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Properties related to star countability and star finiteness

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A B S T R A C T

Two star properties recently studied by Song and a generalization of countable compactness called weak star finiteness by Song and previously, 1-cl-starcompactness by Matveev and Ikenaga, are studied. We show that two of these properties coincide with feeble Lindelöfness and feeble compactness respectively in the class of spaces with a dense set of isolated points. Preservation of one of these properties under products by compact and sequentially compact spaces is also considered.

1. Introduction and notation

If X is a topological space and $\mathcal{U}$ is a family of subsets of X, then the *star* of a subset $A \subseteq X$ with respect to $\mathcal{U}$ is the set $\mathsf{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

Suppose that P is a topological property; a space X is said to be *star* P if whenever $\mathcal{U}$ is an open cover of X, there is a subspace $A \subseteq X$ with property P such that $X = \mathsf{St}(A, \mathcal{U})$. The space X is *weakly star* P (respectively, *almost star* P) if given any open cover $\mathcal{U}$ of X, there is a subspace $A \subseteq X$ with property P and such that $\text{cl}(\mathsf{St}(A, \mathcal{U})) = X$ (respectively, $\bigcup\{\text{cl}(\mathsf{St}(\{x\}, \mathcal{U})) : x \in A\} = X$). The set A will be called a *star*

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kernel (respectively, *weak star kernel, almost star kernel*) of the cover $\mathcal{U}$. A space is *feebly compact* if every family of locally finite non-empty open sets is finite; in the class of Tychonoff spaces feeble compactness is equivalent to pseudocompactness (and here, pseudocompact spaces are always assumed to be Tychonoff).

The term "star P" was coined in [10] but many star properties, including those corresponding to "$P =$ finite" and "$P =$ countable" were introduced by Ikenaga (see for example [7]) and first studied by van Douwen et al. in [3]; generalizations were later introduced by many other authors. A survey of star properties with a comprehensive bibliography can be found in [9]. We note that the properties here called "star countable" and "star finite" have been studied previously under various other names; for instance, in the survey [9] they are called "star-Lindelöf" and "starcompact" respectively. The properties "weakly star finite" and "weakly star countable" are also mentioned in this survey on pages 11 and 89, under the names "1-cl-starcompact" (also "1-H-closed") and "1-cl-star-Lindelöf" respectively, where the first of these terms is attributed to Ikenaga [7]. However, beyond the trivial implications which appear in [9]:

$$\text{Countably compact} \Rightarrow \text{1-cl-starcompact} \Rightarrow \text{pseudocompact}$$

in the class of Tychonoff spaces and

$$\text{Lindelöf} \Rightarrow \text{1-cl-star-Lindelöf},$$

in the class of Hausdorff spaces, little attention seems to have been given to these properties until recently.

The terms "weakly star countable" and "almost star countable" were used by Y-K. Song in [14] and [15] respectively, where a number of results concerning these properties were obtained.

An interesting problem is the subtle relationship between the properties of being weakly star countable, that of being weakly star finite and feeble compactness. Clearly, the second of these properties implies the first and as we mention in Section 3, weakly star finite, weakly regular spaces are feebly compact. However, in [16], a pseudocompact, first countable, locally compact Hausdorff space which is not weakly star countable and hence not weakly star finite, is constructed in ZFC (an earlier example under CH appears as Example 2.3.2 in [3]). We note that in both these last cited papers (although not in [9]), the term "2-starcompact" is equivalent to pseudocompact in the class of Tychonoff spaces and weakly star finite implies "strongly 2-starcompact" (see Diagram 3 in [9], but the notation there is confusingly different). We address the problem of this relationship in Section 3, while Section 2 contains results on weakly and almost star countable spaces.

Although in the survey [9] no separation axioms are assumed unless explicitly stated, all spaces in this paper are assumed to be (at least) Hausdorff unless otherwise specified and all undefined terms can be found in [4].

2. Generalizations of the star countable property

In [9], Matveev defines a space to have the property $DC(\omega_1)$ if has a dense subspace D such that every uncountable subset of D has an accumulation point in X. The following result, cited in [9], seems to be part of the folk-lore of the subject, but for completeness we give a proof.

Theorem 2.1. *If a space X has a dense subspace D such that every uncountable subset of D has an accumulation point in X, then X is weakly star countable.*

Proof. Let $\mathcal{U}$ be an open cover of X and pick $x_0 \in D$. If $\mathrm{cl}(\mathsf{St}(\{x_0\},\mathcal{U})) \supseteq D$, then we are done; otherwise choose $x_1 \in D \setminus \mathrm{cl}(\mathsf{St}(\{x_0\},\mathcal{U}))$. Having constructed discrete sets $A_\alpha = \{x_\gamma : \gamma < \alpha\}$ for each $\alpha < \beta$, if $\mathrm{cl}(\mathsf{St}(A_\alpha,\mathcal{U})) \supseteq D$, then the construction ends. Otherwise pick $x_\alpha \in D \setminus \mathrm{cl}(\mathsf{St}(A_\alpha,\mathcal{U}))$; the set $A_{\alpha+1} = \{x_\beta : \beta \leq \alpha\}$ is clearly discrete. If for all $\alpha \in \omega_1$ we have that $D \nsubseteq \mathrm{cl}(\mathsf{St}(A_\alpha,\mathcal{U}))$, then by the hypothesis, A_{ω_1}

has an accumulation point $p \in X$ and hence there is some $U \in \mathcal{U}$ such that $p \in U$. However, since p is an accumulation point of A_{ω_1}, there is some minimal $\beta \in \omega_1$ such that $x_\beta \in U$ and then by construction, for all $\gamma \neq \beta$, $x_\gamma \notin U$, a contradiction. Thus for some ordinal $\delta \in \omega_1$, $\mathrm{cl}(\mathsf{St}(A_\delta, \mathcal{U})) \supseteq D$, concluding the proof. □

Corollary 2.2. *If a space X has a dense subspace of countable extent, then X is weakly star countable*

Proof. Let S be a dense subspace of X of countable extent, then every uncountable subset of S has an accumulation point in S. □

Corollary 2.3. *Each space with a dense σ-countably compact subspace is weakly star countable.*

Corollary 2.4. *A space with a dense Lindelöf subspace is weakly star countable.*

Recall that a space X is *weakly Lindelöf* if every open cover of X possesses a countable subfamily whose union is dense in X. Since every weakly Lindelöf space is weakly star countable, and each space of countable cellularity can easily be seen to be weakly Lindelöf, the following result is immediate.

Theorem 2.5. *A space X with countable cellularity is weakly star countable.*

Example 2.6. There is a space X with a dense σ-compact subspace which is not almost star countable.

Proof. Let $\lambda(\omega_1)$ denote the set of limit ordinals in $\omega_1 + 1$. The space X is a subset of the product $(\omega_1 + 1) \times (\omega + 1)$ obtained by omitting all points of the form (α, ω), for which $\alpha \in \lambda(\omega_1)$. Let $\mathcal{U}$ be the open cover of X defined as follows:

$$\mathcal{U} = \{\omega_1 \times \{n\} : n \in \omega\} \cup \{\{\beta\} \times (\omega + 1) : \beta \in \omega_1 \setminus \lambda(\omega_1)\}.$$

If A is a countable subset of $\omega_1 \times \omega$, then we may find $\alpha \in \omega_1$ such that $A \subseteq \alpha \times \omega$, and then if $\gamma \geq \alpha$, we have that $(\gamma, \omega) \notin \mathrm{cl}(\mathsf{St}(x, \mathcal{U}))$ for each $x \in A$. □

Question 2.7. Is there a normal example of a space with a dense Lindelöf subspace which is not almost star countable?

Theorem 2.8. *If X is a P-space, then the following conditions are equivalent:*

a) X is weakly star countable,

b) X is almost star countable.

Furthermore, each of the above conditions is implied by

c) X has a dense subspace of countable extent,

and if X is normal, then all three conditions are equivalent to

d) X is star countable.

Proof. The fact that c) ⇒ a) is Theorem 2.1 and that a) implies b) is trivial.

That b) implies a), follows from the fact that if $\{C_n : n \in \omega\}$ is a countable family of sets in a P-space X, then $\bigcup\{\mathrm{cl}_X(C_n) : n \in \omega\} = \mathrm{cl}_X(\bigcup\{C_n : n \in \omega\})$.

Suppose now that X is a normal P-space which has no dense subspace of countable extent. In particular, $e(X) > \omega$ and so we may find a closed discrete subspace $D = \{d_\alpha : \alpha \in \omega_1\} \subseteq X$ of cardinality ω_1. Since X is a P-space it is easy to see that we may find a disjoint open expansion $\mathcal{U} = \{U_\alpha : \alpha \in \omega_1\}$ of D, where $d_\alpha \in U_\alpha$ and since X is normal, we may assume that $\mathcal{U}$ is discrete. For each $\alpha \in \omega_1$, we may find a closed neighbourhood V_α of d_α such that $V_\alpha \subseteq U_\alpha$ and the open cover $\mathcal{F} = \mathcal{U} \cup \{X \setminus \bigcup\{V_\alpha : \alpha \in \omega_1\}\}$ witnesses the fact that X is not weakly star countable.

Finally note that if X is not star countable, then by Corollary 2.7 of [1], X has uncountable extent and the above argument shows that X is not weakly star countable. Thus b) implies d) and since the converse is trivial, the proof is complete. □

A construction similar to that of Example 2.6, shows that normality is a necessary condition for the equivalence of c) and d) in Theorem 2.8: Let $Y = (\omega_2 + 1)_\omega$ be the ω-modification of $\omega_2 + 1$ and $X \subseteq Y \times Y$ the space obtained by deleting all points of the form (α, ω_2), where α is a limit ordinal. The open covering

$$\{(\omega_2 + 1)_\omega \times (\omega_2)_\omega\} \cup \{\{\beta\} \times (\omega_2 + 1)_\omega : \beta \in \omega_2 \text{ is a non-limit ordinal}\}$$

witnesses that X is not star countable, but the subset $(\omega_2 + 1)_\omega \times (\omega_2)_\omega$ has countable extent. To see this, suppose that $D \subseteq (\omega_2 + 1)_\omega \times (\omega_2)_\omega$ is a discrete set of size ω_1, then there is some $\gamma \in \omega_2$ such that $D \subseteq (\gamma + 1)_\omega \times (\gamma + 1)_\omega = Z$. Since $(\gamma + 1)_\omega$ is Lindelöf, it follows from a result of Gewand (see [5]) that $Z \times Z$ is Lindelöf, and hence D is not closed.

Recall that a space is *weakly regular* if each non-empty open set contains a non-empty regular closed set and is *feebly Lindelöf* if every locally finite family of non-empty open sets is countable.

Theorem 2.9. *A weakly regular weakly star countable space is feebly Lindelöf.*

Proof. If a space X is not feebly Lindelöf, then there is an uncountable locally finite family of non-empty, mutually disjoint open sets $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$. Since X is weakly regular, for each $\alpha < \omega_1$, we may find a non-empty open set V_α such that $\mathrm{cl}(V_\alpha) \subseteq U_\alpha$. Now consider the open cover of X given by

$$\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{X \setminus \bigcup\{\mathrm{cl}(V_\alpha) : \alpha < \omega_1\}\}.$$

If $D \subseteq X$ is countable, then since each point of D lies in at most two elements of $\mathcal{U}$, it follows that there is some $\beta \in \omega_1$ such that $D \cap U_\beta = \emptyset$. Clearly, if $p \in V_\beta$, then $p \notin \mathrm{cl}(\bigcup\{\mathsf{St}(x, \mathcal{U}) : x \in D\})$ and so $\mathrm{cl}(\bigcup\{\mathsf{St}(x, \mathcal{U}) : x \in D\}) \neq X$. □

Theorem 2.10. *If X is pseudocompact and almost star countable then each space Y such that $X \subseteq Y \subseteq \beta X$ is almost star countable.*

Proof. Suppose that $\mathcal{U}$ is an open covering of Y; for each $U \in \mathcal{U}$, there is an open subset V_U of βX such that $V_U \cap Y = U$. Then $\mathcal{V} = \{V_U \cap X : U \in \mathcal{U}\}$ is an open cover of X and hence there is a countable set $C \subseteq X$ such that $X = \bigcup\{\mathrm{cl}_X(\mathsf{St}(x, \mathcal{V})) : x \in C\}$. However, $\mathsf{St}(x, \mathcal{V}) = X \cap (\bigcup\{V_U : x \in V_U\})$ and so

$$\mathrm{cl}_{\beta X}(\mathsf{St}(x, \mathcal{V})) = \mathrm{cl}_{\beta X}(\bigcup\{V_U : x \in V_U\}).$$

Since C is countable, it follows that

$$F = \bigcup\{\mathrm{cl}_{\beta X}(\bigcup\{V_U : x \in V_U\}) : x \in C\}$$

is an F_σ-set in βX which contains X. Since X is pseudocompact it follows that $\beta X \setminus X$ contains no non-empty G_δ and so $F = \beta X$. But then, since Y is dense in βX,

$$\mathrm{cl}_{\beta X}(\bigcup\{V_U : x \in V_U\}) = \mathrm{cl}_{\beta X}(\bigcup\{V_U \cap Y : x \in V_U\})$$

and so

$$\begin{aligned}\mathrm{cl}_{\beta X}(\bigcup\{V_U \cap Y : x \in V_U\}) \cap Y =\\ \mathrm{cl}_Y(\bigcup\{V_U \cap Y : x \in V_U\}) = \mathrm{cl}_Y(\mathsf{St}(x,\mathcal{U})).\end{aligned}$$

Finally, note that

$$\begin{aligned}Y &= \bigcup\{\mathrm{cl}_{\beta X}(\bigcup\{V_U : x \in V_U\}) : x \in C\} \cap Y\\ &= \bigcup\{\mathrm{cl}_Y(\mathsf{St}(x,\mathcal{U})) : x \in C\}. \qquad \square\end{aligned}$$

Theorem 2.11. *If X is a pseudocompact space, which is almost star countable and dense in a Tychonoff space Y, then Y is almost star countable.*

Proof. Let $i_\beta : \beta X \to \beta Y$ be the extension of the identity function $i : X \to Y$; we may find $X \subseteq Z \subseteq \beta X$ such that $i_\beta[Z] = Y$. The result now follows from Theorem 2.10 and the fact that the continuous image of an almost star countable space is almost star countable (this is Proposition 3.4 of [15]). $\square$

3. Weakly star finite spaces

It is clear that if P = 'finite', then the properties of being almost star P and weakly star P are identical. In this section we investigate the relationship between the properties of being feebly compact and being weakly star finite. A space is said to be *feebly compact* if every locally finite family of non-empty open sets is finite.

Arguments almost identical to those of Theorems 2.1 and 2.9 can be used to prove the following results. However, a proof of Theorem 3.1 can be found on page 14 of [9], where a space which satisfies the hypothesis is called *weakly countably compact* in [8] and *countably pracompact* in [9]; here we use the former terminology.

Theorem 3.1. *If a space X has a dense subspace D such that every infinite subset of D has an accumulation point in X, then X is weakly star finite.*

Theorem 3.2. *A weakly regular weakly star finite space is feebly compact.*

For the proof of the next result it is necessary to note the following characterization of feeble compactness which can be found in Section 1.11 of [12]: A space X is feebly compact if and only if each countable open cover of X contains a finite subfamily whose union is dense in X.

Theorem 3.3. *If X is feebly compact and star countable then it is weakly star finite.*

Proof. Let $\mathcal{U}$ be an open cover of X; since X is star countable, there is a countable subset $N = \{x_n : n \in \omega\} \subseteq X$ such that $\mathsf{St}(N,\mathcal{U}) = X$. For each $n \in \omega$, let $F_n = \mathsf{St}(\{x_n\},\mathcal{U})$; then $\mathcal{F} = \{F_n : n \in \omega\}$ is a countable open cover of X and since X is feebly compact, there is a finite subcollection $\mathcal{G} \subseteq \mathcal{F}$, such that $\mathrm{cl}(\bigcup\mathcal{G}) = X$. $\square$

Corollary 3.4. *A separable feebly compact space is weakly star finite.*

Theorem 3.5. *A space with a dense set of isolated points is weakly star finite if and only if it is feebly compact.*

Proof. A feebly compact space with a dense set D of isolated points is easily seen to be weakly countably compact and so by Theorem 3.1 (or [9]), such a space is weakly star finite. For the converse, suppose that X is not feebly compact; then there is a locally finite family of non-empty open sets $\mathcal{U} = \{U_n : n \in \omega\}$

and for each $n \in \omega$ we may pick $x_n \in D \cap U_n$ and denote by A the set $\{x_n : n \in \omega\}$. The open cover $\mathcal{F} = \{\{x_n\} : n \in \omega\} \cup \{X \setminus A\}$ witnesses that X is not weakly star finite. □

Corollary 3.6. *A dispersed space is weakly star finite if and only if it is feebly compact.*

An argument analogous to that of Theorem 3.5 but using transfinite recursion to ω_1, can be used to prove the following result.

Theorem 3.7. *A space with a dense set of isolated points is weakly star countable if and only if it is feebly Lindelöf.*

It is well known and easy to prove that being star finite is equivalent to being countably compact and it is easy to see that each H-closed space and each space with a dense countably compact subspace is weakly star finite. However, the spaces Ψ of Theorem 2.20 of [11] are feebly compact and have a dense set of isolated points, hence by the previous theorem, they are weakly star finite, but they need not be star countable. Here (as in [11]) we define a family $\mathcal{M} \subseteq \mathcal{P}(\kappa)$ to be *almost disjoint* if each element of $\mathcal{M}$ is infinite and the intersection of any two distinct elements is finite.

Suppose that $\kappa > \mathfrak{c}$ such that $\kappa^\omega = \kappa$ and let $\mathcal{M}$ be a maximal almost disjoint family on κ of cardinality at least κ. Let $\Psi = \kappa \cup \mathcal{M}$, with the topology in which each point of κ is isolated and a basic open neighbourhood of $M \in \mathcal{M}$ is of the form $\{M\} \cup (M \setminus F)$, where $F \subseteq M$ is finite. Faithfully enumerate the countable subsets of κ as $\{S_\alpha : \alpha < \kappa\}$ and suppose that we have chosen points $M_\beta \in \mathcal{M}$ and basic open neighbourhoods U_β of M_β such that $U_\beta \cap S_\beta = \emptyset$ for each $\beta < \alpha$. If $M \in \mathrm{cl}(S_\alpha)$, then $M \cap S_\alpha$ is infinite and so if $M_1, M_2 \in \mathrm{cl}(S_\alpha)$ are distinct, then $M_1 \cap S_\alpha \neq M_2 \cap S_\alpha$. Thus $|\mathrm{cl}(S_\alpha) \cap \mathcal{M}| \leq |\mathcal{P}(S_\alpha)| = \mathfrak{c}$ and so we may choose $M_\alpha \in \mathcal{M}$ so that $M_\alpha \notin \mathrm{cl}(S_\alpha)$ and $M_\alpha \neq M_\beta$ for all $\beta < \alpha$. Let U_α be a basic open neighbourhood M_α such that $U_\alpha \cap S_\alpha = \emptyset$. The open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \cup \{\{M\} \cup \kappa : M \in \mathcal{M},\, M \neq M_\alpha, \text{ for all } \alpha \in \kappa\}$$

witnesses that Ψ is not star countable.

Even more, if λ is any uncountable cardinal, and $\mathcal{P}$ is a partition of λ into ω_1-many infinite sets, then any Ψ-space constructed with a maximal almost disjoint family $\mathcal{M}$ which contains $\mathcal{P}$ will not be star countable, since the open cover

$$\{P \cup \{P\} : P \in \mathcal{P}\} \cup \{\lambda \cup \{M\} : M \in \mathcal{M} \setminus \mathcal{P}\}$$

has no countable star kernel.

Another example of a pseudocompact space which is not star countable can be found in Section 2.3 of [2], but we do not know whether the space described there (due essentially to Reznichenko in [13]) is weakly star finite or not. It is well known that the product of two countably compact spaces need not be countably compact and so the property of being star finite is not even finitely productive. Using the example found in Section 9.15 of [6] a simple argument shows that the product of two countably compact spaces, each with a dense set of isolated points, may not even be weakly star finite. Although the product of a weakly countably compact space with a compact space is known to be weakly countably compact (see [8] or page 57 of [9]), we do not know if the product of a weakly star finite space and a compact space or an H-closed space is weakly star finite. The next two theorems attempt to address this problem.

We say that a space X is *weakly sequentially compact* if it has a dense subspace D such that every sequence in D has a subsequence which converges in X. Clearly a weakly sequentially compact space is weakly countably compact.

Theorem 3.8. *Suppose that X is weakly countably compact and Y is weakly sequentially compact, then $X \times Y$ is weakly countably compact.*

Proof. Let D denote a dense subset of X with the property that each infinite subset of D has an accumulation point in X and let E be a dense subset of Y such that every sequence in E has a convergent subsequence in Y; Let $S = \{(x_n, y_n) : n \in \omega\}$ be an infinite subset of $D \times E$. If there is some $a \in X$ or some $b \in Y$ such that either $\{n \in \omega : x_n = a\}$ or $\{n \in \omega : y_n = b\}$ is infinite, then clearly S has an accumulation point in $X \times Y$; thus we may assume without loss of generality that $\{y_n : n \in \omega\}$ is injective. Since Y is weakly sequentially compact, we may find a convergent (necessarily injective) subsequence which we denote by $\langle y_{\phi(n)} \rangle$ and which converges to $y \in Y$. The corresponding sequence $\langle x_{\phi(n)} \rangle$ is a sequence of points in D which, since X is weakly countably compact, has an accumulation point $x \in X$. It follows that (x, y) is an accumulation point of the sequence $\langle (x_{\phi(n)}, y_{\phi(n)}) \rangle$ and hence, since this sequence is injective, also of the set S. □

Corollary 3.9. *The product of two weakly countably compact spaces, one of which is sequential, is weakly countably compact.*

Corollary 3.10. *If X is weakly countably compact and Y is weakly sequentially compact, then $X \times Y$ is weakly star finite.*

Theorem 3.11. *If X is a first countable H-closed space and K is countably compact, then $X \times K$ is weakly star finite.*

Proof. For each $x \in X$, let $\mathcal{B}^x = \{B_n^x : n \in \omega\}$ be a countable nested local base of open sets at x and let $\mathcal{U}$ be an open cover of $X \times K$. Since $\bigcup\{\mathcal{B}^x : x \in X\}$ is a base for X, we may assume the elements of $\mathcal{U}$ are of the form $B \times W$, where $B \in \bigcup\{\mathcal{B}^x : x \in X\}$ and W is open in K. For the moment, fix $z \in X$ and let

$$\mathcal{C}_n^z = \{W : V \times W \in \mathcal{U} \text{ and } B_n^z \subseteq V\};$$

then $\{\bigcup \mathcal{C}_n^z : n \in \omega\}$ is an ascending countable open cover of K, and hence there is some $k(z) \in \omega$ such that $\bigcup \mathcal{C}_{k(z)}^z = K$. Thus $\mathcal{C}_{k(z)}^z$ is an open cover of K and again since K is countably compact, there is some finite set $F^z \subseteq K$ such that $\mathsf{St}(F^z, \mathcal{C}_{k(z)}^z) = K$; hence

$$\mathsf{St}(\{z\} \times F^z, \mathcal{U}) \supseteq B_{k(z)}^z \times K.$$

Now let z vary; $\{B_{k(z)}^z : z \in X\}$ is an open covering of X and, since X is H-closed, there is a finite set $G \subseteq X$ such that $\bigcup\{B_{k(z)}^z : z \in G\}$ is dense in X. It is then clear that if we let $H = \bigcup\{\{z\} \times F^z : z \in G\}$, then

$$\mathrm{cl}(\mathsf{St}(H, \mathcal{U})) = X \times K. \qquad \square$$

The product of a feebly compact space and an H-closed space is feebly compact (see [12], 4AF); this result together with the previous theorem leads us to ask:

Question 3.12. Is the product of an H-closed space and a countably compact space weakly star finite?

The following questions also remain open:

Question 3.13. Is a feebly compact (Tychonoff, regular, Hausdorff) space which is weakly star countable space necessarily weakly star finite?

Question 3.14. Is a Hausdorff weakly star countable (respectively, weakly star finite) space necessarily feebly Lindelöf (respectively, feebly compact)?

Question 3.15. If $|\beta X \setminus X| = 1$, is X weakly star finite?

Question 3.16. Is a feebly compact space with countable cellularity or calibre ω_1, weakly star finite?

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