermi creation and annihilation operators

$$q_\ell^* = \sum_{i=1}^{n} (\overline{U_{i,\ell}} \, c_i^* + \overline{W_{i,\ell}} \, c_i), \quad q_\ell = \sum_{i=1}^{n} (U_{i,\ell} \, c_i + W_{i,\ell} \, c_i^*), \tag{31}$$

and the new Fermi operators $q_\ell, q_{\ell'}^*$ satisfying canonical anti-commutation relations. The matrices $U = (u_{i,\ell})$ and $W = (v_{i,\ell})$ appearing in (31) satisfy

$$AU + BW^* = U\Lambda, \qquad -B^*U - A^*W^* = W^*\Lambda, \tag{32}$$

and μ is given by $\mu = -\sum_{i=1}^{n} \lambda_i \operatorname{Tr} W^*W$ and Λ denotes the diagonal $n \times n$ matrix of the $\{\lambda_j\}$. $\square$

  Theorem 4.5. *There exists a finite real number $\mu(\kappa) > 0$ (the vacuum energy renormalization) such that the first renormalized Hamiltonian*

$$\underbrace{H(\kappa) + \mu(\kappa) \cdot \mathbb{1}}_{\equiv H_{1,ren.}(\kappa)} \geq 0, \tag{33}$$

as a quadratic form on $\mathcal{H}_{ferm.} \otimes \mathcal{H}_{bos.}$. Here $H(\kappa)$ is the original Hamiltonian describing qed in the Coulomb gauge introduced in (1).

  Proof. It follows from (20)–(22) that

$$\mathbb{H}(\kappa) \geq \mathbb{H}_{ferm.}^{\circ}(V) + \mathbb{H}_{Coul.}(\kappa) - \mathbb{H}_{Curr.}(\kappa), \tag{34}$$

as operators on $\mathcal{L} \otimes \mathcal{H}_{bos}$. Moreover, by the correspondence (17) and (18), $\mathbb{H}_{Coul.}(\kappa)$ is represented on $\mathcal{H}_{ferm}$ by

$$:H_{Coul.}(\kappa): = H_{Coul.}(\kappa) - \sum_{i=1}^{4} E_i^{tr.},$$

where $E_i^{trunc.}$ arises by replacing the term $\frac{1}{2} \sum_J \sum_i \alpha_i \epsilon_J^{(i)}(k)$ in the explicit expressions for the E_i's appearing in the identity.

  Coming back to the original representation on $\mathcal{H}_{ferm} \otimes \mathcal{H}_{bos}$, taking into account that the operator on the r.h.s. of (34) acts trivially on $\mathcal{H}_{bos}$, and using (24)–(26), (34), and the above-mentioned representation for $\mathbb{H}_{Coul.}(\kappa)$, it follows that

$$H(\kappa) \geq H_{ferm.}^{\circ}(V) - \underbrace{\left(H_{Curr.}(\kappa) - \sum_{i=1}^{4} E_i(\kappa)\right)}_{=: \, :H_{Curr.}(\kappa):} + \underbrace{H_{Coul.}(\kappa) - \sum_{i=1}^{4} E_i^{tr.}(\kappa)}_{=: \, :H_{Coul.}(\kappa):}$$

$$\geq \underbrace{H_{ferm.}^{\circ}(V) + \sum_{i=1}^{4} \left(E_i(\kappa) - E_i^{tr.}(\kappa)\right)}_{=: H_{ferm.}^{mod.}(\kappa) \geq -\mu(\kappa) \cdot \mathbb{1}} + \underbrace{H_{Coul.}(\kappa) - H_{Curr.}(\kappa)}_{=H_{CC} \geq 0}$$

$$\geq -\mu(\kappa) \cdot \mathbb{1}. \tag{35}$$

In the last inequality, we have used Corollary 4.3 and the fact that according to its definition given in (35) the Fermi operator $H_{ferm.}^{mod.}(\kappa)$ is of the general form described in (29). [Note that we have replaced $H_{ferm.}^{\circ}(V)$ by $H_{ferm.}^{\circ}(\kappa)$; the latter is the free fermion Hamiltonian with the restriction to $k \in \Gamma_\kappa$.] $\square$

Remark 4.2. The bound holds for all finite V and Λ. In particular, if $E(\kappa)$ denotes the lowest eigenvalue of $H(\kappa) + \mu(\kappa)$, then the lowest accumulation point of the sequence $\{E(\kappa)\}$ is positive.

Remark 4.3. As remarked above, the form of the representation of $\mathbb{H}_{Coul.}(\kappa)$ on $\mathcal{H}_{ferm}$ is simply due to the Wick dots in (10). This choice is, on the other hand, motivated by the fact that in perturbation theory there are cancellations between the instantaneous Coulomb interaction (10) and the transverse term (4), leading to a final covariant propagator (Ref. 16, pp. 252–253). Since $:H_{Curr}(\kappa):$ is the operator which arises from the transverse term as a consequence of the first unitary transformation, it seems advisable to define the instantaneous Coulomb interaction in an analogous way, in order that a covariant propagator may arise in the limits $\Lambda \to \infty$, followed by $V \to \infty$.

V. THE ISSUE OF TRIVIALITY

There are two different reasons why the present theory might be trivial.

Assume we are given a bounded self-adjoint Hamiltonian on a Hilbert space $\mathcal{H}$ of the form

$$H_\kappa = \sum_{k \in \Gamma_\kappa \backslash \{0\}} C_\kappa(k),$$

where C_κ are bounded self-adjoint operators on $\mathcal{H}$. Since $\sum_{k \in \Gamma_\kappa \backslash \{0\} \cap B} \mathbb{1} = O(V)$ for any fixed ball B in k-space, we say that H_κ is extensive, and it is called locally (infrared) finite if

$$\sup_V \frac{1}{V} \sum_{k \in \Gamma_\kappa \backslash \{0\} \cap B} \|C_\kappa(k)\| \le c < \infty$$

for some c independent of κ and any fixed ball B centered at the origin in k-space. Such a bound is presumably equivalent to a bound of the form d/V in the spacings of the levels of $H_{\Lambda,V}$, with $d > 0$ independent of Λ, V, as $V \to \infty$, and, if it does not hold, it may be conjectured that the whole spectrum of the physical Hamiltonian (excluding the fixed value of the vacuum) diverges to $+\infty$. This is therefore believed to be a source of *triviality* associated with the infrared problem.

In the present case, since the photons have been *decoupled* by applying U_κ, the sums over $k \in \Gamma_\kappa \backslash \{0\}$ in H_{CC} and $H_{ferm.}^{mod.}(\kappa)$ in (29) might lead to divergence (to $+\infty$) in the matrix elements of H_{CC} and $H_{ferm.}^{mod.}(\kappa)$ (as $V \to \infty$). Since $H_{CC} := H_{Coul.}(\kappa) - H_{Curr.}(\kappa)$ is equal to the l.h.s. in (43), and together with the fact that

$$\sup_V \frac{1}{V} \sum_{k \in (B \cap \Gamma_\kappa) \backslash \{0\}} \frac{1}{|k|^2} \le c < \infty, \tag{36}$$

where B is some fixed ball centered at the origin and c is a constant independent of V, H_{CC} is seen to be locally infrared finite according to the definition above.

For $H_{ferm.}^{mod.}(\kappa)$, one should examine E_i, $i = 1, \ldots, 4$ in (4). This is seen to involve a point-wise bound on the polarization vectors (9). This requires some manipulations (we remind the reader that we follow the notation of Sakurai's book[16]). We have

$$\sum_\sigma v_{p,\sigma} v_{p,\sigma}^+ = -\frac{(i\gamma \cdot p + m)}{2m}, \quad \sum_\sigma u_{p,\sigma} u_{p,\sigma}^+ = \frac{(-i\gamma \cdot p + m)}{2m}. \tag{37}$$

Using (37), we may write the matrix summands in (26), which involve the polarization vectors, in the typical form (up to some changes of sign and symbols)

$$(\gamma \cdot \epsilon_{k,\alpha})(\gamma \cdot p_1)(\gamma \cdot \epsilon_{k,\alpha}) = \gamma \cdot \epsilon_{k,\alpha}(2p_1 \cdot \epsilon_{k,\alpha}) - (\gamma \cdot \epsilon_{k,\alpha})(\gamma \cdot p_1)$$
$$= 2p_1 \cdot \epsilon_{k,\alpha}\gamma \cdot \epsilon_{k,\alpha} - \gamma \cdot p_1. \tag{38}$$

We now denote the absolute value of a number a by $\|a\|$ and the components of a three-vector v by v_i, $i = 1, \ldots, 3$. We have

$$\|2p_1 \cdot \epsilon_{k,\alpha}\| \leq 2\,|p_1|\,|\epsilon_{k,\alpha}| = 2\,|p_1|. \tag{39}$$

Recall that the dot denotes the scalar product in Euclidean three-space and |.| denotes the Euclidean norm. By (38) and (39), $H_{ferm.}^{mod.}(\kappa)$ is proved to be locally infrared finite if we succeed in finding a bound

$$\sup_{V,\alpha} \frac{1}{V} \sum_{k \in (B \cap \Gamma_\kappa) \setminus \{0\}} \frac{1}{|k|^2} \max_{i=[1,3]} \|\epsilon_{k,\alpha,i}\| \leq c < \infty, \tag{40}$$

again for some fixed ball centred at the origin. Verification of (40) is straightforward, using (9).

A second possible reason for triviality is *charge renormalization*. Since the cutoff on the fermions should be independent of that in the photon field (as, in fact, adopted in Ref. 9). We now set a different cutoff Λ' on the fermion field (6) for the purpose of discussion. Examination of H_{CC} shows that setting

$$e^2 \equiv e_{bare}^2 = O\left(|\Lambda|^{(-1-\epsilon)/3}\right) e_{ren}^2 \tag{41}$$

(recall that $|\Lambda|$ denotes the number of sites in the set Λ, which for a cube is of the same order as of a sphere of radius $\Lambda^{1/3}$), we would obtain

$$\lim_{\Lambda \to \infty} H_{CC} = 0,$$

and the coefficients in (26) show that

$$\lim_{\Lambda \to \infty} \sum_{i=1}^{4} \left(E_i(\kappa) - E_i^{tr.}(\kappa)\right) = 0$$

[note that $E_i(\kappa)$ and $E_i^{tr.}(\kappa)$ both depend on Λ, which was not explicitly indicated], under the same condition (41). Thus, a suitable "charge renormalization" trivializes the theory, in the sense that the Hamiltonian reduces to the *free* Hamiltonian, when the ultraviolet cutoff is removed (for fixed $m = m_{bare}$). The well-known charge renormalization in qed (see Ref. 16, pp. 279–283 or Ref. 18, pp. 445, 462)

$$e_{bare}^2 = \frac{1}{Z_3} e_{ren}^2, \quad \text{with} \quad Z_3 = 1 - \frac{e^2}{12\pi^2} \log \frac{\Lambda^{2/3}}{m^2}, \tag{42}$$

would *not*, however, trivialize the theory. We conclude that there are no *a priori* reasons leading to suspect that qed, either with bare parameters or (42), is a trivial theory. We emphasize, however, that there is also no *a priori* reason why (42) must be incorporated to show global existence of qed, and, for this reason, we did not investigate in detail the effect of replacing $m = m_{bare}$ by its renormalized version;[16] see also Ref. 7 for a discussion of mass renormalization in non-relativistic qed and references. For instance, a non-trivial S matrix (with no mass or charge renormalizations) would not be contradictory with the same S matrix having an asymptotic, but divergent, asymptotic expansion in terms of the fine structure constant, the latter, however, making sense only with the usual mass and charge renormalisations (Refs. 18 and 16).

On the other hand, it is well known that a finite renormalization of mass and charge (or field strength renormalization) may be necessary to arrive at the physical values of mass and charge, because a nonperturbative definition of renormalized fields exists, which may be also stated in terms of the Källén-Lehmann spectral representation of the two-point function (see Ref. 18, pp. 438–439 and p. 458 *et seq.*). This depends on further study, either of the dynamics (two-point function) or of the non-relativistic limit of the theory, in terms of which both the physical charge (for small momentum transfer; see Ref. 16, p. 279) and the physical mass (Ref. 16, p. 69) are conventionally defined.

VI. CONCLUSION

Our method, using the space $\mathcal{L}$ of Sec. III, permits the use of a unitary transformation to convert the Fermi Hamiltonian of QED_{1+3} in the Coulomb gauge to the form (21), in which fermions interact both through the instantaneous Coulomb force and an additional term incorporating the "photon

cloud." Going back to the original fermion Fock space, the free fermion Hamiltonian is modified by an additional quadratic term,

$$H^{mod.}_{ferm.}(\kappa) = H^{\circ}_{ferm.}(V) - H^{trunc.}_{Curr.}(\kappa) - H^{trunc}_{Coul.}(\kappa),$$

which may be interpreted as a *Bremsstrahlung* term (see Ref. 16, pp. 229–231). Thus, the *photon cloud* also acts to *dress* the electron-positron field. This suggests that the spectrum of the physical Hamiltonian $H_{phys.}$ may be purely absolutely continuous, as suggested by Buchholz's important result[4] (proven under very reasonable assumptions) that in QED_{1+3} there exist no eigenvalues of the mass operator $M^2 = H^2_{phys.} - \vec{P}^2_{phys.}$, where $\vec{P}_{phys.}$ denotes the physical momentum. In this connection, the Källén-Lehmann two-point function of the fields $F_{\mu,\nu}$, which should not depend on the choice of gauge, are expected to have no delta contribution at zero mass (corresponding to $Z_3 = 0$; see Ref. 18, p. 458 *et seq.*), and this is consistent with the fact that the photons are also "dressed," see Remark 4.1.

The most significant point about the unitary operator U_κ is that it generalizes well-known transformations which, in the limits $\Lambda \to \infty$, followed by $V \to \infty$, lead to *inequivalent representations* of the canonical commutation relations, for multiple reasons, reviewed in Wightman's lectures in Ref. 8: infrared and ultraviolet divergences [note that due to $\lambda_k = (\omega_k\sqrt{2\omega_k})^{-1}$, $\int dk\, |\lambda_k|^2$ has both infrared and ultraviolet logarithmic divergences] as well as Euclidean invariance associated with Haag's theorem and vacuum polarization. The fact that non-Fock representations are imperative in the theory without cutoffs explains that we have obtained an apparently more realistic physical picture in the cutoff version, with reasonable properties when the cutoffs are removed (Theorem 4.5).

It is possible that

$$\mu(\kappa) = V(-\rho + \nu(\Lambda)), \tag{43}$$

where $\nu(\Lambda) \to \infty$ as $\Lambda \to \infty$. An indication that $\mu = O(V)$ is given by the fact that it involves a sum over momentum modes (see Proposition 4.1) and a divergent part of type $\nu(\Lambda)V$ would be the analog of the "chemical potential renormalization" mentioned in the introduction in connection with non-relativistic qed, with V replacing N, the number of electrons, which is not a good quantum number in the relativistic case.

It is, however, of crucial importance to investigate the unitary transformation of Proposition 4.1 in greater detail because ρ, in (43), should be related to the ground state energy of positronium (Ref. 16), which would represent a vital test of the theory, as a first experimentally measured number obtained from a non-perturbative relativistic quantum field theory.

A final point connected to $\mu(\kappa)$ is the fact that the Heisenberg picture time evolution is left invariant by a c-number vacuum energy renormalization. The positive renormalized Hamiltonian obtained in Theorem 4.5 is therefore the correct Hamiltonian in the sense of automorphisms of the algebra of observables, in the proper limits.

ACKNOWLEDGMENTS

[1] Berezin, F. A., *The Method of Second Quantization* (Academic Press, 1966).

[2] Bogolubov, N. N. and Bogolubov, Jr., N. N., *Introduction to Quantum Statistical Mechanics*, 2nd ed. (World Scientific Publishing Co., 2010).

[3] Bohr, N. and Rosenfeld, L., "Zur frage der Messbarkeit der elektromagnetischen Feldgrössern," Kgl. Danske Vidensk. Selsk. Mat. Fys. Med. **12**, 8 (1933), http://gymarkiv.sdu.dk/MFM/kdvs/mfm%2010-19/mfm-12-8.pdf.

[4] Buchholz, D., "Gauss' law and the infraparticle problem," Phys. Lett. B **174**, 331 (1986).

[5] *Differential Equations and Mathematical Physics*, edited by Weikard, R. and Weinstein, G. (American Mathematical Society, Cambridge, Massachusetts, 2000).

[6] Dyson, F. J., "Is a graviton detectable?," Int. J. Mod. Phys. A **28**, 133041-1–133041-14 (2013).

[7] Fröhlich, J., "Spin, or actually spin and spin statistics," Sémin. Poincaré **11**, 1–56 (2007).

[8] *High Energy Electromagnetic Interactions and Field Theory*, edited by Levy, M. (Gordon and Breach, 1967).

[9] Jaffe, A., Lanford, O. E., and Wightman, A. S., "A general class of cutoff model field theories," Commun. Math. Phys. **15**, 47–68 (1969).

[10] Lieb, E. H., "Some problems in statistical mechanics that I would like to see solved, IUPAP Boltzmann prize lecture," Phys. A **263**, 491–499 (1999).

[11] Lieb, E. H. and Loss, M., "Stability of a model of relativistic quantum electrodynamics," Commun. Math. Phys. **228**, 561–588 (2002).
[12] Lieb, E. H. and Loss, M., "Existence of atoms and molecules in nonrelativistic quantum electrodyamics," Adv. Theor. Math. Phys. **7**, 667–710 (2003).
[13] Lieb, E. H. and Loss, M., "A note on polarization vectors in quantum electrodynamics," Commun. Math. Phys. **252**, 477–483 (2004).
[14] Lieb, E. H. and Seiringer, R., *The Stability of Matter in Quantum Mechanics* (Cambridge University Press, 2010).
[15] *Niels Bohr and the Development of Physics*, edited by Pauli, W. (Pergamon Press, 1955).
[16] Sakurai, J. J., *Advanced Quantum Mechanics* (Addison Wesley Publishing Co, 1967).
[17] Spohn, H., *Dynamics of Charged Particles and Their Radiation Field* (Cambridge University Press, 2004).
[18] Weinberg, S., *The Quantum Theory of Fields, Volume 1* (Cambridge University Press, 1996).
[19] Wilson, R. W. and Penzias, A. A., "Isotropy of cosmic background radiation at 4080 MHz," Science **156**, 1100–1101 (1967).