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Torsion Units in Integral Group Rings of Metabelian Groups

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Abstract

We prove a special case of the Conjectures of Zassenhaus and Bovdi.

Introduction

Let $V(ZG)$ be the group of units of augmentation one of the group ring ZG. Set $G(k) = \{g \in G : o(g) = k\}$, where $o(g)$ denotes the order of g, and $K(g) =< [g, h \mid h \in G >$. If

$$x = \sum \alpha(g)g \in ZG$$

we define

$$T^{(k)}(x) = \sum_{g \in G(k)} \alpha(g)$$

called the k-th generalized trace of x. Also denote by

$$\tilde{x}(g) = \sum_{h \sim g} \alpha(g)$$

Bovdi proved the fallowing [1, Lemma 1.1]:

Lemma 1: If p is a prime, $x \in V(ZG)$ and $o(x) = p^n$ then $T^{(p^n)}_{(x)} \equiv 1(mod \mid p)$ and $T^{(p^i)}_{(x)} \equiv 0(mod \mid p)$ for $i < n$.

and conjectured that actually:

$$T^{(p^n)}(x) = 1 \text{ and } T^{(p^i)}(x) = 0 \quad \text{for } i < n$$

Due to a result of [2], which we shall state below, the Zassenhaus conjecture on cyclic subgroups of V(ZG) implies Bovdi's conjecture. In [1], the author worked on Bovdi's conjecture for nilpotent metabelian groups. Using a result of [1] we prove the conjecture when G is metabelian and $K(g) = G'$ for every $g \in G - \zeta(G)$.

Results

Theorem A: Let G be a metabelian group such that for all $g \in G - \zeta(G)$, where $\zeta(G)$ denotes the center of G, we have that $K(g) = G'$. If $\alpha \in V(ZG)$, $n = o(\alpha)$, then

$$T^{(k)}(\alpha) = \delta_{nk}$$

Theorem B Let G be as in Theorem A. Furthermore, assume that, for all $g \in G - \zeta(G)$ $gK(g)$ is precisely the conjugacy class of g. If $\alpha \in V(ZG)$, $n = o(\alpha)$. Then there exist $\beta \in Q(G)$ such that $\beta^{-1}\alpha\beta \in G$.

Corollary If $|G'| = p$, a prime and all nontrivial conjugacy classes have order p, then the Zassenhaus conjecture holds.

Proof of Theorem A

By hypoteses $K(g) = G'$, for every $g \in G - \zeta(G)$.

Let $h \in gK(g) - \zeta(G)$ then $K(h) = G'$ so $hK(h) = gK(g)$. By [1, Lemma 2.2] we have $o(g) = o(h)$. Let $\alpha = \sum \alpha(g)g \in V(ZG) - G$. Choose $g_1, \ldots, g_k \in supp(\alpha)$, such that $g_i g_j^{-1} \notin G'$, $i \neq j$ and

$$supp(\alpha) \subseteq \bigcup_{i=1}^{k} g_i G'$$

. Then

$$\alpha = \sum_i \sum_{g \in G'} \alpha(gg_i)gg_i = \sum_i \sum_{t \in g_i K(g_i)} \alpha(t)t$$

By Berman's Theorem, $g_j \notin \zeta(G)$, for $1 \leq j \leq k$.

Since $G \setminus G'$ is abelian, looking at the image of α in $Z(G \setminus G')$ we see that there is a unique $g_0 \in supp(\alpha)$ such that

$$\sum_{t \in g_0 K(g_0)} \alpha(t) = 1 \qquad (1)$$

and

$$\sum_{t \in g_i K(g_i)} \alpha(t) = 0, g_i \neq g_0 \qquad (2)$$

Now $o(h) = o(g_i)$ for all $h \in g_i K(g_i)$, so if we denote by $n_0 = o(g_0)$ we have

$$T^i(\alpha) = \delta_{in_0}$$

If n is a prime power then $n = n_0$, by [1, Lemma 1.1]. If $|G| < \infty$ then by Whitcomb's Argument there exists $g \in G$ such that $\alpha - g \in \Delta G \Delta G'$. So, by [3, Theorem 1.3] $g \notin \zeta(G)$ and $o(g) = o(\alpha)$. Since $\Delta G \Delta G' \subset \Delta(G, G')$ and $\alpha - G_0 \in \Delta(G, G')$, we conclude that $gg_0^{-1} \in G'$. So $gK(g) = gG' = g_0 G' = g_0 K(g_0)$. Hence $n = o(g) = o(g_0) = n_0$.

To prove the infinite case we procede as in [1], using induction on n.

We can assume that two distinct primes divide n. Let $L(ZG)$ denote the Z-module generated by all $gh - hg (g, h \in G)$. It is easy to check that for an element $y \in L(ZG)$,

$$\tilde{y}(g) = 0 \qquad (3)$$

for all $g \in G$.

If p is prime dividing n then

$$\alpha^p \equiv \sum_{t \in G} \alpha_t^p t^p mod(L(ZG) + pZG)$$

Since $o(\alpha^p) = \frac{n}{p}$, by induction we have $T^{(\frac{n}{p})}\alpha = 1$. Therefore, applying (3) we obtain from the last congruence

$$\sum_{t^p \in G(\frac{n}{p})} \alpha_t^p \equiv 1(mod \mid p)$$

which implies

$$\sum_{t^p \in G(\frac{n}{p})} \alpha_t^p = 1. \tag{4}$$

Suppose that $p^2 \not| n$. Then

$$t^p \in G(\frac{n}{p}) \Longrightarrow t \in G(n)$$

Hence

$$T^{(n)}(\alpha) = \sum_{t^p \in G(\frac{n}{p})} \alpha_t$$

and by (4) we get $T^{(n)}(\alpha) \equiv 1(mod \mid p)$. It follows from (1)-(2) that $n = k_0$.

Suppose now that $n = p_1 p_2 \dots p_r$, where p_i are pairwise distinct primes $(i = 1, \dots, r)$, $r \geq 2$. It is easy to see that $t \in G(\frac{n}{p_i})$ implies $t^{p_i} \in G(\frac{n}{p})$ and hence

$$\sum_{t^{p_i} \in G(\frac{n}{p_i})} \alpha_t = T^{(n)}(\alpha) + T^{(\frac{n}{p_i})}(\alpha) \quad (i = 1, 2).$$

Applying (4) for p_1 and p_2 we have $T^{(n)}(\alpha) + T^{(\frac{n}{p_i})}(\alpha) \equiv 1(mod \mid p_i)$ $(i = 1, 2)$

Thus in view of (1)-(2) we conclude that $n = n_0$.

q.e.d.

Proof of Theorem B

By the proof of theorem A there exist a unique $g_0 \in G$ such that

$$1 = \sum_{t \in g_0 K(g_0)} \alpha(t) = \sum_{t \sim g_0} \alpha(t) = \tilde{\alpha}(g_0)$$

We recall from [2, Theorem 2.5] that in this case there exists $\beta \in Q(Z)$ such that $\beta^{-1}\alpha\beta \in G$.

The result follows.

q.e.d

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