GROUP RINGS WHOSE INVOLUTORY UNITS COMMUTE

Victor Bovdi, Michael Dokuchaev

Abstract

Let K be a commutative integral domain of characteristic 2 and G a nonabelian locally finite 2-group. Consider $V(KG)$, the group of units with augmentation 1 in the group algebra KG. An explicit list of groups is given, and it is proved that all involutions in $V(KG)$ commute with each other if and only if G is isomorphic to one of the groups on this list. In particular, this property depends only on G and not at all on K.

1 Introduction

Let FG be the group algebra of a locally finite p-group G over a field F of characteristic p. Then the normalized unit group

$$V(FG) = \Big\{ \sum_{g\in G} \alpha_g g \in FG \,\Big|\, \sum_{g\in G} \alpha_g = 1 \Big\}$$

is a locally finite p-group.

An interesting way to study $V(FG)$ is to construct embeddings of important groups into it. In this paper we answer the question when dihedral groups can not be embedded into $V(FG)$. Clearly, only the case $p = 2$ has to be considered. At the same time we obtain the list of the locally finite 2-goups such that $V(FG)$ does not contain a subgroup isomorphic to a wreath product of two groups (for the case of odd p see [7]). Let C_{2^n}, C_{2^∞} and Q_8 be the cyclic group of order 2^n, the quasicyclic group of type 2^∞ and the quaternion group of order 8, respectively. An *involution* is a group element of order 2. Our main result is the following:

Theorem *Let K be a commutative integral domain of characteristic 2, and G a locally finite nonabelian 2-group. Then all involutions of $V(KG)$ commute if and only if G is one of the following groups:*

(i) $S_{n,m} = \langle a, b \mid a^{2^n} = b^{2^m} = 1, a^b = a^{1+2^{n-1}} \rangle$ *with* $n, m \geq 2$, *or* $G = Q_8$;

(ii) $G = Q_8 \times C_{2^n}$ *or* $G = Q_8 \times C_{2^\infty}$;

(iii) *the semidirect product of the cyclic group* $\langle d \mid d^{2^n} = 1\rangle$ *with a quaternion group* $\langle a, b \mid a^4 = 1, a^2 = b^2 = [a,b]\rangle$ *such that* $[a,d] = d^{2^{n-1}}$ *and* $[b,d] = 1$;

(iv) $H_{32} = \langle x, y, u \mid x^4 = y^4 = 1,\ x^2 = [y,x], y^2 = u^2 = [u,x],\ x^2y^2 = [u,y] \rangle$.

The subscripts are the serial numbers of these groups in the CAYLEY library of groups of order dividing 128 described by M. F. Newman and E. A. O'Brien in [5]. It is a mere coincidence that H_{32} has order 32.

For a 2-group G we denote by $\Omega(G)$ the subgroup generated by all elements of order 2 of G. We set $\Omega(V) = \Omega(V(KG))$. Clearly, $\Omega(G)$ is a normal subgroup of G. As usual, denote $\exp(G)$, $C_G(\langle a,b\rangle)$ the exponent of a group G and the centralizer of the subgroup $\langle a,b\rangle$ in G, respectively. For any $a, b \in G$ we define $[a,b] = a^{-1}b^{-1}ab$, $a^b = b^{-1}ab$. Let

$$\mathcal{L}(KG) = \langle xy - yx \mid x, y \in KG\rangle$$

be the commutator submodule of KG.

For an element g of a finite order $|g|$ in a group G, let $\overline{g}$ denote the sum (in KG) of the distinct powers of g:

$$\overline{g} = \sum_{i=0}^{|g|-1} g^i.$$

For an arbitrary element $x = \sum_{g\in G}\alpha_g g \in KG$ denote $\chi(x) = \sum_{g\in G}\alpha_g \in K$.

Let KG be the group algebra of an abelian 2-group G over a commutative integral domain K of characteristic 2. If $x \in KG$, then $x^2 = 0$ if and only if $x \in \mathfrak{I}(\Omega(G))$, where $\mathfrak{I}(\Omega(G))$ is the ideal generated by all elements of form $g - 1$ with $g \in \Omega(G)$. Moreover,

$$\Omega(V) = 1 + \mathfrak{I}(\Omega(G)) \qquad (1)$$

It will be convenient to have a short temporary name for the locally finite 2-groups G such that all elements of order 2 of $V(KG)$ form an abelian subgroup. Let us call the groups G with this property *good*.

2 Preliminary results

Lemma 1 *Let G be a finite nonabelian good 2-group. Then all involutions of G are central, $G' \subseteq \Omega(G)$ and either G is the quaternion group of order 8 or $\Omega(G)$ is a direct product of two cyclic groups.*

Proof. Let G be a nonabelian good 2-group. Clearly $\Omega(V) \cap G = \Omega(G)$ is a normal abelian subgroup of G.

Suppose that $|\Omega(G)| = 2$. Then by Theorem 12.5.2 in [8]

$$G = \langle a, b \mid a^{2^m} = 1, b^2 = a^{2^{m-1}}, a^b = a^{-1}\rangle$$

with $m \geq 2$ is a generalized quaternion 2-group. If $|G| > 8$, then we choose $c \in \langle a\rangle$ of order 8. Then $1 + (1+c^2)(c+b)$ and $1 + (1+c^2)(c+cb)$ are noncommuting involutions of $V(KG)$, which is impossible. Therefore G is the quaternion group of order 8.

Let $|\Omega(G)| > 2$. The normal subgroup $\Omega(G)$ contains a central element a in G and $x = 1+(a+1)g$ is an involution for every $g \in G$. If $b \in \Omega(G)$ then $bx = xb$ and $(a+1)(1+[b,g]) = 0$. It implies that $[g,b] = 1$ or $[g,b] = a$. Let $[g,b] = a$ and $|g| = 2^t$. Then $z = 1 + (g+1)\overline{g^2}$ is an involution and $zb = bz$. From this we get that $a \in \langle g^2\rangle$ and since $a \in \Omega(G)$ we conclude that

$a = g^{2^{t-1}}$. Clearly, if $t = 2$, then $\langle b, g\rangle$ is the dihedral group of order 8, which is impossible. If $t > 2$, then $x = 1 + g(1 + g^{2^{t-2}})(1 + b)$ is an involution which does not commute with b, which is a contradiction. Therefore, $[g, b] = 1$ and $\Omega(G)$ is a central subgroup of G. Let a and b be arbitary elements of $\Omega(G)$, $g, h \in G$ and $[g, h] \neq 1$. Then $x = 1 + (a + 1)g$ and $y = 1 + (b + 1)h$ are involutions and $[x, y] = 1$. From this we conclude that the commutator subgroup G' is a subgroup of $\langle a, b\rangle$ in $\Omega(G)$ and $\exp(G') = 2$.

Now, let $|\Omega(G)| \geq 8$ and let a, b, c be linearly independent elements of $\Omega(G)$. Then by the above reasoning we have $[g, h] \in \langle a, b\rangle \cap \langle a, c\rangle \cap \langle b, c\rangle = 1$, which is impossible. Therefore, $|\Omega(G)| = 4$ and $G' \subseteq \Omega(G)$.

Lemma 2 *A two-generator finite nonabelian 2-group is good if and only if it is either the quaternion group of order 8 or*

$$S_{n,m} = \langle a, b \mid a^{2^n} = b^{2^m} = 1, a^b = a^{1+2^{n-1}}\rangle \tag{2}$$

with $n, m \geq 2$.

Proof. Suppose that G is not the quaternion group of order 8. By Lemma 1 $\Omega(G)$ is a direct product of two cyclic groups and $G' \subseteq \Omega(G)$. Then the Frattini subgroup $\Phi(G) = \{g^2 \mid g \in G\}$ is central and by theorem 3.3.15 in [4], $|G/\Phi(G)| = 4$. Since $\Phi(G)$ is a subgroup of the centre $\zeta(G)$ and the factor group $G/\zeta(G)$ can not be cyclic, this implies $\Phi(G) = \zeta(G)$.

It is easy to verify that a two-generator good group G is metacyclic. Indeed, every maximal subgoup M of G is abelian and normal in G, because $\Phi(G) = \zeta(G) \subset M$ and $|M/\zeta(G)| = 2$. Clearly, $\Omega(M) \subseteq \Omega(G)$ and in case $|\Omega(M)| = 2$ the subgroup M is cyclic and we conclude that G is metacyclic.

Now let $|\Omega(M)| = 4$ for every maximal subgroup M of G. Then G and M are two-generator groups. It is easy to see that all such groups of order 16 are metacyclic. If $|G| = 2^n$ $(n \geq 5)$ then G and all maximal subgroups of G are two generator groups and by theorem 3.11.13 in [4] G is metacyclic too. Since $G' \subseteq \Omega(G)$ by Lemma 1, it follows from theorem 3.11.2 in [4], that G is defined by

$$G = \langle a, b \mid a^{2^n} = 1, b^{2^m} = a^{2^l}, a^b = a^{1+2^{n-1}}\rangle$$

with $n, m \geq 2$. If $n \neq l + m$, then $a^{2^{n-l-m}}$ is central and

$$(a^{2^{n-l-m}}b)^{2^m} = a^{2^{n-l}}b^{2^m} = 1.$$

Then G can be generated by a, $a^{2^{n-l-m}}b$ and defined by (2). Suppose $n = l + m$ and $m = 1$. Since G is not the quaternion group of order 8, $n > 2$. Then $a^{-2^{n-2}}b$ is an involution and $[a^{-2^{n-2}}b, a] \neq 1$, which is a contradiction with Lemma 1. Further, in the last case when $n = l + m$ and $m > 1$, we have $(ab)^{2^m} = (a^2b^2[a, b])^{2^{m-1}} = 1$ and G can be generated by a, ab and defined by (2).

Lemma 3 (E. A. O'Brien, see Lemma 4.1 in [10]) *The groups H of order dividing 128 such that $\Phi(H)$ and $\Omega(H)$ are equal, central, and of order 4, are precisely the following: $C_4 \times C_4$, $C_4 \rtimes C_4$, $C_4 \rtimes Q_8$, $Q_8 \times C_4$, $Q_8 \times Q_8$, the central product of the group $S_{2,2} = \langle a, b \mid$*

$a^4 = b^4 = 1,\ a^2 = [b, a]\,\rangle$ *with a quaternion group of order* 8, *the nontrivial element common to the two central factors being* a^2b^2,

$$H_{245} = \langle\, x, y, u, v \mid \quad \begin{aligned} & x^4 = y^4 = [v, u] = 1, \\ & x^2 = v^2 = [y, x] = [v, y], \\ & y^2 = u^2 = [u, x], \\ & x^2y^2 = [u, y] = [v, x]\,\rangle \end{aligned}$$

and the groups named in parts (iii), (iv) *of Theorem.*

The group H_{245}, is one of the two Suzuki 2-groups (see Higman [6]) of order 64.

2.1 Proof of the necessity of the theorem

Let G be a finite nonabelian good 2-group. Therefore $\Omega(V)$ is abelian.

By Lemma 1 all involutions of G are central, $G' \subseteq \Omega(G)$ and $\Omega(G)$ is either a group of order 2 or a direct product of two cyclic groups. Clearly, $\Phi(G) \subseteq \zeta(G)$ and if $|\Omega(G)| = 2$ then by Lemma 1, G is a quaternion group of order 8. Thus, we can suppose that $|\Omega(G)| = 4$.

First let $\Phi(G)$ be cyclic. Since all involutions are central, by [11], Theorem 2, G is the direct product of a group of order 2 and the generalized quaternion group of order 2^{n+1}. By Lemma 1, G is a Hamiltonian 2-group of order 16.

We may suppose that $\Phi(G)$ is the direct product of two cyclic groups. Let the exponent of G be 4. Then $\Phi(G) = \Omega(G)$ and by a result of N. Blackburn ([9], Theorem VIII.5.4), $|G| \leq |\Omega(G)|^3$. Therefore the order of G divides 64. Then by O'Brien's Lemma G is precisely of one of the following types: $C_4 \rtimes C_4$, $C_4 \rtimes Q_8$, $Q_8 \times C_4$, the groups named in parts (iii), (iv) of Theorem and $Q_8 \times Q_8$, H_{245}, $Q_8 \curlyvee S_{2,2}$.

Now we shall find noncommuting involutions z_1, z_2 in $V(KG)$ if G is one of the last three groups listed above.

Let G be the central product of the group

$$S_{2,2} = \langle\, a, b \mid a^4 = b^4 = 1,\ a^2 = [b, a]\,\rangle$$

with the quaternion group of order 8, the nontrivial element common to the two central factors being a^2b^2. Then

$$G \cong \langle a, b, d, f \mid \quad \begin{aligned} & a^4 = d^4 = 1, b^2 = a^2 = [a, b], f^2 = d^2 = [d, f], \\ & [a, d] = [b, d] = [b, f] = 1, [a, f] = a^2 \rangle \end{aligned}$$

and we put $z_1 = 1 + d^2a + b + a^3d + bd + f + abf + df + abdf$ and $z_2 = 1 + b(1 + d^2)$. If $G \cong H_{245}$ then

$$H_{245} \cong \langle\, a, b, d, f \mid \quad \begin{aligned} & a^4 = b^4 = 1, b^2 = d^2 = a^2, [a, b] = 1, \\ & [a, d] = [b, f] = [d, f] = b^2, [b, d] = a^2, [a, f] = a^2b^2\,\rangle, \end{aligned}$$

and put $z_1 = 1 + a + ab + d + a^2bd + f + bf + ab^2df + a^3b^3df$ and $z_2 = 1 + (b + b^{-1})$. Now let G be a direct product of two quaternion groups $\langle a, b\rangle$ and $\langle c, d\rangle$ of order 8. Then we set $z_1 = 1 + a + bc^2 + c + abc + a^2d + abd + acd + bcd$ and $z_2 = 1 + b(1 + c^2)$.

It is easy to verify that in all three cases $z_1^2 = z_2^2 = 1$, $z_1 z_2 \neq z_2 z_1$.

Now, let the exponent of G be larger than 4. Using Lemma 2, we conclude that G contains a two-generator nonabelian subgroup H which is either Q_8 or $S_{n,m}$.

We wish to prove that if $\exp(G) > 4$ and $G = H \cdot C_G(H)$ for every two-generator nonabelian subgroup H, then

$$G = Q_8 \times \langle d \mid d^{2^n} = 1, n > 1\rangle.$$

First, let $H = Q_8 = \langle a, b\rangle$ be a quaternion subgroup of order 8 of G. Then $G = Q_8 \cdot C_G(Q_8)$ and $C_G(Q_8)$ does not contain an element c of order 4 with the property $c^2 = a^2$, because ac would be a noncentral involution of G, which is impossible. If $C_G(Q_8)$ is abelian and $|\Omega(C_G(Q_8))| = 4$ then $C_G(Q_8)$ is the direct product of $\langle a^2\rangle$ and $\langle d \mid d^{2^n} = 1, n > 1\rangle$, and $G = Q_8 \times \langle d\rangle$.

We can suppose that $C_G(Q_8)$ is nonabelian and does not contain an element u such that $u^2 = a^2$. Since $\exp(C_G(Q_8)) > 4$, there always exists a subgroup

$$S_{n,m} = \langle c, d \mid c^{2^n} = d^{2^m} = 1, c^d = c^{1+2^{n-1}}\rangle$$

of $C_G(Q_8)$ which is of exponent larger than 4. Then $\zeta(S_{n,m}) = \langle c^2, d^2\rangle$ and as $\exp(S_{n,m}) > 4$, one of the generators c or d has order larger than 4. Therefore every $u \in \Omega(\zeta(S_{n,m}))$ is the square of one of the elements from $S_{n,m}$. Thus, since we assume that $C_G(Q_8)$ does not contain an element of order 4 whose square is a^2, we get that $S_{n,m} \cap Q_8 = 1$ and $S_{n,m} \times Q_8$ is a subgroup of G with the property $|\Omega(S_{n,m} \times Q_8)| = 8$, which is impossible.

Now let $H = S_{n,m} = \langle a, b \mid a^{2^n} = b^{2^m} = 1, a^b = a^{1+2^{n-1}}\rangle$ with $n, m \geq 2$ be a subgroup of G. Then $G = S_{n,m} \cdot C_G(S_{n,m})$. As $|\Omega(G)| = 4$ we can choose $d \in C_G(S_{n,m})$ such that $d \notin S_{n,m}$ but $d^2 \in S_{n,m}$. Then $d^2 = a^{2i}b^{2j}$. If i or j is even then $d^{-1}a^ib^j \in \Omega(G) = \Omega(S_{n,m})$ and $d \in S_{n,m}$, which is impossible. If i and j are odd and $n > 2$ then $d^{-1}a^{i+2^{n-2}}b^j \in \Omega(G) = \Omega(S_{n,m})$ which is a contradiction. Therefore, $n = 2$ and $\langle a^i, d^{-1}b^j\rangle$ is a quaternion subgroup and by assumption $G = Q_8 \cdot C_G(Q_8)$. We obtained the previous case.

It is easy to check that if the commutator subgroup G' is of order 2, then $G = H \cdot C_G(H)$ for every two-generator nonabelian subgroup of G. Indeed, $G' = H' = \langle c\rangle$, this implies that $H = \langle a, b\rangle$ is a normal subgroup of G and let $[a, b] = c$, $[a, g] = c^k$, $[b, g] = c^l$, where $0 \leq k, l \leq 1$, $g \in G$. Then at least one of the elements g, ag, bg, abg belongs to $C_G(H)$ and $g \in H \cdot C_G(H)$. Therefore $G = H \cdot C_G(H)$.

It follows that we can suppose that $\exp(G) > 4$, the commutator subgroup $G' = \Omega(G)$ has order 4 and G contains a two-generator nonabelian subgroup L such that $G \neq L \cdot C_G(L)$.

Let $L = \langle b, d \mid [b, d] \neq 1\rangle$ and $a \in G \setminus (L \cdot C_G(L))$. Then $[d, a]$ or $[b, a]$ is not equal to $[b, d]$. Indeed, in the opposite case, from $[a, b] = [d, b]$ and $[a, d] = [b, d]$ we get $bda \in C_G(L)$ and $a \in L \cdot C_G(L)$ which is impossible.

Now we want to prove that we can choose $a \in G \setminus (L \cdot C_G(L))$ and $b, d \in L$ such that $\langle b, d\rangle = L$, $[a, b] = 1$, and with the following property:

$$[a, d] \neq [b, d], [a, d] \neq 1, [b, d] \neq 1. \qquad (3)$$

If $[a, d] = 1$ then we can put $a' = a$, $b' = d$ and $d' = b$.

We consider the following cases:

Case 1. Let $[b,d]=[b,a]\neq[a,d]$. Then $[b,ad]=1$ and we put $a'=ad$, $b'=b$ and $d'=d$. If $[a',d']=1$ then $ad\in C_G(L)$ which implies that $a\in L\cdot C_G(L)$, a contradiction.

Case 2. Let $[b,d]=[a,d]\neq[a,b]$. Then $[ab,d]=1$ and put $a'=ab$, $b'=d$ and $d'=b$. If $[a',d']=1$ then $ab\in C_G(L)$ which gives a contradiction again.

Case 3. Let $[a,b]\neq[a,d]\neq[b,d]\neq[a,b]$. Suppose that all these commutators are not trivial. Since $|\Omega(G)|=4$, one of these commutators equals to the product of two others and

$$[ab,bd]=[a,b]\cdot[b,d]\cdot[a,d]=1.$$

Put $a'=ab$, $b'=bd$ and $d'=d$.

In what follows we suppose that $L=\langle b,d\rangle$ and $a\in G\setminus(L\cdot C_G(L))$ such that $[a,b]=1$ and the condition (3) is satisfied.

It is easy to see that if $\langle a,b\rangle=\langle u\rangle$ is cyclic then from $[a,d]\neq 1\neq[b,d]$ we have $a=u^{2k+1}$ and $b=u^{2t+1}$ for some $k,t\in N$, because the squares of all elements in G are central. Then $ab\in C_G(L)\subseteq L\cdot C_G(L)$ and $a\in L\cdot C_G(L)$, which is impossible. Therefore, $\langle a,b\rangle$ is not cyclic.

Consider, $W=\langle a,b,d\rangle$. Then the commutator subgroup of W has order 4 and

$$H=\langle a,b\rangle=\langle a_1\rangle\times\langle b_1\rangle.$$

Clearly $W=\langle a_1,b_1,d\rangle$ and $|W'|=4$. It is easy to see that a_1 and b_1 can be chosen such that condition (3) is satisfied and $\langle a_1\rangle\cap\langle b_1\rangle=1$.

Let $a,b,d\in G$ with the property (3), $[a,b]=1$ and $\langle a\rangle\cap\langle b\rangle=1$. Put $H=\langle a\mid a^{2^n}=1\rangle\times\langle b\mid b^{2^m}=1\rangle$ and $W=\langle a,b,d\rangle$. Then

$$G'=\Omega(G)=\Omega(H)=\langle a^{2^{n-1}}\rangle\times\langle b^{2^{m-1}}\rangle$$

and H is a normal subgroup of G.

First, we will prove that $g^2\in H$ for every $g\in G\setminus H$. There exists $c=g^{2^{k-1}}$ such that $c\notin H$ and $c^2\in H$. If $k>1$, then $c\in\Phi(G)\subset\zeta(G)$ and we obtain that $c^2\in\zeta(G)\cap H$ and $c^2=a^{2i}b^{2j}$. Thus, $(c^{-1}a^ib^j)^2=1$ and $c^{-1}a^ib^j\in\Omega(G)=\Omega(H)$. This implies that $c\in H$, which is impossible. Therefore, $k=1$ and $g^2=a^{2i}b^{2j}$ for some i and j, and we have shown that $g^2\in H$ for all $g\notin H$.

We have $[g,a^ib^j]=a^{s2^{n-1}}b^{r2^{m-1}}$ for some $r,s\in\{0,1\}$. It is easy to see that the case $n>2$ and $m>2$ is impossible. Indeed, if $n>2$ and $m>2$ then $a^{i+s2^{n-2}}\cdot b^{j+r2^{m-2}}\cdot g^{-1}\in\Omega(G)=\Omega(H)$ and $g\in H$, which is a contradiction.

Since $\exp(G)>4$ and for any $g\in G$, $g^2=a^{2i}b^{2j}$ for some i,j it follows that $\exp(H)>4$. Thus we may suppose that $n>2$ and b is an element of order 4.

Now we describe the group $W=\langle a,b,d\rangle$ and we distinguish a number of cases according to the form of the element d^2.

Case 1. Let $d^2=a^{2i}$. Since $d\notin H$ and a^id^{-1} is not of order 2, we have that $i=2k+1$ is odd. Then as $(a^{i+2^{n-2}}d^{-1})^2\neq 1$ we have

$$(a^{i+2^{n-2}}d^{-1})^2=[a^i,d]\cdot a^{2^{n-1}}=[a^{2k+1},d]\cdot a^{2^{n-1}}=[a,d]\cdot a^{2^{n-1}}\neq 1.$$

We conclude that $[a,d]\neq a^{2^{n-1}}$ and $[a,d]=a^{s2^{n-1}}b^2$, $s\in\{0,1\}$, and by property (3) $[b,d]=a^{(1+s)2^{n-1}}b^2$ or $[b,d]=a^{2^{n-1}}$. If $[b,d]=a^{2^{n-1}}$ then $a^{i+(1+s)2^{n-2}}bd^{-1}$ has order 2 and does not belong to $\Omega(H)$, and this case is impossible.

Let $[a,d] = a^{s2^{n-1}}b^2$ and $[b,d] = a^{(1+s)2^{n-1}}b^2$. Then

$$W = \langle a, b, d\rangle = \langle a^{i+2^{n-2}}d^{-1}, a^{(1+s)2^{n-2}}b, ab\rangle$$

and $\langle a^{i+2^{n-2}}d^{-1}, a^{(1+s)2^{n-2}}b\rangle$ is a quaternion subgroup of order 8. Moreover, ab has order 2^n and $[a^{i+2^{n-2}}d^{-1}, ab] = a^{2^{n-1}} = (ab)^{2^{n-1}}$. This shows that W satisfies (iii) of the Theorem. Observe that $\langle a^{(1+s)2^{n-2}}b, ab\rangle = H$.

Case 2. Let $d^2 = b^2$. As before,

$$(a^{2^{n-2}}bd^{-1})^2 = a^{2^{n-1}}[b,d] \neq 1.$$

Therefore, $[b,d] \neq a^{2^{n-1}}$, and we obtain $[b,d] = a^{s2^{n-1}} \cdot b^2$, and by property (3) $[a,d] = a^{(1+sr)2^{n-1}} \cdot b^{2r}$, where $r, s \in \{0,1\}$. It is easy to see that $\langle bd^{-1}, a^{s2^{n-2}}b\rangle$ is a quaternion group of order 8 and $W = \langle bd^{-1}, a^{s2^{n-2}}b, ab^r\rangle$ is defined as in case 1 and satisfies condition (iii) of the Theorem. Moreover, $\langle a^{s2^{n-2}}b, ab^r\rangle = H$.

Case 3. Let $d^2 = a^{2i}b^2$. If i is even then $a^i \in \zeta(G)$ and $(da^{-i})^2 = b^2$. Then $W = \langle a, b, da^{-i}\rangle$ and if we replace d by $d' = da^{-i}$ we obtain $d'^2 = b^2$ which is case 2.

Now let $d^2 = a^{2i}b^2$ and assume i is odd. If $[a,d] = a^{s2^{n-1}} \cdot b^2$, then $a^{i+s2^{n-2}} \cdot d^{-1} \in \Omega(H)$ and $d \in H$, which is impossible. Therefore, $[a,d] = a^{2^{n-1}}$ and $(a^{i+2^{n-2}} \cdot d^{-1})^2 = b^2$.

If we replace d by $d' = a^{i+2^{n-2}}d^{-1}$, we obtain $W = \langle a, b, d'\rangle$ and $(d')^2 = b^2$, which is case 2 and we have that W satisfies (iii) of the Theorem.

This proves that the subgroup W has a system of generators u, v, w such that

$$W = \langle w, u, v \mid w^4 = 1, w^2 = v^2, v^w = v^{-1}, u^{2^n} = 1, u^w = u^{1+2^{n-1}}, [u,v] = 1\rangle,$$

with $n > 2$ and $H = \langle u, v\rangle$.

Suppose that there exists $g \in G \setminus W$. Clearly, $G' \subseteq W$ and W is normal in G. Above we proved that the squares of all elements of G outside W belong to H and they are central in W. Therefore, by the above argument we conclude that $g^2 = u^{2s}v^{2t}$ for every $g \in G \setminus W$, where $t \in \{0,1\}$, $s \in N$. It is easy to see that

$$(g^{-1}u^s)^2 = [g, u^s]g^{-2}u^{2s} = [g, u^s]v^{-2t}.$$

If $(g^{-1}u^s)^2 = 1$ then $g^{-1}u^s \in \Omega(W) \subseteq W$ and $g \in W$, which is impossible. Clearly, the order of elements $[g, u^s]$ and v^{-2t} divide 2 and $g^{-1}u^s$ is an element of order 4. Then $M = \langle g^{-1}u^s, v, w, u^{2^{n-2}}\rangle$ is a subgroup of exponent 4 and $\Omega(M) = \Omega(G)$. Clearly, $M/\Omega(M)$ is an elementary 2-subgroup of order 16. Therefore, M is a group with four generators. By O'Brien's Lemma, M is isomorphic either to $Q_8 \times Q_8$ or to H_{245} or to $S_{2,2} \curlyvee Q_8$. It is impossible, because the centres of these groups have exponent 2 but in M there exists a central element $u^{2^{n-2}}$ of order 4. Thus the description of finite good 2-groups is completed.

Now we suppose that G is an infinite good group. We shall prove that G is the direct product of the quaternion group of order 8 and the quasicyclic 2-group.

It is easy to see that if G has exponent 4 then G is finite. Indeed, if G is abelian then its finiteness follows from the first Prüfer's Theorem ([3], p.173) and the condition $|\Omega(G)| \leq 4$. If G is non-abelian then take an ascending chain $G_1 \subset G_2 \subset \cdots$ of finite subgroups of G. It follows from the description of finite good 2-groups given above that this chain is finite, that is $G_n = G_{n+1} = \cdots$ for some $n \in N$. As G is a locally finite, $G = G_n$, a contradiction.

Thus, we may suppose that $\exp(G) > 4$. We show now that $\zeta(G)$ contains a divisible subgroup. Let $T_2(G) = \langle g \in G \mid g^4 = 1\rangle$ and $\Omega_2(G) = \langle g^4 \mid g \in G\rangle$. Then $(gh)^4 = g^4h^4$ for all $g, h \in G$ and the map $g \to g^4$ is a group homomorphism of G onto $T_2(G)$ with kernel $\Omega_2(G)$. As $\Omega_2(G)$ has exponent 4, by the above paragraph, $\Omega_2(G)$ is finite. If $\zeta(G)$ does not contain a divisible group, $\zeta(G)$ is finite. Hence, $T_2(G) \subseteq \zeta(G)$ is finite too which implies the finiteness of G, a contradiction.

We have that $\zeta(G) = R \times P$, where $1 \neq P$ is divisible and R does not contain a divisible subgroup. Observe now that $R \neq 1$ is cyclic, P is quasicyclic and for every non-central $g \in G$ there exists a non-central element $g_1 \in G$ such that

$$g_1 \equiv g \pmod{P} \text{ and } \langle g_1^2\rangle = R. \tag{4}$$

Indeed, we have that $g^2 = cc'$, where $c \in R, c' \in P$. Taking $g_1 = gd^{-1}$ with $d^2 = c'$ we get $g_1^2 = c$, $g_1 \equiv g \pmod{P}$. As g_1 is non-central, $c \neq 1$. It follows that $R \neq 1$ and as $|\Omega(G)| = 4$, R is cyclic and P is a quasicyclic. Let $R = \langle z \mid z^{2^n} = 1\rangle$ with $n \geq 1$ and $P = \langle c_1, c_2, \ldots, c_k, \ldots \mid c_1^2 = 1, c_{k+1}^2 = c_k\rangle$ with $k = 1, 2, \ldots$. If $c = z^i$ with even i, then $g_1z^{-i/2}$ is a noncentral element of order 2, which is impossible. Hence, i is odd and $\langle g_1^2\rangle = R$, as desired in (4).

Next we observe that R has order 2 (i.e., $n = 1$). Let g and t be two non-commuting elements in G. We have that $[g, t] = z^{2^{n-1}i}c_1^j$ $(i, j \in \{0, 1\})$ and by (4) we can suppose that $g^2 = z^{i_1}$, $t^2 = z^{i_2}$ with $i_1 \equiv i_2 \equiv 1 \pmod{2}$. Choose m_1 and m_2, such that $i_1m_1 \equiv i_2m_2 \equiv 1 \pmod{2^n}$. If $n > 1$, then $x = g^{m_1}t^{-m_2}z^{i2^{n-2}}c_2^j$ has order 2 and $[x, t] \neq 1$, a contradiction. Thus R has order 2 and $g^2 = t^2 = z$. If $i = 0$ then $gt^{-1}c_2^j$ is a non-central element of order 2, a contradiction. Therefore, $[g, t] = z^ic_1^j$ and $Q = \langle gc_2^j, tc_2^j\rangle$ is isomorphic to the quaternion group of order 8.

Now we show that $G = Q \times P$ and this will complete the proof of necessity of the Theorem. Fix a noncentral element x of G. There exists an element $c \in P$ such that $(xc)^2 = (gc_2^j)^2$. Indeed, if $j = 0$ this follows from (4). In case $j = 1$ by (4) take xc' such that $(xc')^2 = z$. Then $(xc'c_2)^2 = zc_1$ as we need.

It is enough to show that $x_1 = xc \in W = Q \times \langle c_3\rangle$. Suppose that $x_1 \notin W$. Then $W_1 = \langle x_1, W\rangle$ has order 128. As W_1 contains an element of order 8, by the description of the finite good 2-groups given above, W_1 is one of the finite groups listed in (ii) and (iii) of the Theorem.

If $W_1 = Q_1 \times \langle d \mid d^{16} = 1\rangle$, where Q_1 is isomorphic to the quaternion group of order 8, then $W = Q_1 \times \langle d^2\rangle$ and as $x_1 \notin W$, $x_1 = vd$ for some $v \in W$. Hence, $x_1^8 = d^8 \neq 1$, contradicting the fact that $x_1^4 = 1$.

If W_1 is given by (iii) of the Theorem, then $W = \langle a, b\rangle \times \langle d^2\rangle$ and $x_1 = vd$ for some $v \in W$. Then we have again that $x_1^8 = d^8 \neq 1$ which is impossible.

Thus, $x_1 \in W$ and, consequently, $G \cong Q \times C_{2^\infty}$.

We prove the sufficiency by verifying directly that all involutory units commute in KG where G is one of the groups listed in the theorem.

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Victor Bovdi
Bessenyei Teachers College
4401 Nyíregyháza,
Sóstoí ut. 31/b, Hungary
E-mail vbovdi@math.klte.hu

Michael Dokuchaev
Departamento de Matemática
IME, USP, Cx. P. 66281,
CEP 05389-970, São Paulo, Brazil
E-mail dokucha@ime.usp.br