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The mod k chromatic index of random graphs

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Abstract
The *mod k chromatic index* of a graph G is the minimum number of colors needed to color the edges of G in a way that the subgraph spanned by the edges of each color has all degrees congruent to 1 (mod k). Recently, the authors proved that the mod k chromatic index of every graph is at most $198k - 101$, improving, for large k, a result of Scott. Here we study the mod k chromatic index of random graphs. We prove that for every integer $k \geq 2$, there is $C_k > 0$ such that if $p \geq C_k n^{-1} \log n$ and $n(1-p) \to \infty$ as $n \to \infty$, then the following holds: if k is odd, then the mod k chromatic index of $G(n, p)$ is asymptotically almost surely (a.a.s.) equal to k, while if k is even, then the mod k chromatic index of $G(2n, p)$ (respectively, $G(2n+1, p)$) is a.a.s. equal to k (respectively, $k+1$).

KEYWORDS
chromatic index, decomposition, edge coloring, mod

1 | INTRODUCTION

Throughout this paper, all graphs are simple and $k \geq 2$ is a fixed integer. If $G = (V, E)$ is a graph and $F \subset E$, then the subgraph of G *spanned* by F is $G[F] = (W, F)$, where W is the set of vertices of G that are incident to at least one edge in F. Note that, in particular, $G[F]$ has no isolated vertices. A χ'_k*-coloring* of a graph G is a coloring of the edges of G in which the subgraph spanned by the edges of each color has all degrees congruent to 1 (mod k). The *mod k chromatic index* of G, denoted $\chi'_k(G)$, is the minimum number of colors in a χ'_k-coloring of G. Note that isolated vertices play no role in these definitions, and hence $\chi'_k(G) = \chi'_k(G - v)$ if v is an isolated vertex in G. Since a proper coloring of the edges of G is a χ_k'-coloring, we have $\chi_k'(G) \le \chi'(G)$, where, as usual, $\chi'(G)$ denotes the chromatic index of G. It turns out that $\chi_k'(G)$ is usually much smaller than $\chi'(G)$. In 1991, Pyber [8] proved that $\chi_2'(G) \le 4$ for every graph G, and in 1997 Scott [9] proved that $\chi_k'(G) \le 5k^2 \log k$ for every graph G. Recently, Botler et al. [4] proved that $\chi_k'(G) \le 198k - 101$ for any G,[1] which improves Scott's bound for large k and is sharp up to the multiplicative constant, as every graph with a vertex of degree k requires at least k colors in any χ_k'-coloring. In this paper, we study the behavior of the mod k chromatic index of the random graph $G(n,p)$ for a wide range of $p = p(n)$. More specifically, we prove the following result.

Theorem 1. *For every integer $k \ge 2$, there is a constant C_k such that if $p \ge C_k n^{-1} \log n$ and $n(1-p) \to \infty$, then the following holds as $n \to \infty$.*

(i) *If k is even, then*

$$\lim_{n\to\infty} \mathbb{P}\Big(\chi_k'(G(2n,p)) = k\Big) = 1$$

and

$$\lim_{n\to\infty} \mathbb{P}\Big(\chi_k'(G(2n+1,p)) = k+1\Big) = 1.$$

(ii) *If k is odd, then*

$$\lim_{n\to\infty} \mathbb{P}\Big(\chi_k'(G(n,p)) = k\Big) = 1.$$

Theorem 1 extends a result of Botler et al. [2] that dealt with the case $k = 2$ and $C\sqrt{n^{-1}\log n} < p < 1 - 1/C$ for some C.

In Section 2, we present some technical lemmas that we shall need and in Section 3 we prove Theorem 1. While Theorem 1 tells us that the typical value of the mod k chromatic index is at most $k+1$ for a wide range of $p = p(n)$, in Section 4, we give a sequence $G_2, G_3, \ldots$ of graphs for which $\chi_k'(G_k) \ge k+2$ for every $k \ge 2$ (see Proposition 14). Section 5 contains some remarks on other ranges of p.

A property P holds for $G(n,p)$ *asymptotically almost surely* (a.a.s.) or *with high probability* if the probability that $G(n,p)$ satisfies P tends to 1 as $n \to \infty$. The asymptotic notation $o(1)$, $\ll$ and $\gg$ will always be with respect to $n \to \infty$.

The main results of this work were announced in the extended abstract [3].

2 | TECHNICAL LEMMAS

Given a graph G and $i \in \{1, \ldots, k\}$, let $V_i = V_i(G)$ be the set of vertices v of G with $\deg_G(v) \equiv i \pmod k$. Let $n_i(G) = |V_i|$ and let $G_i = G[V_i]$. The sets $V_1, \ldots, V_k$ are the *degree classes* of G.

[1] By making use of a recent result of Hasanvand (see the comments following Theorem 1.4 in [5]), the proof in [4] yields $\chi_k'(G) \le 38k - 37$.

In this section, after presenting some auxiliary probabilistic and random graph results, we prove that $G = G(n, p)$ in a wide range of p with high probability satisfies (a) $n_i(G)$ do not deviate much from n/k (Lemma 10), (b) every vertex of G has about $pn_i(G)$ neighbors in G_i (Lemma 11), (c) the G_i are all connected (Lemma 12), and (d) any induced balanced bipartite subgraph of G with high minimum degree contains a k-*factor* (Lemma 13), that is, a k-regular spanning subgraph.

2.1 | Preliminaries

We shall use the following Chernoff bound (see [1, Corollary A.1.14] or [7, Theorem 2.8]).

Lemma 2 (Chernoff bound). *Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with $\mathbb{P}(X_i = 1) = p_i$ for all i. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}(X) = \sum_{i=1}^{n} p_i$. Then, for every $\varepsilon > 0$, there is $c_\varepsilon > 0$ such that $\mathbb{P}(|X - \mu| > \varepsilon\mu) \leq 2e^{-c_\varepsilon \mu}$.*

The next lemma asserts that a binomial random variable $\mathrm{Bin}(n, p)$ with parameters n and p is well distributed among the congruence classes $\bmod k$ as long as $np(1 - p) \to \infty$.

Lemma 3. *Let $n \geq 1$, $0 < p < 1$ and $k \geq 2$ be given. There is a positive constant a_k that depends only on k such that, for any integer t,*

$$\left| \mathbb{P}(\mathrm{Bin}(n, p) \equiv t \pmod k) - \frac{1}{k} \right| \leq e^{-a_k np(1-p)}. \tag{1}$$

Proof. Let α be a primitive kth root of unity. The kth roots of unity are precisely $1 = \alpha^0, \alpha^1, \ldots, \alpha^{k-1}$. Now, for every $j \in \{0, \ldots, k-1\}$, we have

$$\alpha^{-jt}(1 + (\alpha^j - 1)p)^n = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \alpha^{j(i-t)}. \tag{2}$$

Fix any $t \in \{0, \ldots, k-1\}$. For any integer i,

$$\sum_{j=0}^{k-1} (\alpha^j)^{i-t} = \begin{cases} (\alpha^{k(i-t)} - 1)/(\alpha^{i-t} - 1) = 0 & \text{if } k \nmid i - t, \\ k & \text{otherwise.} \end{cases}$$

Thus, summing (2) for $j = 0, \ldots, k-1$, we get

$$\begin{aligned} \sum_{j=0}^{k-1} \alpha^{-jt}(1 + (\alpha^j - 1)p)^n &= k \cdot \sum_{i \equiv t \pmod k} \binom{n}{i} p^i (1-p)^{n-i} \\ &= k \cdot \mathbb{P}(\mathrm{Bin}(n, p) \equiv t \pmod k). \end{aligned} \tag{3}$$

Note that, if $j = 0$, then $\alpha^{-jt}(1 + (\alpha^j - 1)p)^n = 1$, and if $j \neq 0$, then

$$\begin{aligned} |(1 + (\alpha^j - 1)p)^n| &= |(1 - p) + \alpha^j p|^n \\ &= (1 - 2(1 - Re(\alpha^j))p(1 - p))^{n/2} \\ &\leq e^{-(1 - Re(\alpha^j))p(1-p)n}. \end{aligned} \tag{4}$$

Inequality (1) follows from (3) and (4). □

Given a graph G and U, $W \subset V(G)$ with $U \cap W = \varnothing$, let $e(U, W) = e_G(U, W)$ be the number of edges in G with one endpoint in U and the other endpoint in W.

Definition 4 (p, α)-bijumbled. Let p and α be given. We say that a graph G is *weakly* (p, α)*-bijumbled* if, for all $U, W \subset V(G)$ with $U \cap W = \varnothing$ and $1 \leq |U| \leq |W| \leq pn|U|$, we have

$$|e(U, W) - p|U||W|| \leq \alpha\sqrt{|U||W|}. \tag{5}$$

If (5) holds for all pairs of disjoint sets U, $W \subset V(G)$, then we say that G is (p, α)-*bijumbled*.

Fact 5. If G is weakly (p, α)-bijumbled, then for every $U \subset V(G)$ we have

$$\left| e(G[U]) - p\binom{|U|}{2} \right| \leq \alpha|U|. \tag{6}$$

Proof sketch. Let $u = |U|$. Double counting shows that

$$2e(G[U])\binom{u-2}{\lfloor u/2 \rfloor - 1} = \sum_{U'} e_G(U', U \setminus U'), \tag{7}$$

where the sum ranges over $U' \subset U$ with $|U'| = \lfloor u/2 \rfloor$. Inequality (6) follows from (7). We omit the details. □

Lemma 6 (Haxell et al. [6, Lemma 3.8]). *For any* $0 < p = p(n) \leq 1$, *the random graph* $G(n, p)$ *is a.a.s. weakly* $(p, A\sqrt{pn})$*-bijumbled for a certain absolute constant* A.

In [6], Lemma 6 is proved with $A = e^2\sqrt{6}$.

Corollary 7. *Let* $G = G(n, p)$, *where* $0 < p = p(n) \leq 1$. *If* $t^2 > A^2 n/p$, *then a.a.s.* $e(U, W) > 0$ *for any pair of sets* U, $W \subset V(G(n, p))$ *with* $U \cap W = \varnothing$ *and* $\min\{|U|, |W|\} \geq t$.

Corollary 8. *Suppose* $pn \geq C \log n$ *for some constant* $C > 3$. *Then a.a.s.* $G(n, p)$ *is* $(p, A\sqrt{pn})$*-bijumbled for some* $A \geq 2$.

Proof sketch. Lemma 6 tells us that $G(n, p)$ is a.a.s. weakly $(p, A\sqrt{pn})$-bijumbled for some A. We may assume that $A \geq 2$. Now let U and W be disjoint, with $|W| > pn|U|$. Then $A\sqrt{pn|U||W|} > Apn|U|$. In particular, $p|U||W| - A\sqrt{pn|U||W|} \leq p|U|n - Apn|U| \leq 0 \leq e(U, W)$. As $np \geq C \log n$ and $C > 3$, we have that $\Delta(G(n, p)) \leq 2pn$ almost surely. Therefore $e(U, W) \leq 2pn|U| \leq Apn|U| \leq p|U||W| + A\sqrt{pn|U||W|}$, and we conclude that $G(n, p)$ is indeed $(p, A\sqrt{pn})$-bijumbled. □

We shall also need the following fact. A result as in the lemma below can be proved by considering balanced bipartitions chosen uniformly at random and by applying a Chernoff bound for hypergeometric distributions, but we give the result below, which can be proved by applying Lemma 2 ($c_{1/5}$ below is the constant given by Lemma 2 for $\varepsilon = 1/5$).

Lemma 9. *Let J be a graph on $2u \le n$ vertices and suppose that $\delta(J) \ge 10c_{1/5}^{-1}(\log n + \omega)$, where $\omega = \omega(n) \to \infty$ as $n \to \infty$. Then, if n is large enough, there is $U \subset V(J)$ with $|U| = u$ such that the bipartite graph $J[U, W]$ induced between U and $W = V(J)\backslash U$ is such that $\delta(J[U, W]) \ge 2\delta(J)/5$.*

Proof. Let $\{x_1, y_1\}, \ldots, \{x_u, y_u\}$ be an arbitrary partition of $V(J)$ into pairs, and let $U = \{z_i : 1 \le i \le u\}$, where each z_i is chosen uniformly at random from $\{x_i, y_i\}$, independently for each i. Let $W = V(J)\backslash U$ and put $J' = J[U, W]$. For each $v \in V(J)$, let P_v be the number of pairs $\{x_i, y_i\}$ contained entirely in $N_J(v)$ and let Q_v be the number of $\{x_i, y_i\}$ with $|\{x_i, y_i\} \cap N_J(v)| = 1$. Clearly, $\deg_J(v) = 2P_v + Q_v$. Let $A = \{v : P_v < 2\deg_J(v)/5\}$. By the definition of A, if $v \notin A$, then $\deg_{J'}(v) \ge 2\deg_J(v)/5$. In what follows, we deal with the vertices in A. Fix a vertex $v \in A$. A moment's thought tells us that $\deg_{J'}(v) = P_v + d'(v)$, where $d'(v) \sim \mathrm{Bin}(Q_v, 1/2)$. Let $\mu = \mathbb{E}(d'(v)) = Q_v/2$. Since $v \in A$, we have $\mu = (\deg_J(v) - 2P_v)/2 > \deg_J(v)/10 \ge \delta(J)/10$. We have

$$\begin{aligned}
\mathbb{P}\left(\deg_{J'}(v) < \frac{2}{5}\deg_J(v)\right) &= \mathbb{P}\left(P_v + d'(v) < \frac{2}{5}\deg_J(v)\right) = \mathbb{P}\left(d'(v) < \frac{2}{5}\deg_J(v) - P_v\right) \\
&\le \mathbb{P}\left(d'(v) < \frac{2}{5}(\deg_J(v) - 2P_v)\right) = \mathbb{P}\left(d'(v) < \frac{2}{5}Q_v\right) \\
&\le \mathbb{P}\left(|d'(v) - \mu| > \frac{1}{5}\mu\right),
\end{aligned}$$

which, by Lemma 2 and our hypothesis on $\delta(J)$, is at most $2e^{-c_{1/5}\mu} = 2e^{-c_{1/5}\delta(J)/10} = o(n^{-1})$. Thus, by the union bound, the probability that $\deg_{J'}(v) < (2/5)\deg_J(v)$ for some $v \in A$ is $o(1)$, showing that, for large n, most choices of U will do (recall that the vertices $v \notin A$ are never a problem). □

2.2 | Degree classes of $G(n, p)$

We first show that the degree classes of $G(n, p)$ are typically of cardinality about n/k. This is assertion (a) given at the beginning of Section 2. In Lemmas 11–13, we prove assertions (b)–(d).

Lemma 10. *Let $k \ge 2$ be a fixed integer and let $p = p(n)$ with $np(1 - p) \to \infty$ as $n \to \infty$ be given. Then, with probability at least $1 - o(1/n)$, for every $1 \le i \le k$ we have*

$$\frac{n}{2k} \le n_i(G(n, p)) \le \frac{3n}{2k}. \tag{8}$$

Proof. Fix i ($1 \le i \le k$). We show that (8) holds with probability $1 - o(1/n)$. The result then follows from the union bound.

Let $G = G(n,p)$. Fix $U \subset V(G)$ with $|U| = \lceil (1-1/4k)n \rceil$ and let $W = V(G)\backslash U$. Let $m = |W|$. Let $F \subset \binom{U}{2}$ and condition on $E(H) = F$, where $H = G[U]$. For every $u \in U$, let X_u be the indicator function of the event $\{\deg_G(u) \equiv i \pmod{k}\}$. Since $\deg_G(u) = \deg_H(u) + e_G(\{u\}, W)$ and we are conditioning on $E(H) = F$, we have that $p_u = \mathbb{P}(X_u = 1) = \mathbb{P}(\mathrm{Bin}(m,p) \equiv t \pmod{k})$, where $t = i - \deg_H(u)$. Lemma 3 tells us that $|p_u - 1/k| \le e^{-a_k p(1-p)m} = o(1)$. Let $X = \sum_{u \in U} X_u$ and note that $\mathbb{E}(X) = |U|(1/k + o(1))$. Lemma 2 then tells us that, for some absolute constant $c > 0$,

$$\mathbb{P}\left(\left|X - \frac{1}{k}|U|\right| > \frac{n}{4k}\right) \le 2e^{-c|U|/k} = o\left(\frac{1}{n}\right).$$

Also, note that $X \le n_i(G) \le X + m$, and that

$$\frac{1}{k}|U| - \frac{n}{4k} \ge \frac{1}{k}\left\lceil \left(1 - \frac{1}{4k}\right)n \right\rceil - \frac{n}{4k} \ge \frac{n}{2k}$$

and

$$\frac{1}{k}|U| + \frac{n}{4k} + m \le \frac{1}{k}\left\lceil \left(1 - \frac{1}{4k}\right)n \right\rceil + \frac{n}{4k} + \frac{n}{4k} \le \frac{3n}{2k}.$$

Therefore

$$\mathbb{P}\left(\frac{n}{2k} \le n_i(G) \le \frac{3n}{2k} \;\middle|\; E(H) = F\right) = 1 - o\left(\frac{1}{n}\right).$$

Since this holds for arbitrary F, the result follows. □

Recall that $V_i = V_i(G)$ is the set of vertices v of G with $\deg_G(v) \equiv i \pmod{k}$ and $G_i = G[V_i]$.

Lemma 11. *For every integer $k \ge 2$ there is a positive constant C such that if $p = p(n) \ge Cn^{-1}\log n$ and $n(1-p) \to \infty$ as $n \to \infty$, then a.a.s. $G = G(n,p)$ is such that, for every $v \in V(G)$ and every $1 \le i \le k$,*

$$|N(v) \cap V_i| \ge \frac{pn}{3k}.$$

Proof. Let $c_{1/4} > 0$ be as given by Lemma 2 and let $C = 3k/c_{1/4}$. We prove that this choice of C will do. Fix $1 \le i \le k$ and $v \in V = V(G)$. Let $U = V\backslash\{v\}$. We first generate the edges of $G = G(n,p)$ in $H = G[U]$. Since $p \ge Cn^{-1}\log n$, our assumption that $n(1-p) \to \infty$ implies that $np(1-p) \to \infty$ as well. Hence Lemma 10 applies and we see that, with probability $1 - o(1/n)$, we have

$$n_j(H) \ge \frac{n-1}{2k} \tag{9}$$

for all $1 \le j \le k$. Let us suppose that (9) does hold for every j. We now generate the edges between v and U in G. Clearly, $N(v) \cap V_i = N(v) \cap V_{i-1}(H)$, where, of course, we consider the indices modulo k. Also, $|N(v) \cap V_i| \sim \mathrm{Bin}(n_{i-1}(H), p)$. Note that

$$\mathbb{E}(|N(v) \cap V_i|) = pn_{i-1}(H) \ge p\,\frac{n-1}{2k} \ge \frac{4pn}{9k}$$

and also that $(3/4)pn_{i-1}(H) \ge pn/3k$ for all large enough n. Lemma 2 then gives that, with probability $1 - 2\exp(-4c_{1/4}pn/9k) = 1 - o(1/n)$, we have $|N(v) \cap V_i| \ge (3/4)pn_{i-1}(H) \ge pn/3k$. It now suffices to take the union bound considering all $1 \le i \le k$ and $v \in V$. □

Lemma 12. *Let $k \ge 2$ be an integer and let C and $p = p(n)$ be as in Lemma 11. Then a.a.s. $G = G(n,p)$ is such that G_i is connected for every $1 \le i \le k$.*

Proof. Fix i. Lemma 11 tells us that a.a.s.

$$\delta(G_i) \ge \frac{pn}{3k}. \tag{10}$$

We suppose (10) holds and that G is $(p, A\sqrt{pn})$-bijumbled for some $A \ge 2$ (recall Corollary 8) and deduce that G_i is connected if n is large enough. Suppose for a contradiction that J is a component of G_i with $t = |V(J)| \le |V(G_i)|/2$. The number $e(J)$ of edges in J satisfies

$$\frac{1}{2}t\,\frac{pn}{3k} \le \frac{1}{2}t\delta(J) \le e(J) \overset{\text{Fact (5)}}{\le} p\binom{t}{2} + A\sqrt{pn}\;t \le p\frac{t^2}{2} + A\sqrt{pn}\;t,$$

whence

$$\frac{pn}{6k} \le \frac{1}{2}pt + A\sqrt{pn}. \tag{11}$$

Since $pn \to \infty$, it follows from (11) that, say, $t \ge 2n/7k$ for any large enough n. By the choice of J, we have $|V(G_i) \backslash V(J)| \ge t$. Therefore, by Corollary 7, we have $e(V(J), V(G_i) \backslash V(J)) > 0$, as $t^2 \gg A^2n^2/C \log n \ge A^2n/p$. Since J is a component of $G_i = G[V_i]$ this is a contradiction. We conclude that G_i is indeed connected. □

Lemma 13. *Let $k \ge 1$, $c > 0$, $A > 0$ and $0 < p = p(n) < 1$ be given. Suppose G is a $(p, A\sqrt{pn})$-bijumbled graph of order n and $p \gg 1/n$. Then, if n is large enough, for any U and $W \subset V(G)$ with $U \cap W = \varnothing$, $|U| = |W| \ge cn$ and $\delta(G[U,W]) \ge p|U|/8$, the graph $G[U,W]$ contains a k-factor.*

Proof. Let U and W be as in the statement of the lemma and let $m = |U| = |W|$. We prove that $G[U,W]$ contains a k-factor by induction on k. Fix $k \ge 1$ and suppose $G[U,W]$ contains a $(k-1)$-factor F. It suffices to prove that $B = G[U,W] - F$ contains a perfect matching. We check Hall's condition: for every $S \subset U$, we have $|N(S)| \ge |S|$.

Let $\delta = \delta(B)$. A simple argument shows that if $|S| \leq \delta$ or $|S| > m - \delta$, then $|N(S)| \geq |S|$. We therefore assume that $\delta < |S| \leq m - \delta$ and suppose for a contradiction that $|N(S)| < |S|$. Let $s = |S|$. Then

$$\left(\frac{1}{8}cpn - k\right)s \leq \left(\frac{1}{8}pm - k\right)s \leq \delta s \leq e(S, N(S)) \leq ps^2 + A\sqrt{pn}\ s,$$

whence

$$\frac{1}{8}cpn - k \leq ps + A\sqrt{pn}\,. \tag{12}$$

Since $pn \gg 1$, it follows from (12) that if n is large enough, then, say, $cpn/9 \leq ps$ and hence $|S| = s \geq cn/9$.

Let $T = W \setminus N(S)$ and note that $N(T) \subset U \setminus S$. Hence, $|N(T)| \leq |U| - |S| < |U| - |N(S)| = |W| - |N(S)| = |T|$. Arguing as above, we get that $|T| \geq cn/9$. Using that $pn \gg 1$, we see by Corollary 7 that $e(S, T) > 0$, which contradicts the definition of T. We conclude that $B = G[U, W] - F$ satisfies Hall's condition. This concludes the induction step and the result follows. □

3 | MAIN THEOREM

We now prove Theorem 1. In what follows, we say that a graph G is a *mod k graph* if all its nonisolated vertices have degrees congruent to 1 mod k. The reader may find it useful to recall the notation and terminology introduced at the beginning of Section 2.

The key idea of this proof when k and n are even or k is odd is (1) to use Lemma 11 to find a set of vertex-disjoint stars forming a star forest F with $\chi'_k(F) \leq k - 1$ so that $G' = G - E(F)$ has an even number of vertices in each degree class $V_i(G')$ with $i > 1$, and then (2) use Lemma 13 to find, for each $i > 1$, a bipartite $(i-1)$-factor B_i in G'_i and then let $G'' = G' - E(B)$, where $B = \bigcup_{2 \leq i \leq k} B_i$ (see Figure 1). By construction, $\chi'_k(F \cup B) \leq k - 1$ and $G'' = G - E(F \cup B)$ is a mod k graph, which can be colored monochromatically. It follows that $\chi'_k(G) \leq k$. The remaining case, namely, when k is even and n is odd, then follows by using the n even case to color $G - v$ with k colors for some $v \in V(G)$, and then coloring most of the edges incident to v with the $(k+1)$st color.

Proof of Theorem 1. Let $C_k = \max\{C, 41c_{1/5}^{-1}k\}$, where C is the constant given by Lemma 11 and $c_{1/5}$ is the constant given by Lemma 2 for $\varepsilon = 1/5$. Let $p = p(n)$ be as in the statement of the theorem and let $G = G(n, p)$. Below, we tacitly assume that n is large enough whenever necessary.

We start by observing that $\chi'_k(G) \geq k$ holds a.a.s. regardless of the parity of k. Indeed, owing to our hypothesis on p, Lemma 10 tells us that $n_k(G(n,p)) \geq n/2k$ with probability at least $1 - o(1/n)$. Noting that G a.a.s. has no isolated vertices, we deduce that G a.a.s. has a vertex v of nonzero degree with $\deg_G(v) \equiv 0 \pmod{k}$. It is clear that such a vertex v forces $\chi'_k(G) \geq k$, regardless of the parity of k. We also have to prove that, for even k and odd n, we a.a.s. have $\chi'_k(G) \geq k + 1$. This is done below.

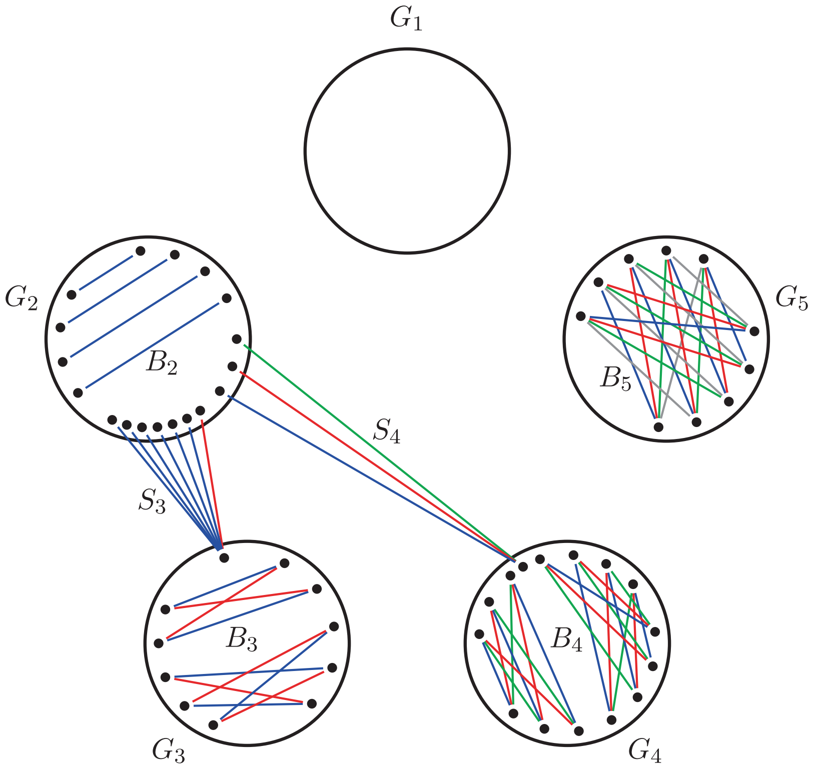

FIGURE 1 Subgraphs $F = S_3 \cup S_4$ and $B = B_2 \cup \cdots \cup B_5$. Here, $k = 5$, $n_2(G)$ and $n_5(G)$ are even and $n_3(G)$ and $n_4(G)$ are odd. [Color figure can be viewed at wileyonlinelibrary.com]

We divide the remainder of the proof according to the parity of k.

Case 1: *k even*. We consider the even n and odd n cases separately.

Case 1.1 *n even*: We shall prove that, in this case, $\chi'_k(G) \leq k$ holds a.a.s. by following the strategy outlined at the beginning of Section 3.

We first claim that, a.a.s., there are vertex-disjoint stars $S_3, \ldots, S_k$ such that, for each $i \in \{3, 4, \ldots, k\}$, the star S_i has $i - 1$ edges, is centered in G_i and its leaves belong to G_2. This can be seen by applying Lemma 11 successively, to obtain each of the S_i ($i = 3, 4, \ldots$) in turn. Let $F = \bigcup_{i \in I} S_i$, where $I = \{i : n_i(G) = |V(G_i)| \text{ is odd}\}$.

Let $G' = G - E(F)$. Note that the vertices of G incident to the edges of F have their degrees changed by the removal of F: they all become of degree 1 (mod k) in G', and therefore each of them "moves" from G_i for some $i \geq 2$ to G'_1. Furthermore, note that $n_{i(G')} = |V(G'_i)|$ is even for all $i \geq 3$, because $G'_i = G_i$ if $n_i(G)$ is even and $G'_i = G_i - v_i$ if $n_i(G)$ is odd, where v_i is the center of the star S_i.

Since n is even, the number n'_{even} of vertices of even degree in G' has the same parity as n'_{odd}, the number of vertices of odd degree in G', which is even. As k is even, $n'_{\text{even}} = \sum\{n_i(G') : i \text{ even}\}$ and the fact that $n_i(G')$ is even for all $i \geq 3$ implies that n'_{even}

and $n_2(G')$ have the same parity, and hence $n_2(G')$ is even. We conclude that $n_i(G')$ is even for all $i \geq 2$.

Fix $2 \leq i \leq k$. In view of Lemmas 10 and 11, we assume that (8) holds and that we have $\delta(G_i) \geq pn/3k$, and consequently $\delta(G_i') \geq \delta(G_i) - 1 \geq pn/4k \geq (C_k/4k)\log n \geq (41/4c_{1/5})\log n$. We now apply Lemma 9 with $J = G_i'$ and obtain U and $W \subset V(G_i')$ with $|U| = |W| = n_i(G')/2 \leq n_i(G)/2 \leq 3n/4k$ and $U \cap W = \varnothing$ such that

$$\delta(G[U, W]) = \delta(G_i'[U, W]) \geq \frac{2}{5}\delta(G_i') \geq \frac{pn}{10k} \geq \frac{1}{8}p|U|.$$

Note that $|U| = |W| = n_{i(G')}/2 \geq (n(G_i) - 1)/2 \geq (n/2k - 1)/2 \geq n/5k$. We are now in a position to apply Lemma 13 and obtain an $(i-1)$-factor B_i in $G[U, W] = G_i'[U, W]$.

Let $B = \bigcup_{2 \leq i \leq k} B_i$ and $G'' = G' - E(B)$. Note that G'' is a mod k graph, which can be entirely colored with color 1, say. Furthermore, $\chi_k'(F \cup B) \leq k - 1$. We conclude that $\chi_k'(G) \leq k$. This finishes the proof of Case 1.1.

Before we proceed, for later reference, we observe the following: since G'' contains every edge of G incident to the vertices in G_1, the coloring we have obtained in this case is such that every vertex in G_1 is incident only to edges of a certain fixed color (color 1 above).

Case 1.2 *n odd*: Fix a vertex u in $G = G(n, p)$ and let $H = G - u$. Lemma 11 tells us that we may suppose that u has at least $pn/3k$ neighbors in H_1.

By Case 1.1, with high probability H can be colored with colors $1, \ldots, k$ so that all edges incident to vertices with degree 1 (mod k) in H are colored with the same color, say 1 (see the last paragraph of Case 1.1). We now color the edges incident to u. Suppose $\deg(u) \equiv d \pmod{k}$ where $d \in \{1, 2, \ldots, k\}$. If $d \neq 1$, then we assign each of the colors $2, \ldots, d$ once to an edge joining u to vertices in H_1 (this is possible since there are at least $k - 1$ such edges), leaving a number congruent to 1 (mod k) of uncolored edges incident to u. We assign to these uncolored edges a new color. We thus obtain a χ_k'-coloring of G with $k + 1$ colors.

Suppose now that G admits a χ_k'-coloring with k colors. This implies that all edges incident to any given vertex in G_1 must get the same color. By Lemma 12, the graph G_1 is connected with high probability, and hence a.a.s. all the edges of G incident to vertices of G_1 must be colored with the same color, say 1. Moreover, by Lemma 11, the set $V(G_1)$ is a.a.s. a dominating set, that is, every vertex of G not in $V(G_1)$ is adjacent to some vertex in $V(G_1)$. This implies that a.a.s. the edges of color 1 induce a spanning subgraph of G. Let J be this spanning subgraph. Since J is a mod k graph and k is even, every vertex of J has odd degree. This is a contradiction as J has n vertices and n is odd. This argument shows that $\chi_k'(G) \geq k + 1$ with high probability.

Case 2: *k odd*. We proceed as in Case 1.1, except that, to produce $G' = G - E(F)$ so that $n_i(G')$ is even for every $i \geq 2$, we have to argue a little more.

Recall $I = \{i : n_i(G) \text{is odd}\}$. If $I \neq \varnothing$, then we can use the stars S_i $(i \in I)$ as in Case 1.1 to define $F = \bigcup_{i \in I} S_i$, except that, if doing so we obtain $G' = G - E(F)$ with $n_2(G')$ odd, then we replace the star S_i with $i - 1$ rays by a star S_i' with $k + i - 1$ rays for an arbitrary $i \in I$. Since $i - 1$ and $k + i - 1$ have opposite parities, we can thus force $n_2(G')$ to be even.

If $I = \varnothing$ and $n_2(G)$ is odd, we can take two stars S_3 and S_3' with centers in $V_3(G)$ and with 2 rays and $k + 2$ rays, respectively, and define $F = S_3 \cup S_3'$. Then $G' = G - E(F)$ is such that $n_i(G')$ is even for every $i \geq 2$. If $I = \varnothing$ and $n_2(G)$ is even, we simply let $G' = G$.

The rest of the proof follows Case 1.1 *mutatis mutandis*. □

4 | A LOWER BOUND FOR $max_G \chi_k'(G)$

In this section, we present a lower bound for the maximum mod k chromatic index of graphs. We clearly have $\max_G \chi_k'(G) \geq k$, because any graph G that contains a vertex v with $\deg_G(v) > 0$ and $\deg_G(v) \equiv 0 \pmod{k}$ is such that $\chi_k'(G) \geq k$. In 1991, Pyber [8] showed that his upper bound of 4 for the mod 2 chromatic index of graphs is tight because the 4-wheel (the graph obtained from a cycle of length 4 by adding a new vertex adjacent to all of its vertices) has mod 2 chromatic index equal to 4. Note that the 4-wheel is precisely the complete 3-partite graph $K_{1,k,k}$ with $k = 2$. The proposition below generalizes this observation: $\chi_k'(K_{1,k,k}) = k + 2$ for every $k \geq 2$; in particular, $\max_G \chi_k'(G) \geq k + 2$.

Proposition 14. *For every $k \geq 2$, we have $\chi_k'(K_{1,k,k}) = k + 2$.*

Proof. Let $G = K_{1,k,k}$ be the complete 3-partite graph with vertex classes $\{u\}$, A and B. Suppose for a contradiction that G has a χ_k'-coloring with c colors, where $c \leq k + 1$. Note that some color, say 1, must be used to color precisely $k + 1$ edges incident to u, and hence, every other edge incident to u must be colored with a distinct color. In particular, this implies that $c \geq k$. On the other hand, given a vertex $v \neq u$, there are only two ways of coloring the $k + 1$ edges incident to v: (a) by coloring all the $k + 1$ edges with the color used on uv, or (b) by coloring each of the $k + 1$ edges with a distinct color. Vertices of type (a) are called *monochromatic* and vertices of type (b) are called *rainbow*. Clearly, since we only have $k + 1$ colors, every color occurs at every rainbow vertex.

Claim 1: Every vertex $v \in A \cup B$ is rainbow.

Proof. Let us first note that, since there are $k + 1$ edges incident to u with color 1, we may assume without loss of generality that there are two vertices x and y in A and a vertex z in B for which ux, uy, and uz have color 1. We now fix $v \in A \cup B$ and show that it is rainbow.

Case 1: *uv has color* 1 *and $v \in B$*. Suppose v is monochromatic. Then v is monochromatic of color 1, as uv is of color 1. Note that both x and y are then incident to at least two edges of color 1, and hence they are both monochromatic of color 1. It follows that every vertex in B is monochromatic of color 1. Since $k \geq 2$, this implies that every vertex in A is also monochromatic of color 1. We conclude that u is also monochromatic of color 1, and this is a contradiction. Hence v is rainbow.

Case 2: *uv has color* 1 *and $v \in A$*. Suppose v is monochromatic. Then v is monochromatic of color 1. Note that z is then incident to two edges of color 1 and hence is monochromatic of color 1. We are now as at the beginning of Case 1 above (we have a vertex in B monochromatic of color 1), and hence we again have a contradiction. Thus v must be rainbow.

Case 3: *uv has a color different from* 1 *and $v \in B$.* Let uv have color 2. Suppose v is monochromatic. The argument in Case 1 shows that z is rainbow. The edge zu has color 1, whence there is some $w \in A$ such that zw has color 2. Since we are supposing that v is monochromatic of color 2, the edge vw is of color 2. This implies that w is monochromatic of color 2, giving another edge of color 2 incident to u. This is a contradiction, showing that v is rainbow.

Case 4: *uv has a color different from* 1 *and $v \in A$.* It suffices to repeat the argument in Case 3, replacing the vertex z in that argument by the vertex x or y. This concludes the proof of the claim. □

Let 2 be a color different from 1 that occurs at u. We know that 2 occurs exactly once at u. Since every vertex in $A \cup B$ is rainbow, color 2 occurs at every vertex in $A \cup B$ and it clearly occurs exactly once at every such vertex. This means that the edges of color 2 form a perfect matching, but this is impossible as $G = K_{1,k,k}$ has an odd number of vertices. This shows that $\chi'_k(K_{1,k,k}) > k + 1$.

We now show that $\chi'_k(K_{1,k,k}) \leq k + 2$. Suppose $A = \{a_i : 1 \leq i \leq k\}$ and $B = \{b_i : 1 \leq i \leq k\}$. Let $A' = A \cup \{a_{k+1}\}$ and $B' = B \cup \{b_{k+1}\}$, where a_{k+1} and b_{k+1} are two new vertices, and consider the complete bipartite graph G^+ with vertex classes A' and B'. Let us color the edges of G^+ properly with colors 1, ..., $k + 1$ (the chromatic index of $G^+ = K_{k+1,k+1}$ is $k + 1$). We now omit the vertices a_{k+1} and b_{k+1} from G^+ and add a new vertex u adjacent to all the vertices in $A \cup B$. We thus obtain a $K_{1,k,k}$. It remains to color the edges uv ($v \in A \cup B$). Let m_j be the "missing color" at b_j ($1 \leq j \leq k$): this is the color of $a_{k+1}b_j$ in the proper coloring of G^+. Note that all the m_j ($1 \leq j \leq k$) are distinct. We now color the $k + 1$ edges ua_i ($1 \leq i \leq k$) and the edge ub_1 with color $k + 2$, and color the edges ub_j ($2 \leq j \leq k$) with color m_j. It is then clear that the edges of color c ($1 \leq c \leq k + 1$) form a matching and the edges of color $k + 2$ form a star with $k + 1$ rays. Thus $\chi'_k(K_{1,k,k}) \leq k + 2$. □

We put forward the following rather optimistic conjecture (see [4, Conjecture 6]).

Conjecture 15. *There is an absolute constant C such that $\chi'_k(G) \leq k + C$ for every graph G.*

5 | CONCLUDING REMARKS AND FUTURE WORK

In this paper we determined the mod k chromatic index of $G(n, p)$ for $p = p(n)$ such that $p \geq C_k n^{-1} \log n$ and $n(1 - p) \to \infty$. It is natural to investigate the remaining ranges of p. For instance, if G is a forest, it is not hard to prove that $\chi'_k(G) = \max\{r \in \{1, ..., k\}$: $r \equiv d(v) \pmod k$ for some $v \in V(G)\}$. This observation settles the case $p \ll n^{-1}$, since in this range $G(n, p)$ is a.a.s. a forest. The next step would be to consider $p = cn^{-1}$ for some constant $c \in (0, 1)$, in which case the components of $G(n, p)$ are a.a.s. trees and unicyclic graphs. Unfortunately, the formula above for $\chi'_k(G)$ does not extend to all unicyclic graphs: it is not hard to prove that if G is any graph that contains a cycle of length $\ell \geq 3$ in which $\ell - 1$ vertices have degree precisely $k+1$, and one vertex has degree at most k, then $\chi'_k(G) \geq k+1$. Quite possibly, the most challenging range would be $n^{-1} \leq p \leq cn^{-1}\log n$, where c is a smallish constant.

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