

Scalarization of asymptotically anti-de Sitter black holes with applications to holographic phase transitions

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We study the spontaneous scalarization of spherically symmetric, static and asymptotically anti-de Sitter (aAdS) black holes in a scalar-tensor gravity model with nonminimal coupling of the form $\phi^2(\alpha\mathcal{R} + \gamma\mathcal{G})$, where α and γ are constants, while $\mathcal{R}$ and $\mathcal{G}$ are the Ricci scalar and Gauss-Bonnet term, respectively. Since these terms act as an effective “mass” for the scalar field, nontrivial values of the scalar field in the black hole space-time are possible for *a priori* vanishing scalar field mass. In particular, we demonstrate that the scalarization of an aAdS black hole requires the curvature invariant $-(\alpha\mathcal{R} + \gamma\mathcal{G})$ to drop below the Breitenlohner-Freedman bound close to the black hole horizon, while it asymptotes to a value well above the bound. The dimension of the dual operator on the AdS boundary depends on the parameters α and γ and we demonstrate that—for fixed operator dimension—the expectation value of this dual operator increases with decreasing temperature of the black hole, i.e., of the dual field theory. When taking backreaction of the space-time into account, we find that the scalarization of the black hole is the dual description of a phase transition in a strongly coupled quantum system, i.e., corresponds to a holographic phase transition. A possible application are liquid-gas quantum phase transitions, e.g., in ^4He . Finally, we demonstrate that extremal black holes with $\text{AdS}_2 \times S^2$ near-horizon geometry *cannot support regular scalar fields on the horizon* in the scalar-tensor model studied here.

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I. INTRODUCTION

With the advent of high precision, multimessenger observations of black holes in a broad interval of masses and sizes (see e.g., [1–3]), it will soon be possible to test theoretical predictions of general relativity related to these objects. One of the most interesting questions is whether classical black holes are, indeed, very simple, structureless objects that can be described by a very small number of parameter—the mass, charge and angular momentum—and as such fulfill the *No hair conjecture*. Theoretical black holes in asymptotically flat, 4-dimensional Einstein-Maxwell theory have been proven to possess this feature [4], but the existence of black holes with primary [5,6] and secondary hair [7] in models with nonlinear matter sources demonstrates clearly that it is far from obvious that this conjecture holds true in other settings. In fact, while a number of theorems for scalar fields in black hole space-times exist [8], black holes can carry nontrivial scalar fields on their horizon when they are e.g., nonminimally coupled to the curvature of the space-time. This has been investigated extensively over the past years in the context of Horndeski scalar-tensor gravity models [9–11]. Indeed, in these models, static, asymptotically flat black holes that carry scalar hair can be constructed [12,13]. “Spontaneous

scalarization” is a phenomenon that appears typically in models that contain nonminimal coupling terms of the form $f(\phi)\mathcal{I}(g_{\mu\nu}, \Sigma)$, where $f(\phi)$ is a function of the scalar field, while $\mathcal{I}$ depends on the metric $g_{\mu\nu}$ and/or other fields Σ . The scalar field then gets “sourced” by $\mathcal{I}$ and spontaneously scalarized black holes can be constructed for sufficiently large couplings. Recent examples include the scalarization of static, uncharged, asymptotically flat black holes using $\mathcal{I} = \mathcal{G}$, where $\mathcal{G}$ is the Gauss-Bonnet term and $f(\phi) = \phi^2$ [14], different other forms of $f(\phi)$ with a single term in $f(\phi)$ [15–17] or a combination of different powers of ϕ [18,19]. These studies have been extended to include charge [20–22] as well as a positive cosmological constant [23–25].

Modifications of general relativity are usually motivated by assuming general relativity to be only a classical, low energy limit of a (more general) quantum theory of gravity that should be applicable as well at and close to the Planck scale. One of the best candidates for such a theory remains string theory. One remarkable prediction of string theory is the so-called gauge/gravity duality, a conjecture that relates gravity theories in $(d + 1)$ space-time dimensions to gauge theories in d dimensions [26]. The best tested and well studied example is the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [27], which connects a gravity

theory in $(d + 1)$ -dimensional AdS space-time to an $SU(N)$ gauge theory on the d -dimensional boundary of AdS. This duality is a weak-strong coupling duality such that “weakly coupled”, classical gravity theories in AdS can be used to describe strongly coupled quantum systems on the conformal boundary of AdS. With a black hole present in the bulk of AdS, the quantum system can be studied at a given temperature. Holographic phase transitions typically appear when lowering the black hole temperature and correspond to nontrivial matter fields forming on the black hole below a certain critical temperature [28]. These ideas have mainly been applied to the description of high-temperature superconductivity in the framework of holographic superconductors [29,30] as well as the description of the quark-gluon plasma (see e.g., [31] and references therein). The first studies have typically been conducted using scalar fields. In fact, as shown in [32], a scalar field becomes unstable in AdS if its mass m drops below the so-called Breitenlohner-Freedman (BF) bound, i.e., if $m^2 \leq m_{\text{BF}}^2$. This bound is dimension-dependent and can be shown to lead to formation of nontrivial scalar hair in a number of settings within asymptotically AdS (aAdS) space-time.

In this paper, we discuss a scalar-tensor gravity model in $(3 + 1)$ -dimensional aAdS space-time. In this model, the scalar field is *a priori* massless, but couples nontrivially to the Ricci scalar and Gauss-Bonnet term, respectively, of the space-time. As we will demonstrate in the following, this nontrivial coupling corresponds to an “effective mass” for the scalar field and generates the formation of nontrivial scalar hair on the black hole for sufficiently low temperature of the latter. When taking backreaction of the space-time into account, this “spontaneous scalarization” can be interpreted as a holographic phase transition, i.e., a phase transition on a spatially 2-dimensional surface with order parameter given by a real-valued scalar field and appearing in a strongly coupled quantum system. Note that uncharged, static, spherically symmetric, aAdS black holes in a scalar-Gauss-Bonnet model have been studied briefly in [23], however, not with the emphasis on solutions that posses a power law fall-off on the AdS boundary, a requirement that will be crucial for us in the following.

Our paper is organized as follows: in Sec. II, we discuss the model, while Sec. III deals with the probe limit, i.e., the limit of vanishing backreaction of the space-time. In Sec. IV, the inclusion of backreaction is presented, while Sec. V contains our conclusions.

II. THE MODEL

In this paper, we study a scalar-tensor gravity model with the following action

$$S = \int d^4x \sqrt{-g} \left[\frac{\mathcal{R}}{2} - \Lambda + \phi^2 (\alpha \mathcal{R} + \gamma \mathcal{G}) - \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad (1)$$

where $\mathcal{R}$ is the Ricci scalar, $\mathcal{G}$ the Gauss-Bonnet term, $\Lambda < 0$ the cosmological constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the field strength tensor of a $U(1)$ gauge field A_μ and ϕ a real-valued, massless scalar field that is coupled to the Ricci scalar $\mathcal{R}$ as well as the Gauss-Bonnet term $\mathcal{G}$ given by

$$\mathcal{G} = (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2) \quad (2)$$

via the couplings α and γ , respectively. Variation of the action with respect to the metric, scalar field and $U(1)$ gauge field leads to a set of coupled differential equations that have to be solved numerically given appropriate boundary conditions. Variation with respect to the scalar field, $U(1)$ gauge field and metric, respectively, leads to the following set of coupled, nonlinear differential equations:

$$\begin{aligned} \square \phi + (\alpha \mathcal{R} + \gamma \mathcal{G}) \phi &= 0, & \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) &= 0, \\ G_{\mu\nu} + \Lambda g_{\mu\nu} &= T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{EM})}, \end{aligned} \quad (3)$$

where

$$\begin{aligned} T_{\mu\nu}^{(\phi)} &= \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\sigma \phi \partial^\sigma \phi \\ &\quad + 4\alpha [D_\mu (\phi \partial_\nu \phi) - g_{\mu\nu} D_\sigma (\phi \partial^\sigma \phi)] \\ &\quad - 2\gamma (g_{\rho\mu} g_{\lambda\nu} + g_{\lambda\mu} g_{\rho\nu}) \eta^{\kappa\lambda\beta\delta} \eta^{\rho\iota\sigma\tau} R_{\sigma\tau\beta\delta} D_\iota (\phi \partial_\kappa \phi) \end{aligned} \quad (4)$$

and

$$T_{\mu\nu}^{(\text{EM})} = F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}. \quad (5)$$

In the following, we will consider spherically symmetric, static black hole solutions. The Ansatz for the metric reads:

$$ds^2 = -N\sigma^2 dt^2 + \frac{1}{N} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (6)$$

with $N \equiv N(r)$ and $\sigma \equiv \sigma(r)$ depending on the radial coordinate r only. The explicit form of the curvature invariants is then:

$$\begin{aligned} \mathcal{R} &= \frac{1}{r^2 \sigma} (-N'' r^2 \sigma - 3N' \sigma' r^2 - 4N' r \sigma - 2N \sigma'' r^2 - 4N \sigma' r \\ &\quad - 2N \sigma + 2\sigma), \end{aligned} \quad (7)$$

$$\begin{aligned} \mathcal{G} &= \frac{4}{r^2 \sigma} (N'' N \sigma - N'' \sigma + (N')^2 \sigma + 5N' \sigma' N - 3N' \sigma' \\ &\quad + 2\sigma'' N^2 - 2\sigma'' N). \end{aligned} \quad (8)$$

Assuming the symmetries of the $U(1)$ gauge field and the scalar field, respectively, to be equivalent to those of the space-time, we choose $A_\mu dx^\mu \equiv V(r) dt$ and $\phi \equiv \phi(r)$. Inserting the Ansätze into the equations of motions leads

to a coupled set of nonlinear ordinary differential equations of the form

$$\begin{aligned} N' &= \mathcal{F}_1(N, \sigma, V, V', \phi, \phi'), & \sigma' &= \mathcal{F}_2(N, \sigma, V, V', \phi, \phi'), \\ V'' &= \mathcal{F}_3(N, \sigma, V, V', \phi, \phi'), & \phi'' &= \mathcal{F}_4(N, \sigma, V, V', \phi, \phi'), \end{aligned} \quad (9)$$

where the prime now and in the following denotes the derivative with respect to r and $\mathcal{F}_i$, $i = 1, 2, 3, 4$ are functions of the arguments. The system (9) has to be solved according to appropriate boundary conditions. At the regular horizon $r = r_h$ with $N(r = r_h) = 0$ the condition for the scalar field reads:

$$\left(\frac{\phi'}{\phi}\right)_{r=r_h} = -\left(\frac{\alpha\mathcal{R} + \gamma\mathcal{G}}{N'}\right)_{r=r_h}, \quad (10)$$

while for the U(1) gauge field we have $V(r = r_h) = 0$. As long as $\mathcal{R}$ and $\mathcal{G}$ are well behaved on the horizon, we would expect ϕ and ϕ' to be also well behaved there except when N' becomes zero. This corresponds to the extremal limit, where the Hawking temperature of the black hole given by

$$T_H = \frac{1}{2\pi}(\sigma N')|_{r=r_h} \quad (11)$$

tends to zero. For $r \rightarrow \infty$, we assume the space-time to be asymptotically AdS and the black hole to possess an electric charge Q , i.e.,

$$\begin{aligned} \sigma(r \rightarrow \infty) &\rightarrow 1, & N(r \rightarrow \infty) &\rightarrow 1 - \frac{\Lambda}{3}r^2, \\ V(r \rightarrow \infty) &\rightarrow \frac{Q}{r_h} - \frac{Q}{r} \equiv \mu - \frac{Q}{r}, \end{aligned} \quad (12)$$

where μ corresponds to the value of the U(1) gauge field on the conformal boundary $r \rightarrow \infty$ and can be interpreted as chemical potential in gauge/gravity applications. Finally, for $\Lambda < 0$, the scalar field has a power-law fall-off of the form:

$$\phi(r \rightarrow \infty) \rightarrow \frac{\phi_+}{r^{\lambda_+}} + \frac{\phi_-}{r^{\lambda_-}} \quad (13)$$

with

$$\lambda_{\pm} = \frac{3 \pm \sqrt{\Delta}}{2}, \quad \Delta \equiv 9 + 48\alpha + 32\gamma\Lambda, \quad (14)$$

i.e., the couplings α and γ together with Λ determine the dimension of the dual operator on the conformal boundary in gauge/gravity applications and $\phi_{\pm}$ can be interpreted as the expectation value of this operator. Note that $\Lambda = 0$, i.e., the asymptotically flat case, is explicitly excluded here and is not a “smooth limit” of (14). In fact, as shown

in [14,15,22] the scalar field always falls off like $\phi(r \rightarrow \infty) \sim \phi_0/r$ for $\Lambda = 0$ (independent of α and γ), where ϕ_0 is interpreted as the scalar charge of the solution. This interpretation is no longer possible in aAdS and we will hence in the following refer to $\phi_{\pm}$ as “the expectation value of the dual operator on the conformal boundary”—using gauge/gravity terminology. Note that although our calculation is done in Schwarzschild-like coordinates, the result for the power of the scalar-field fall-off agrees with that obtained in the Fefferman-Graham construction, see Appendix B.

Let us mention as well that there is a scaling symmetry in the model, which reads (including all parameters and field):

$$\begin{aligned} r &\rightarrow \beta r, & t &\rightarrow t, & M &\rightarrow \beta M, & Q &\rightarrow \beta Q, & \Lambda &\rightarrow \frac{\Lambda}{\beta^2}, \\ \alpha &\rightarrow \alpha, & \gamma &\rightarrow \beta^2 \gamma, & \phi_{\pm} &\rightarrow \beta^{\lambda_{\pm}} \phi_{\pm}, & \mu &\rightarrow \beta \mu, \end{aligned} \quad (15)$$

which scales the metric by β^2 and $A_{\mu}dx^{\mu}$ by β . This allows to set one parameter to a fixed value without loss of generality. In our numerical construction (see below), we will often fix the horizon radius to $r_h = 1$.

III. THE PROBE LIMIT

For vanishing scalar field $\phi(r) \equiv 0$, the model has explicit solutions: the Schwarzschild-anti-de Sitter (SAdS) solution for vanishing electric charge and the Reissner-Nordström-anti-de Sitter (RNAdS) solution for nonvanishing electric charge, respectively. These solutions read:

$$\begin{aligned} N(r) &= 1 - \frac{2M}{r} + \frac{Q^2}{2r^2} - \frac{\Lambda}{3}r^2, & \sigma &\equiv 1, \\ V(r) &= Q\left(\frac{1}{r_h} - \frac{1}{r}\right) \equiv \mu - \frac{Q}{r}. \end{aligned} \quad (16)$$

M is the mass and Q the electric charge of the solution. The event horizon r_h is the largest root of the equation $N(r_h) = 0$ and leads to the following relations

$$\begin{aligned} M &= \frac{1}{2}r_h - \frac{\Lambda}{6}r_h^3 + \frac{Q^2}{4r_h}, & \Lambda &= 3\left(\frac{1}{r_h^2} - \frac{2M}{r_h^3} + \frac{Q^2}{2r_h^4}\right), \\ Q &= \pm\sqrt{-2r_h^2 + \frac{2}{3}\Lambda r_h^4 + 4Mr_h}. \end{aligned} \quad (17)$$

The Hawking temperature (11) of the RNAdS black hole reads

$$2\pi T_H = -\Lambda r_h - \frac{Q^2}{2r_h^3} + \frac{1}{r_h} \quad (18)$$

and becomes zero for the extremal solution, which fulfills $N(r_{h,\text{ex}}) = N'(r_{h,\text{ex}}) = 0$. This latter condition gives:

$$r_{h,\text{ex}} = \sqrt{\frac{1 - \sqrt{1 - 2\Lambda Q_{\text{ex}}^2}}{2\Lambda}},$$

$$Q_{\text{ex}} = \pm r_{h,\text{ex}} \sqrt{2(1 - \Lambda r_{h,\text{ex}}^2)}. \quad (19)$$

Hence, the extremal possible charge Q_{ex} increases (decreases) from $+\sqrt{2}r_{h,\text{ex}}$ (from $-\sqrt{2}r_{h,\text{ex}}$) at $\Lambda = 0$ when decreasing Λ , which is related to the additional attractive nature of the negative cosmological constant.

In the following, we will assume this space-time background to be fixed and will consider a scalar field that does not backreact onto the space-time. The linear scalar field equation then reads:

$$\frac{1}{r^2} (r^2 N \phi')' + (\alpha \mathcal{R} + \gamma \mathcal{G}) \phi = 0, \quad (20)$$

where the explicit form of the Gauss-Bonnet term and the Ricci scalar, respectively, for the RNAdS solution are:

$$\mathcal{G} = \frac{8}{3}\Lambda^2 + \frac{2}{r^8} (24r^2 M^2 - 24rMQ^2 + 5Q^4), \quad \mathcal{R} = 4\Lambda. \quad (21)$$

Moreover, as is obvious from these expressions, both $\mathcal{R}$ and $\mathcal{G}$ are constant on the AdS boundary at $r \rightarrow \infty$. Hence, all arguments related to the holographic interpretation, regularization and renormalization of a scalar field model with massive scalar field in asymptotically AdS apply also here (see e.g., [33] for a discussion on these issues).

A. The Breitenlohner-Freedman bound and parameter restrictions

The $(d+1)$ -dimensional asymptotically AdS (aAdS) space-time with AdS radius $\ell = \sqrt{-d(d-1)/(2\Lambda)}$ possesses a classical instability for a massive, real scalar field with equation $(\square - m^2)\phi = 0$ if the mass m is below the so-called Breitenlohner-Freedman (BF) bound [32], i.e., for $m^2 < m_{\text{BF}}^2 = -d^2/(4\ell^2)$, a nontrivial scalar field will form. In our case, the “mass” is given by the term $m^2 \equiv m_{\text{eff}}^2 = -(\alpha \mathcal{R} + \gamma \mathcal{G})$, i.e., we would expect $(d+1)$ -dimensional aAdS to form a nontrivial scalar for

$$\alpha \mathcal{R} + \gamma \mathcal{G} = 4\alpha\Lambda + \gamma \mathcal{G} \geq \frac{d^2}{4\ell^2} = -\frac{d\Lambda}{2(d-1)}. \quad (22)$$

In the following, we will require that $(3+1)$ -dimensional aAdS is stable with respect to the formation of a scalar field, i.e., that asymptotically, the space-time is “pure” AdS. We hence obtain the following restriction on the parameters when using the asymptotic forms of the curvature tensors

$$4\alpha\Lambda + \frac{8}{3}\gamma\Lambda^2 \leq \frac{9}{4\ell^2}. \quad (23)$$

A quick inspection of (14) demonstrates that the requirement $\Delta \geq 0$ is exactly (23).

On the other hand, we would like a nontrivial scalar field to form close to the black hole horizon. In the following, we will demonstrate that (22) is fulfilled close to the event horizon of the scalarized black holes that we present in this paper. Note that the scalarization of black holes for $\alpha > 0$, $\gamma = 0$ is impossible due to the above argument.

B. Numerical results

Equation (20) has to be solved numerically since—to our knowledge—no analytical solutions to this equation exist for generic values of α and γ . We have solved the equations using an adaptive grid collocation solver [34]. We have also chosen $\phi_- = 0$ for all our calculations. Moreover, we can choose $\phi(r_h) = 1$ as well as $r_h = 1$ without losing generality. The latter condition fixes the mass M in terms of Q and Λ . In the following we will only discuss cases with $\Lambda > -3$ in order to ensure that the AdS radius $\ell = \sqrt{-\Lambda/3}$ is larger than the horizon radius r_h .

1. $\alpha = 0$

In the limit $Q = 0$, the black hole is given by the Schwarzschild-AdS (SAdS) solution which has vanishing chemical potential $\mu = 0$. As is well known from the $\Lambda = 0$ case, spontaneous scalarization of the Schwarzschild black hole appears for a very specific value of $\gamma = \gamma_{\text{cr}}$, in our choice of couplings and prefactors for $\gamma_{\text{cr}}(\Lambda = 0) \approx 0.18$ [14]. We would hence expect something similar to appear in an asymptotically AdS space-time. Indeed, we find that spontaneous scalarization appears for $\gamma = \gamma_{\text{cr}}(\Lambda)$. We show the dependence of γ_{cr} on Λ in Fig. 1 (black solid curve). Decreasing Λ , γ_{cr} first increases slightly and then decreases again, but shows little dependence on the value of Λ . This means that the value of the curvature radius of the space-time, i.e., the AdS radius, has little influence on the scalarization process of the black hole as long as this radius is (much) larger than the horizon radius of the black hole. This seems reasonable since the scalarization happens on and close to the horizon. Λ can only be decreased to a minimal value of $\Lambda_* \approx -1.65$ corresponding to an AdS radius of $\ell \approx 1.35$ which is comparable in size to the horizon radius $r_h = 1.0$. At this value of Λ , the power-law fall off of the scalar field function is no longer possible since Δ (black dashed) tends to zero. Inserting Λ_* into the equation $\Delta = 0$ gives $\gamma_* \approx 0.17$, which is in perfect agreement with our numerical results. Accordingly, the power of the scalar field fall-off λ_+ (equivalent to the dimension of the dual operator on the conformal boundary, black dotted-dashed) ranges from three at $\Lambda = 0$ to $3/2$ at Λ_* .

The case for $\Lambda = 0$ and $Q \neq 0$ has been studied in [22]. In particular, it was found that two independent and disjoint branches of charged, spontaneously scalarized black holes exist: one is the solution that tends to the

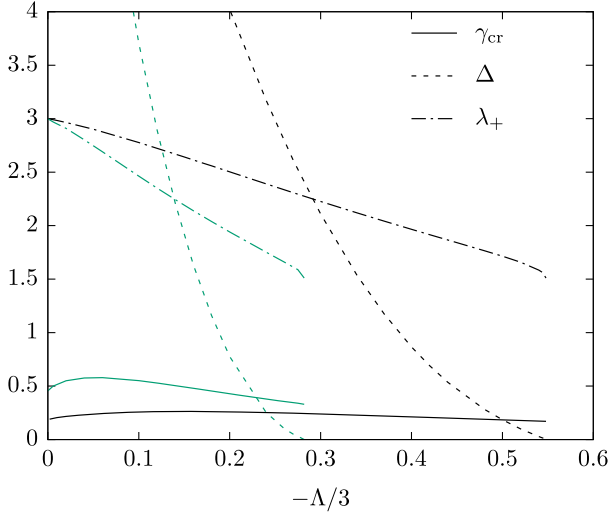


FIG. 1. We give the value of γ (solid) for which nontrivial solutions of the scalar field equation (20) exist in dependence on $-\Lambda/3 \equiv 1/\ell^2$ for a SAdS background ($Q = 0$, black) and a RNAdS background with $Q = 1$ (green). We also give Δ (dashed) as well as the value of λ_+ (dotted-dashed). The latter corresponds to the dimension of the operator on the conformal boundary in the gauge/gravity duality interpretation. Note that all curves stop at a given value of Λ because $\Delta \rightarrow 0$ there.

uncharged solution of [14–16] and exists for $\gamma_{\text{cr}} > 0$, while the second branch appears close to extremality of the RN solution and requires $\gamma_{\text{cr}} < 0$. In order to understand the pattern, we have first fixed $Q = 1$ and compared the results with those for $Q = 0$, see Fig. 1. Qualitatively, the results are very similar to those of the uncharged case when considering only $\gamma_{\text{cr}} > 0$. Quantitatively, we observe that γ_{cr} (solid green) is always larger than in the uncharged case and that the value of Λ_* at which $\Delta \rightarrow 0$ increases with charge. For $Q = 1$ we find $\Lambda_* \approx -0.84$, which—using $\Delta = 0$ —gives the value $\gamma_* \approx 0.33$, again in excellent agreement with our numerics (see green dashed curve). We observe that again $\lambda_+ \in [\frac{3}{2}; 3]$ (green dotted-dashed

curve), but that this variation is related to a smaller variation of Λ as in the uncharged case.

In Fig. 2 we show some typical solutions for $\Lambda = -0.006$ and different charges Q . Following the discussion in III A, we would expect $-\gamma_{\text{cr}}\mathcal{G}$ to drop below the BF bound close to the horizon of the black hole. For our choice of Λ , the value of BF bound is $m_{\text{BF}}^2 = -0.0045$. Inspection of Fig. 2 (left) demonstrates that close to the horizon $r_h = 1$, we find $-\gamma_{\text{cr}}\mathcal{G} < -0.0045$, while asymptotically the requirement $\Delta \geq 0$ ensures stability of AdS. Correspondingly, nontrivial scalar fields appear close to the horizon, see Fig. 2 (right). We also observe that the larger the charge Q , the larger we have to choose γ_{cr} to find scalarized black holes. E.g., for the charges given in Fig. 2 we find $\gamma_{\text{cr}} = 0.48, 0.60, 1.00$ and 2.00 for $Q^2 = 0.9946, 1.1032, 1.3083, 1.8106$, respectively. To state it differently: charged black holes require a stronger scalar-tensor coupling in order to be scalarized as compared to their uncharged counterparts.

When increasing Q , we observe a phenomenon that exists also for $\Lambda = 0$ and was first discussed in [22]: the GB term becomes negative close to the black hole horizon due to the approach of extremality, i.e., the approach to a solution with near-horizon geometry $\text{AdS}_2 \times S^2$. For $\Lambda = -0.006$ and $r_h = 1$, the value of the extremal charge is $Q_{\text{ex}}^2 = 1.988$, but we see the appearance of a negative valued GB term already at $Q^2 = 1.3083$ and $Q^2 = 1.8106$.

To evaluate the influence of the cosmological constant on this phenomenon in more detail, we have chosen $\Lambda = -0.6$ (with $Q_{\text{ex}}^2 = 3.2$) and compared the two γ_{cr} branches with those present for $\Lambda = 0$ (with $Q_{\text{ex}}^2 = 2.0$). The results are shown in Fig. 3. We observe that while the positive γ_{cr} branches exists for $Q^2 \in [0; Q_{\text{ex}}^2]$ for $\Lambda = 0$, this is no longer true for $\Lambda = -0.6$. The positive γ_{cr} branch stops when $\Delta = 0$, i.e., for $\gamma_{\text{cr}} = 0.46875$. We find the corresponding value of the charge to be $Q^2 \approx 1.44$, well below the extremal charge $Q_{\text{ex}}^2 = 3.2$. To state it differently, the requirement of asymptotically AdS space-time being stable, i.e., the BF bound, imposes a stronger restriction on the solutions—at

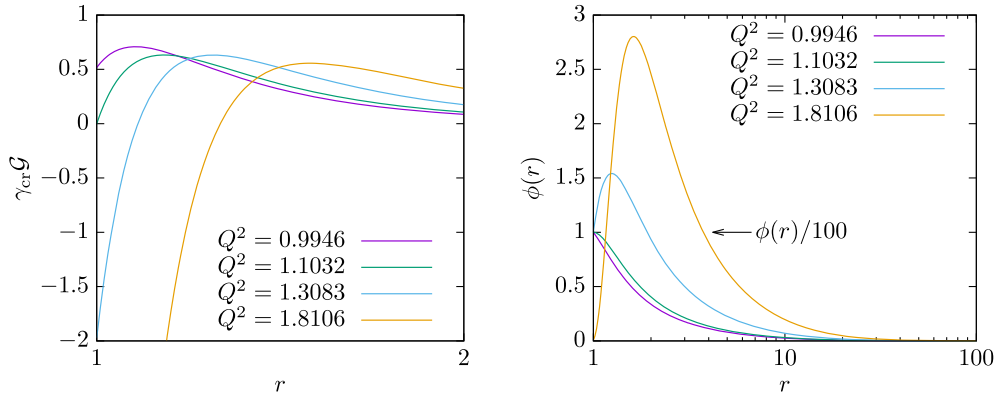


FIG. 2. We show the Gauss-Bonnet term $\gamma_{\text{cr}}\mathcal{G}$ for the RNAdS solutions with $r_h = 1$, $\Lambda = -0.006$ and different values of Q^2 (left) as well as the scalar field solution $\phi(r)$ in the corresponding background RNAdS space-time (right).

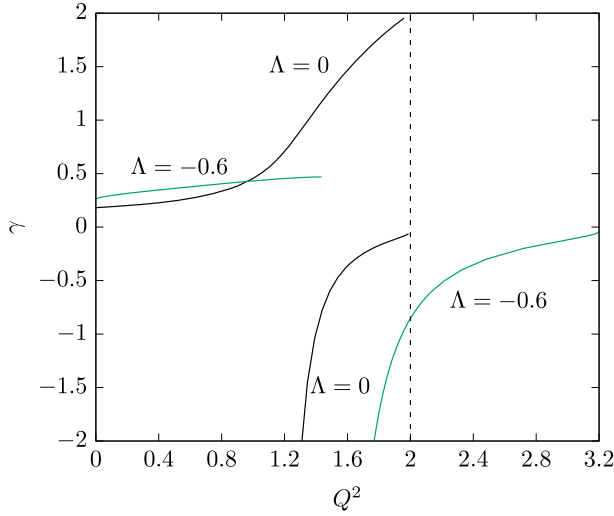


FIG. 3. We give the value of γ_{cr} for which nontrivial solutions of the scalar field equation (20) exist in dependence on Q^2 and for $\Lambda = 0$ (black) and $\Lambda = -0.6$ (green), respectively. The dashed vertical line indicates the extremal limit $Q_{\text{ex}}^2 = 2.0$ of the charged background solution for $\Lambda = 0$, i.e., the RN solution, while $Q_{\text{ex}}^2 = 3.2$ for $\Lambda = -0.6$.

least for sufficiently small Λ —than the requirement for the existence of a black hole horizon. This also leads to the observation that close to the extremal limit, aAdS black holes can only be scalarized for $\gamma_{\text{cr}} < 0$. While for $\Lambda = 0$ an interval of Q^2 exists, for which black holes can be scalarized for positive and negative γ_{cr} , this is no longer the case for $\Lambda = -0.6$. In fact, we find that $Q^2 \rightarrow 1.6$ for $\gamma \rightarrow -\infty$. Hence, nontrivial scalar field solutions to (20) exist either for γ positive or γ negative if $-\Lambda$ is sufficiently large.

2. $\alpha \neq 0$

We have studied the case $\Lambda = -0.006$ and our results are shown in Fig. 4, where we present the domain of existence of scalarized RNAdS black holes in the $\gamma - \alpha$ -plane. This clearly shows that solutions for $\gamma = 0$ are not possible and that the $\alpha\mathcal{R}$ term always requires the presence of the $\gamma\mathcal{G}$ term as well in order to achieve scalarization. The domain of existence is limited by two phenomena: Δ tending to zero, which gives a lower bound on α and $Q = 0$, which gives an upper bound on α , respectively. From $\Delta = 0$ using $\Lambda = -0.006$, we find $\alpha_{\text{cr}}^{(\text{min})} = 0.004\gamma - 0.1875$, hence the $\Delta = 0$ curve in Fig. 4 shows little dependence on γ . Since the presence of the charge Q leads to the presence of negative valued terms in $\mathcal{G}$, we would expect that decreasing Q allows to increase α . This can be done until $Q = 0$. In this limit, the requirement for scalarized black holes to exist is [see (23)]:

$$4\alpha\Lambda + \gamma\left(\frac{8}{3}\Lambda^2 + \frac{48M^2}{r^6}\right) \geq -\frac{3\Lambda}{4} \quad (24)$$

close to the black hole horizon. Using $r_h = 1.0$, $\Lambda = -0.006$ this becomes

$$\alpha \leq -0.01875 + \gamma\left(0.0004 + \frac{50.2002}{r^6}\right). \quad (25)$$

Now, we would need this bound to be fulfilled somewhere outside and close to the horizon in order to observe scalarization, i.e., the bound depends on the actual value of r (or better: range of r), which can only be found numerically. But (25) clearly demonstrates that when increasing γ , we can increase α , in agreement with our numerical results.

In summary, we can scalarize the SAdS solution and then increase the charge Q up to the point where $\Delta = 0$. In Fig. 4 (right), we show the approach to $Q = 0$ for $\alpha = 0.0$. We observe that close to $Q = 0$, the value of the derivative of the scalar function on the horizon changes sign and becomes negative.

3. Fixing the operator dimension

In gauge/gravity duality applications it is often useful to fix the dimension of the dual operator. This corresponds to fixing λ_+ , i.e., fixing Δ . We can then express the Hawking temperature (18) in terms of λ_+ and Q^2 as follows

$$2\pi T_H = \frac{[(3 - \lambda_+)\lambda_+ + 12\alpha]r_h^4 + 4(2r_h^2 - Q^2)\gamma}{8\gamma r_h^3} \quad (26)$$

where

$$\gamma = -\frac{12\alpha + \lambda_+(3 - \lambda_+)}{8\Lambda}. \quad (27)$$

We have investigated this case for $\alpha = 0$, $\gamma > 0$ for which λ_+ can be chosen to lie within the interval $\lambda_+ \in [\frac{3}{2}, 3]$. Note that we cannot reach $\lambda_+ = 3$, because this would require either $\gamma = 0$, in which the scalar field would always be trivial due to existing no-hair theorems, or $\Lambda = 0$, in which case the space-time is asymptotically flat and the scalar field falls off like $\sim 1/r$.

Our results for various fixed values within the given interval for λ_+ are shown in Fig. 5. We find that Q^2 is a decreasing function of $-\Lambda$ (or equivalently decreasing AdS radius ℓ)—independent of the value of λ_+ , see Fig. 5 (left). The largest possible $-\Lambda$ on each individual branch is reached at the $Q = 0$ solution, i.e., when the background is given by the Schwarzschild-Anti-de Sitter (SAdS) solution. To state it differently: in order for charged black holes in aAdS to scalarize, we have to choose the AdS radius larger (as compared to the horizon radius) than for an uncharged aAdS black hole. The larger the charge Q , the larger we have to choose ℓ in order to achieve scalarization.

When decreasing λ_+ , we find that for a fixed value of the charge Q , we have to increase $-\Lambda$ (decrease ℓ) in order for

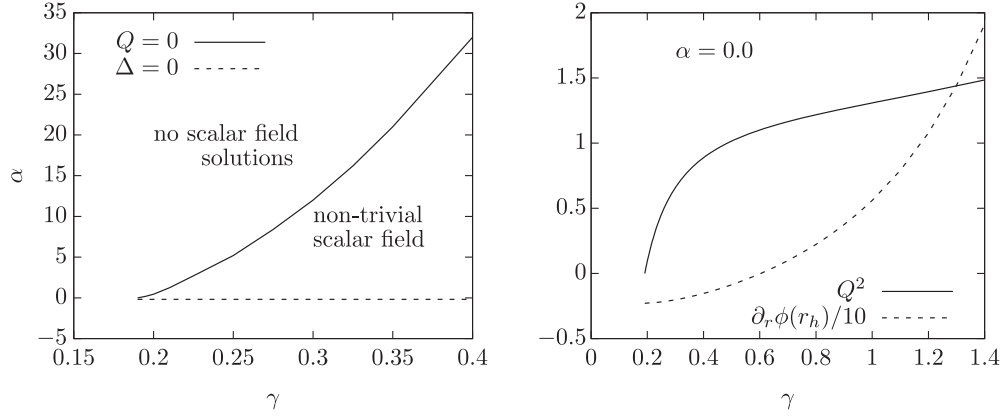


FIG. 4. We show the domain of existence of nontrivial solutions to (20) in the $\gamma - \alpha$ -plane for $\Lambda = -0.006$ (left). We demonstrate the approach to $Q = 0$ for $\alpha = 0.0$ (right). Clearly, the derivative $\partial_r \phi(r_h)$ changes sign close to the approach, but stays finite.

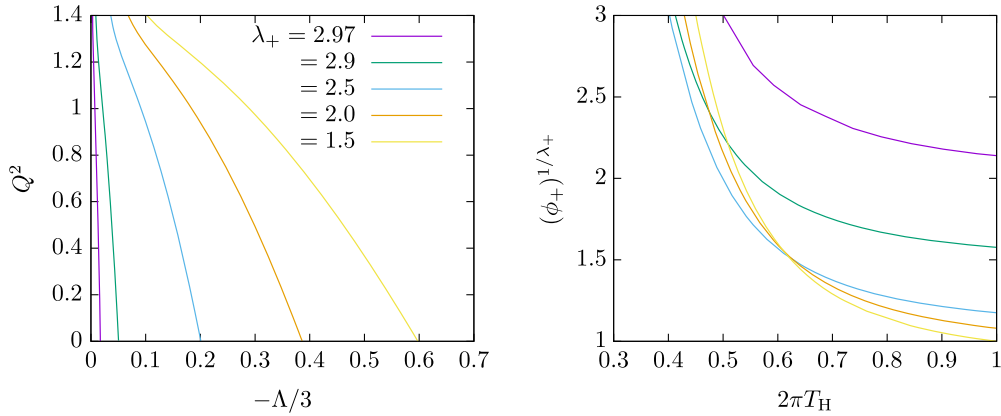


FIG. 5. We show the value of Q^2 in dependence on $-\Lambda/3 = 1/\ell^2$ for which nontrivial solutions to (20) exist for various values of fixed λ_+ , i.e., dimension of the dual operator on the conformal boundary, and $\alpha = 0$ (left). We also give the value of the expectation value of the dual operator with dimension λ_+ , $(\phi_+)^{1/\lambda_+}$, on the boundary in function of the Hawking temperature T_H (right, same color coding as left).

the black hole to scalarize, i.e., the smaller the dimension of the dual operator on the conformal boundary, the smaller we have to choose the AdS radius ℓ (in comparison to the horizon radius) in order to find nontrivial scalar fields. In Table I, we give the values of $-\Lambda/3$ (or equivalently ℓ) as well as the corresponding value of γ [see (27)] for which nontrivial solutions to (20) exist in the $Q = 0$ background in dependence on λ_+ . For these values, the scalar field behaves according to (13) for $\phi_- \equiv 0$. Using the terminology of the gauge/gravity duality, we can then interpret ϕ_+ as the expectation value of the corresponding dual operator with dimension λ_+ . In Fig. 5 (right) we show the value of $(\phi_+)^{1/\lambda_+}$ in dependence on the Hawking temperature T_H of the black hole, which is equal to the temperature of the dual field theory on the AdS boundary. We find that for all fixed λ_+ that we have studied, $(\phi_+)^{1/\lambda_+}$ increases with decreasing temperature, which is equivalent to increasing charge Q . In fact, the branches shown here start at $Q = 0$ (large T_H) and

end at the extremal solution with $T_H = 0$. We find that—except for $\lambda_+ = 2.97$, i.e., a value of λ_+ close to the limiting value $\lambda_+ = 3$ —the strongest increase of $(\phi_+)^{1/\lambda_+}$ happens approximately at the same T_H .

TABLE I. Values of $-\Lambda$, ℓ and γ in dependence on λ_+ for which nontrivial solutions to (20) exist in the background of a Schwarzschild-Anti-de Sitter (SAdS) solution, i.e., for $Q = 0$, and for $\alpha = 0$. We also give the corresponding value of the expectation value of the dual operator on the boundary, ϕ_+ .

λ_+	$-\Lambda/3$	ℓ	γ	ϕ_+
2.97	0.0172	7.625	0.216	9.380
2.90	0.0504	4.454	0.240	3.535
2.50	0.2000	2.236	0.260	1.249
2.00	0.3860	1.610	0.216	0.917
1.50	0.5900	1.302	0.159	0.816

IV. INCLUDING BACKREACTION

With the knowledge of the parameters for which non-trivial solutions to (20) exist, we can treat the full backreacted problem and construct spontaneously scalarized black holes as solutions to the set of coupled, nonlinear differential equations (9). In fact, while $\phi|_{r=r_h} \equiv \phi_h$ is fixed in the linearized problem, it can now be varied continuously and is a free parameter. We have hence chosen the set of parameters $(\Lambda, \alpha, \gamma)$ for which a solution to (20) exists (again choosing $r_h = 1$) and have numerically constructed a branch of solutions characterized by ϕ_h . Our numerical results indicate that $\phi_h \in [0, \tilde{\phi}_h]$. While the Hawking temperature T_H , the mass M and the electric charge Q depend very weakly on ϕ_h , we observe that the derivative of the metric function σ —which itself is no longer trivially equal to unity—at the horizon, $\partial_r \sigma|_{r=r_h} \equiv \partial_r \sigma(r_h)$, diverges when $\phi_h \rightarrow \tilde{\phi}_h$ indicating that the branch of spontaneously scalarized black holes tends to a singular solution. Equally, the Ricci scalar $\mathcal{R}$ [see (7)] and the Gauss-Bonnet term $\mathcal{G}$ [see (8)] diverge on the horizon. Our numerical results indicate that we reach this singular solution before reaching the extremal limit with $T_H = 0$. The fact that black holes with strong curvature cannot be scalarized to give regular black holes with hair has recently been discussed in [25] and our results indicate that this is also true in aAdS. To state it differently: the nonminimal coupling between curvature and scalar field does not allow to find an extremal, spontaneously scalarized black hole—a statement that we prove explicitly in the Appendix A. In fact, this is shown in Fig. 6 (left) for $\alpha = 0, \gamma = 0.3$ and different fixed dimensions of the dual operator λ_+ (or equivalently different values of Λ). In the limit $\phi_h \rightarrow 0$, the solution tends to the RNAdS solution with $\phi(r) \equiv 0$ and $\sigma(r) \equiv 1$. Increasing ϕ_h from zero, the solution becomes a (only numerically known) charged black hole with scalar hair for which $\partial_r \sigma(r_h)$ increases. The value of ϕ_+ also increases

with increasing ϕ_h , see Fig. 6 (right). We find that the larger the λ_+ , the larger ϕ_+ for a given value of ϕ_h .

A. Holographic phase transitions

In the case without backreaction, the scalar field equation (20) is linear and hence the value of the scalar field $\phi(r)$ has no physical meaning. In our model it is thence possible to describe holographic phase transitions only when including the backreaction of the metric. Note that this is in contrast with the study of holographic superconductors [29,30], where the scalar field equation is nonlinear already for a fixed background metric due to a minimal coupling to a U(1) gauge field [28].

When solving the full set of nonlinear equations, we find that the model in the bulk can be interpreted as the dual description of a phase transition in a strongly coupled quantum system that “lives” on the boundary of AdS. This is shown in Fig. 7 (left), where we give the dimensionless value of the condensate $\phi_+^{1/\lambda_+}/T_c$ in dependence of the temperature $T \equiv T_H$, which we now interpret as the temperature of the dual field theory and hence omit the index “H”. Close to the critical temperature $T = T_c$, the condensate curve shows the typical Ginzburg-Landau type behavior $(\phi_+)^{1/\lambda_+} \sim \sqrt{1 - T/T_c}$, which signals a second order phase transition at $T = T_c$. Increasing the operator dimension λ_+ , the value of the condensate increases at a given temperature T . We have also studied the behavior of the condensate $\phi_+^{1/\lambda_+}/\mu_c$ in dependence of the value of the U(1) gauge field on the AdS boundary, i.e., the chemical potential μ . Our results are shown in Fig. 7 (right). For $\mu \geq \mu_c$, we find that nontrivial condensates exist and, again, we find that increasing the operator dimension increases the value of the condensate for a fixed value of μ . The described phase transition, however, cannot be extended all the way down to $T = 0$ because of the reasons mentioned above. Rather, we find

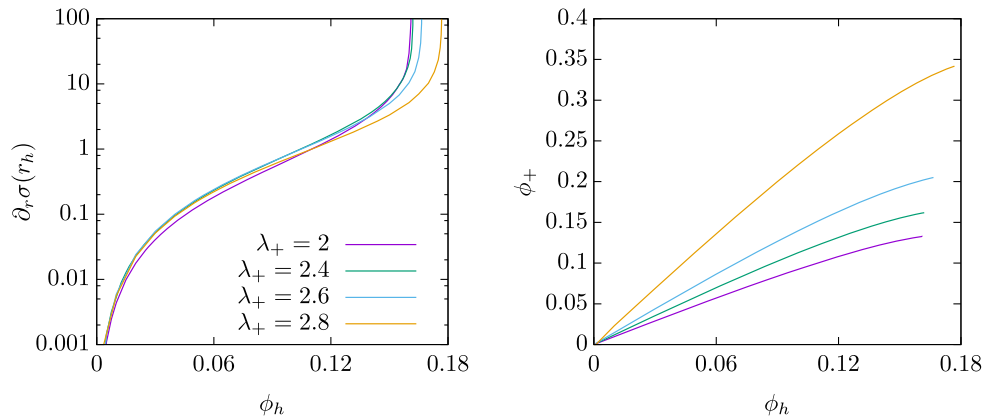


FIG. 6. We show the derivative of the metric function $\sigma(r)$ at the horizon, $\partial_r \sigma(r_h)$, in dependence on the value of the scalar field on the horizon, ϕ_h , for spontaneously scalarized black holes including backreaction, $\gamma = 0.3$ and different values of λ_+ (left). We also give the value of the condensate ϕ_+ in dependence on ϕ_h for these solutions (right, same color coding as left).

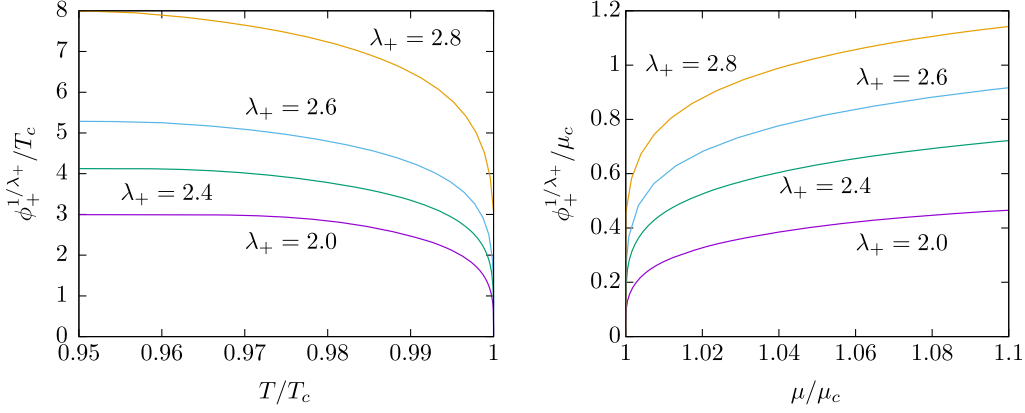


FIG. 7. We give the dimensionless condensate $\phi_+^{1/\lambda_+}/T_c$ ($\phi_+^{1/\lambda_+}/\mu_c$) in dependence on the temperature T divided by the critical temperature T_c (in dependence on the chemical potential μ divided by the critical chemical potential μ_c) for scalarized black holes including backreaction with $\gamma = 0.3$ and several values of λ_+ [left (right)].

that below a given temperature (above a given chemical potential) the value of the condensate becomes constant, i.e., is practically independent of the temperature (chemical potential), when decreasing (increasing) it further.

Let us mention that in our numerical study here, we have fixed $r_h = 1$, i.e., the entropy of the black hole $\mathcal{S} = \pi r_h^2$ to be constant. In gauge/gravity applications it is often more interesting to study the case of fixed charge Q . Our data can easily be transformed to this latter case by using the scaling symmetry (15) that leaves λ_+ unchanged. The relevant quantities (denoted by a hat, $\hat{Q} = 1$) then are

$$\hat{T}_H = QT_H, \quad \hat{\phi}_\pm = \frac{\phi_\pm}{Q^{\lambda_\pm}}, \quad \hat{\gamma} = \frac{\gamma}{Q^2}, \quad \hat{\mu} = \frac{\mu}{Q} \quad (28)$$

where we have to exclude the $Q = 0$ case. Since the uncharged case is of importance in our study and scalarization happens due to the nonminimal coupling (in contrast to e.g., studies of holographic superconductors, where the presence of the U(1) gauge field is essential for the minimally coupled scalar field to form, see e.g., [28,29]), we have presented our results for fixed entropy $\mathcal{S}$ of the black hole.

Due to the fact that the scalar field in our model is real, applications of our results are phase transitions that involve real-valued order parameters rather than complex valued ones. Examples are the difference between the density of the liquid and the density of the gas in a liquid-gas phase transition, the difference in concentrations in binary liquids or the magnetization in uniaxial magnets. Since the AdS/CFT correspondence is designed to describe quantum rather than thermal phase transitions on the one hand and strong coupling rather than weak coupling scenarios on the other, the question remains whether such transitions appear in nature. One example is the investigation of liquid-gas quantum phase transitions in bosonic fluids as discussed recently e.g., in [35]. Since ^4He is known to be a strongly

interacting system, it would be interesting to compare our results with those obtained using other approaches.

V. CONCLUSIONS

In this paper, we have studied spontaneously scalarized, static, spherically symmetric (un)charged black holes in asymptotically AdS space-time. We find that the scalar field power law fall-off on the AdS boundary depends on the coupling constants α and γ —very different to the asymptotically flat case—and that this fall-off is possible when the asymptotic form of the scalar-tensor coupling term fulfills the Breitenlohner-Freedman bound. We also find that while black holes can be scalarized with only the Gauss-Bonnet term $\mathcal{G}$ present, this is not the case for the Ricci scalar $\mathcal{R}$. This latter term needs “the support” of the Gauss-Bonnet term for scalarization to appear. Very similar to the asymptotically flat case, RNAdS black holes close to extremality can only be scalarized for negative values of γ . Including backreaction of the space-time leads to the observation that increasing the value of the scalar field on the horizon ϕ_h from zero that the branch of solutions terminates in a singular solution for which σ' on the horizon and with it $\mathcal{R}$ and $\mathcal{G}$ diverge. The value of the scalar field on the boundary of AdS can then be interpreted as the expectation value of a dual operator in a quantum field theory. This value increases with decreasing temperature and the system describes a phase transition of the Ginzburg-Landau form. However, since the temperature cannot be decreased to zero, the transition stops at finite temperature. In order to study temperatures closer to zero, we would have to study solutions for negative values of γ . However, our results indicate that in this case, the condensate increases with increasing temperature. For the moment, we have no proper dual interpretation of this fact, but will report more details including a detailed study of phase transitions in this system in the future.

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APPENDIX A: SPONTANEOUS SCALARIZATION OF EXTREMAL BLACK HOLES

In the following, we will use the so-called *attractor formalism* developed in [36] for higher derivative gravity, in [37] for higher derivative gravity in AdS and including scalar fields in [38], respectively. For that, we assume that the near horizon geometry of a near-extremal spherically symmetric, static black holes is of the form $\text{AdS}_2 \times S^2$ and¹ hence can be written as

$$ds^2 = v_1 \left(-\rho^2 d\tau^2 + \frac{1}{\rho^2} d\rho^2 \right) + v_2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (\text{A1})$$

where v_1 and v_2 are two positive constants and $r = r_{h,\text{ex}} + \rho$ such that the extremal horizon is located at $\rho = 0$. Furthermore, we have $F_{\rho\tau} = -F_{\tau\rho} = \text{constant} \equiv e$, which follows from the Maxwell equation using (A1).

The entropy function F is given by: $F(v_1, v_2, e, Q, \phi_h) \sim f(v_1, v_2, e, \phi_h) - eQ$, where f reads:

$$f(v_1, v_2, e, \phi_h) = \int_{S^2} d^2x \sqrt{-g} \mathcal{L} \quad (\text{A2})$$

and

$$\mathcal{L} = \left[\frac{\mathcal{R}}{2} - \Lambda + \phi^2 (\alpha \mathcal{R} + \gamma \mathcal{G}) - \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]. \quad (\text{A3})$$

Inserting $F_{\rho\tau} = -F_{\tau\rho} = e$ and the metric (A1), we find:

$$F = 4\pi \left[v_1 - v_2 - \frac{\Lambda}{v_1 v_2} + \phi_h^2 (2\alpha(v_1 - v_2) - 8\gamma) + \frac{1}{2} \frac{e^2 v_2}{v_1} \right] - eQ. \quad (\text{A4})$$

The attractor conditions then read:

¹Strictly speaking, this is an assumption. However, we know that for $\phi \equiv 0$ this is true and assume here that the presence of the scalar field would not change the geometry.

$$\begin{aligned} \frac{\partial F}{\partial v_1} = 0 &\Rightarrow 4\alpha\phi_h^2 v_1^2 v_2 - e^2 v_2^2 + 2\Lambda + 2v_1^2 v_2 = 0 \\ \frac{\partial F}{\partial v_2} = 0 &\Rightarrow -4\alpha\phi_h^2 v_1 v_2^2 + e^2 v_2^2 + 2\Lambda - 2v_1 v_2^2 = 0 \\ \frac{\partial F}{\partial e} = 0 &\Rightarrow 4\pi e \frac{v_2}{v_1} = Q \\ \frac{\partial F}{\partial \phi_h} = 0 &\Rightarrow \phi_h (\alpha(v_1 - v_2) - 4\gamma) = 0 \end{aligned} \quad (\text{A5})$$

The latter expression clearly demonstrates that for $\phi_h \neq 0$ the choice $\alpha = 0$ implies $\gamma = 0$ (and vice versa) and hence scalarization of black holes with near horizon geometry given by $\text{AdS}_2 \times S^2$ is excluded if either term is absent, unless $v_1 = v_2$ in the case $\alpha \neq 0, \gamma = 0$. Setting $v_1 = v_2$, however, leads to $\Lambda = 0$ [using the first two relations in (A5)], a case we want to exclude here.

Now, for the general case, $\alpha \neq 0, \gamma \neq 0$ and $\phi_h \neq 0$, let us consider the scalar field equation. Using the metric (A1), this reads:

$$8\gamma\phi - 2\alpha\phi(v_1 - v_2) - 2\phi' v_2 \rho - \phi'' v_2 \rho^2 = 0. \quad (\text{A6})$$

Implementing the condition $\alpha(v_1 - v_2) = 4\gamma$ from (A5), this equation can be integrated to give

$$\phi' \sim \rho^{-2}. \quad (\text{A7})$$

Hence, ϕ' diverges when approaching the extremal horizon, i.e., for $\rho \rightarrow 0$, such that *regular, extremal black holes with scalar hair do not exist in scalar-tensor gravity with coupling of the form $\phi^2(\alpha\mathcal{R} + \gamma\mathcal{G})$.*

APPENDIX B: FEFFERMAN-GRAHAM CONSTRUCTION

In the Fefferman-Graham construction [39] the bulk metric is of the form:

$$ds^2 = \frac{\ell^2}{4\rho^2} d\rho^2 + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j, \quad (\text{B1})$$

where ρ is the radial coordinate that drives the RG flow in the holographic interpretation. Anti-de Sitter space-time (AdS) corresponds to the choice

$$g_{ij}(x, \rho) dx^i dx^j = -dt^2 + dx^2 + dy^2, \quad (\text{B2})$$

where x, y, z are standard Cartesian coordinates, and is a solution to the Einstein equation with negative cosmological constant $\Lambda = -\frac{3}{\ell^2}$. Since we are dealing with a scalar-tensor gravity model here, we have checked that the space-time, indeed, tends to AdS asymptotically. For that, we have chosen

$$g_{ij}(x, \rho) = -(1 + a(\rho))dt^2 + (1 + b(\rho))(dx^2 + dy^2),$$

$$\text{with } a(\rho) = \sum_{k=1}^{\infty} a_k \rho^k, \quad b(\rho) = \sum_{k=1}^{\infty} b_k \rho^k \quad (\text{B3})$$

where the a_k and b_k are constants. This seems a suitable Ansatz given the symmetries of the space-time. Moreover, we assume $\phi \sim \rho^\beta$ with $\beta \geq 1$ at $\rho \rightarrow 0$. Inserting this into the gravity equation [see rhs of (3)] it is straightforward to show that the dominant term in $T_{\mu\nu}^{(\phi)}$ is the scalar field term $\sim \rho^{2(\beta-1)}$. Moreover, this can never be matched with the

terms from $a(\rho)$ and $b(\rho)$ appearing in $G_{\mu\nu}$. Hence, we conclude that $a_k = 0$ and $b_k = 0$ for all k when $\rho \rightarrow 0$ and that the space-time does tend to “pure” AdS, i.e., (B2), in our scalar-tensor gravity model. Note that this is also in agreement with our numerical results. Now, using this fact, it is straightforward to show using the scalar field equation [lhs of (3)] that the scalar field behaves like (13) with r replaced by ρ^{-1} for $\rho \rightarrow 0$. This means that the power of the fall-off that we have derived in Schwarzschild-like coordinates agrees with that in the Fefferman-Graham construction.

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