THE ULTRABORNOLOGICALNESS OF THE SPACE OF DENJOY-PERRON-KURZWEIL INTEGRABLE FUNCTIONS AND OF OTHER NATURAL NON-COMPLETE SUBSPACES OF $C([a,b],X)$ AND OF $\prod E_i$

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Abstract. We introduce two new classes of linear topological spaces (lts) which are both ultrabornological and either Baire-like or what was called ultra-Baire-like by Kakol[9], and prove some theorems which provide many examples of lts which belong to these new classes and which are either metrizable or subspaces of a product of a family of lts indexed on $[a,b]$. A particular example is that of the title, under the Alexiewicz norm, which till now was only known to be barrelled. We believe that these new examples of non-complete normed ultrabornological spaces are the simplest known so far.

Introduction. The departure point of this paper was the study of the works of Sargent [14] and of Thomson [17], on the space of scalar-valued functions on $[a,b]$ which are integrable in the sense of Denjoy and Perron. Let us get a perspective about them in a few words.

In 1912 Denjoy introduced a process of integration more general than that of Lebesgue. In 1914 Perron gave a definition of integral based in principles different from those of Denjoy. In 1921 Hake proved that Perron's integral is more general than Denjoy's, while Aleksandrov(1924) and Looman (1925) proved independently that Denjoy's integral is more general (hence coincide) than Perron's. Thus, this has been called the Denjoy-Perron integral or the Denjoy integral in the restricted sense. A still different definition of integral was given by Kurzweil in 1957 and independently by Henstock in 1961, which coincides with the Denjoy-Perron integral, hence now called the Denjoy-Perron-Kurzweil integral or simply the gauge integral. Kurzweil's definition has the advantage of being simpler and of making sense for Banach-valued functions. It is somewhat surprising that 3 so different definitions give rise to the same class of integrable functions. Moreover, the gauge integrable functions which are positive coincide with the positive

Lebesgue integrable ones. We should also mention that in 1916 Denjoy and Khintchin defined independently a still more general process of integration than that first given by Denjoy, usually called the Denjoy-Khintchin integral or the Denjoy integral in the wide sense.

The major drawback of this class of gauge integrable functions relatively to that of the Lebesgue integrable is that the later constitute a Banach space while the first not, hence the methods of Functional Analysis could not be applied to its study at Banach's time. However, there is a natural norm, introduced by Alexiewicz, that may be considered on such space (or better, on the associated Hausdorff space obtained by the identification of functions which have the same integral on each subinterval of $[a,b]$). If f is gauge integrable on $[a,b]$, let $F(t)=\int_a^t f(s)ds$ for each $t\in[a,b]$. Then, F is continuous on $[a,b]$ and we may put $|||f|||=\|F\|$, the later being the sup-norm of F. In this way this space, which will be denoted by $K([a,b])$ (or by $K([a,b],X)$ if f is X-valued), become isomorphic to a (non-complete) subspace of $C([a,b])$ (or of $C([a,b],X)$).

The concept of β-space was introduced in [14], where it was proved that: i) $K([a,b])$ and many other subspaces of $C([a,b])$ are β-spaces and, ii) β-spaces satisfy several properties of Banach-Steinhaus type. Later, it was shown in [17] that β-spaces are barrelled, thus explaining within a more modern point of view the reason for the Banach-Steinhaus properties that they have. More recently, the barrelledness of the space of gauge integrable functions in 2 variables was proved in [12].

As we studied [14] and [17], we realized that one could leave aside the set-up of continuous additive interval functions on $[a,b]$, replacing it by the assumption of the existence of a family of continuous projections of a Banach space, indexed on $[a,b]$, thus generalizing those results (which became applicable, for instance, to subspaces of $G([a,b])$, the space of regulated functions). Later, we realized that by essentially the same methods one could prove that most of such spaces were in fact ultrabornological, and that an adaptation could be made to subspaces of the product of a family of lts.

In §1 we make a quick recapitulation of barrelled spaces, giving some examples of non-barrelled subspaces of C([a,b]) in order to compare them with the examples of barrelled or ultrabornological ones obtained in §4.

In §2 we consider several types of Baire properties on the class $\mathcal{C}$ of locally convex spaces (lcs), some of which arise more naturally in the category $\mathcal{L}$ of linear topological spaces (lts). We precede here by an "$\mathcal{L}$" (as in [1]) the name of the class of lts analogous to a given class of lcs. For instance, we call $\mathcal{L}$-barrelled the lts called "ultrabarrelled" in [13]. The important point is that the class of $\mathcal{L}$-barrelled spaces which are lcs is a proper subclass of the barrelled lcs, the same happenning (at least in all known cases) to the other Baire properties in $\mathcal{L}$. However, 2 different definitions have appeared as candidates for $\mathcal{L}$-Baire-like spaces: the U-BL (here denoted u-BL) spaces of Kakol [9] and the *-BL of Carreras [2]. It seems to be unknown wether the classes of u-BL and *-BL spaces coincide or if one of them contains the other (although it is known that metrizable *-BL are u-BL).

We show in §2 that the β-spaces of [14] are exactly the u-BL spaces of [9], and introduce 2 new classes of lts: the ultrabornologically BL and the u-ultrabornologically BL, which are related to ultrabornological spaces in the same way as the BL and the u-BL are related to barrelled spaces.

The main results of this paper are in §3, where it is proved that a class of subspaces of C([a,b],X) and of other metrizable complete lts and a class of subspaces of $\prod\{E_i : i \in [a,b]\}$ are u-ultrabornologically BL, and that others are at least ultrabornological, thus generalizing and strenghtenning both the results of [14] and [17]. The classes mentioned in §2 provide then some natural questions (left open) about the possibility of these spaces to have or to have not stronger properties of Baire type.

In §4 we give several concrete examples of application of §3, showing in particular, that K([a,b],X) is a u-ultrabornologically BL normed space. By the way, some examples of §4 seem to be the simplest ones of non-complete ultrabornological normed spaces.

Since K([a,b],X) is ultrabornological, it satisfies a closed graph theorem (resp. open mapping theorem) provided the range space (resp. domain space) belongs to the class of $\mathcal{C}$-webbed spaces of DeWilde (see [11] for a good account of such spaces), which is a very large class of lcs, with several stability properties. Thus, all the basic theorems of Functional Analysis may be applied to $K([a,b],X)$.

As a matter of fact, "in practice" we never find a discontinuous linear functional or even a discontinuous seminorm defined on a ultrabornological space. We prove such fact, with a precise meaning, in §5, following the ideas of [6] (see also [16]).

It is clear that analogous results should hold for the gauge integrable functions in 2 or more variables if, as in [12], we work directly with this space. We intend, however, in the near future, to adapt this work so as to include that space as a particular example of more general results.

We are grateful to Prof. Chaim S. Honig for asking us to give a lecture on the barrelledness of $K([a,b])$ following the lines of [14] and [17], which aroused our interest on that subject, and for drawing our attention to [6].

Notations. Besides lcs and lts, already explained above, which will be always Hausdorff spaces, we abbreviate "locally convex" by "lc", "absolutely convex " by "ac" and "neighbourhood of 0" by "0-nghb". If E or $(E,\mathcal{T})$ is a lts, we denote by E' or $(E,\mathcal{T})'$ the dual of E, i.e., the set of all continuous linear functionals on E, while E^* will denote the set of all linear functionals on E. If A is a subset of a vector space E, by $sp(A)$ will be denoted the vector subspace of E generated by A. By $\mathbb{K}$ we will denote either $\mathbb{R}$ or $\mathbb{C}$, and $\mathbb{K}^{(\mathbb{N})}$ will be a countable sum of copies of $\mathbb{K}$, with the usual lc topology of the direct sum.

1. REMARKS ON BARRELLED SPACES AND EXAMPLES OF NON-BARRELLED SUBSPACES OF $C([a,b])$

We recall that a barrel in a lts E is an ac closed and absorbent subset of E, that a lts E is barrelled if every barrel of E is a 0-nghb in E and that a lts is Baire-like (shortly BL) if, whenever A_n is a sequence of closed ac subsets of E which is increasing (i e, $A_n \subset A_{n+1}$ for every n) and which covers E (i e. $\bigcup A_n = E$), then there is an integer p such that A_p is a 0-nghb in E. Usually, one is mainly interested in the barrelled or BL spaces which are lcs. A <u>Frechet</u> space is a metrizable and complete lcs.

<u>Remark 1.0</u> If E is a lts which has a barrelled (resp. BL) dense subspace, then E is barrelled (resp. BL).

Most properties of barrelled lts are included in the following theorem, stated for reference, which is well known, at least for lcs.

<u>Theorem 1.1</u> Let $E=(E,\mathcal{T}_0)$ be a lts and consider the conditions:

a) E is BL

b) if F is a Frechet space and $f: E \longrightarrow F$ a closed linear mapping, then f is continuous

b')the same as b), when F is a Banach space

c_1)E is barrelled

c_2)if $\mathcal{T}_1$ is any lc topology on E with a base of $\mathcal{T}_0$-closed 0-nghb, then $\mathcal{T}_1 \subset \mathcal{T}_0$

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d) if $\{F_\alpha\}_{\alpha\in A}$ is a family of normed spaces, $f_\alpha : E \longrightarrow F_\alpha$ linear continuous mappings for every $\alpha\in A$ and $\sup\{\|f_\alpha(x)\| : \alpha\in A\}<\infty$ for every $x\in E$, then there is a 0-nghb V in E (which may be taken ac, if needed) such that $\sup\{\|f_\alpha(x)\| : \alpha\in A, x\in V\}<\infty$.

e) if F is a lcs, if $f_\alpha : E \longrightarrow F$, $\alpha\in A$, is a family of linear continuous mappings such that $\{f_\alpha(x) : \alpha\in A\}$ is a bounded subset of F for every $x\in E$, then $\{f_\alpha : \alpha\in A\}$ is equicontinuous.

f) if F is a normed space, if $f_\alpha : E \longrightarrow F$, $\alpha\in A$, is a family of linear continuous mappings and $\sup\{\|f_\alpha(x)\| : \alpha\in A\}<\infty$ for every $x\in E$, then there is a 0-nghb V in E such that $\sup\{\|f_\alpha(x)\| : \alpha\in A, x\in V\}<\infty$.

g) if x'_α, $\alpha\in A$ is a family of elements of E' and if $\sup\{|x'_\alpha(x)| : \alpha\in A\}<\infty$ for every $x\in E$, then there is a 0-nghb V in E such that $\sup\{|x'_\alpha(x)| : \alpha\in A, x\in V\}<\infty$.

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$\tilde{d}$)
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the same as the corresponding ones without "~", but for sequences instead of families of functions.

h) if F is a lcs, if $f_n : E \longrightarrow F$, $n\in \mathbb{N}$ is a sequence of linear continuous mappings such that there exists $\lim_{n\to\infty} f_n(x)$ for every $x\in E$, then f is cont.

i) if x'_n is a sequence of elements of E' and $x^*\in E^*$ is such that $\lim_{n\to\infty} x'_n(x) = x^*(x)$ for every $x\in E$, then x^* is continuous.

Then: 1) the conditions of group I are equivalent among them, the same happenning with those of group II; 2)

$$\begin{tikzcd}
b) \arrow[r, Rightarrow] & b') \arrow[r, Rightarrow] & c_1) \arrow[r, Leftrightarrow] & c_2) \arrow[r, Rightarrow] & I \arrow[r, Rightarrow] & II \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & g) \arrow[r, Rightarrow] & i) \\
 & a) \arrow[ur, Rightarrow] & & & & & h) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

; 3) in the case E is a lcs, then b), b'), c_1), c_2) and I are equivalent; 4) in the case E is a metrizable lcs, then all the conditions are equivalent.

<u>Proposition 1.2</u> Let E be a lts, $\{F_\alpha\}_{\alpha\in A}$ a family of normed spaces, $f_\alpha : E\to F_\alpha$ linear continuous mappings for every $\alpha\in A$. Let M be the space of $x\in E$ such that $\sup\{\|f_\alpha(x)\| : \alpha\in A\}<\infty$ and let us endow M with the topology induced by E. Then, a) M satisfies at least one of the following conditions:
i) M=E, or ii) M is not dense in E, or iii) M is not barrelled.
b) if, moreover, $\sup\{\|f_\alpha(x)\| : \alpha\in A, x\in V\}=\infty$ for every 0-nghb V in E, then one of the conditions ii) or iii) must hold.

<u>Proof</u>. Let $T_n=\{x\in E: \|f_\alpha(x)\|\leqslant n$ for every $\alpha\in A\}$. It is clear that $T_n= nT_1$ and that T_1 is ac and closed in E. Since $M=\bigcup T_n=\bigcup nT_1$, T_1 must be absorbent in M, hence a barrel of M. Suppose that M were both barrelled and dense in E: then, T_1 would be a 0-nghb in M, hence its closure in E (which is T_1 itself) would be a 0-nghb in E, hence absorbent in E, so that M=E. This proves part a) of the proposition and also shows that, if neither ii) nor iii) would hold, then M=E, so that E should be barrelled. This would contradict the implication $c_1)\Rightarrow d)$ of Th. 1.1, under the hypothesis of part b), hence part b) of the proposition is also proved. ∎

The following reformulation of a particular case of Prop. 1.2 is useful to prove the non-barrelledness of some normed spaces.

<u>Corollary 1.3</u> Let X be a dense subspace of a normed space E and suppose that there is a sequence x'_n in E' such that: 1) $\sup\{\|x'_n\| : n\in \mathbb{N}\}=\infty$, 2) $\sup\{|x'_n(x)| : n\in\mathbb{N}\}<\infty$ for every $x\in X$. Then, X is not barrelled.

<u>Proof</u>. Let M be the set of all $x\in E$ such that $\sup\{|x'_n(x)| : n\in\mathbb{N}\}<\infty$, so that $X\subset M$, hence M is dense in E. By Prop.1.2b) and our hypothesis 1) it follows that ii) or iii) holds, and since ii) is false, we conclude iii), i e M is not barrelled. By Remark 1.0, it follows that X is not barrelled, either. ∎

We will now apply Corol.1.3 to show that some important subspaces of $C([a,b])$ are not barrelled.

If $f:[a,b]\longrightarrow\mathbb{R}$ and $x\in[a,b[$, let us denote $D^+f(x)$ (resp. $D_+f(x)$) to the lim sup (resp. lim inf) of $(f(x+h)-f(x))/h$ when $h\to 0$ by positive values. If $D^+f(x)=D_+f(x)$ and are finite, we will denote that value by $f'(x+)$.

If $f:[a,b]\longrightarrow \mathbb{K}$ is a function of bounded variation on $[a,b]$, we denote $V_a^b(f)$ or simply $V(f)$ the variation of f in $[a,b]$. We denote by $BV([a,b])$ the space of functions of bounded variation on $[a,b]$; by $BVC([a,b])$ the set of functions which are continuous and of bounded variation on $[a,b]$; by $AC([a,b])$ the set of functions which are absolutely continuous on $[a,b]$. Let us recall that, if $h\in BV([a,b])$ then $F_h:C([a,b])\longrightarrow \mathbb{K}$, defined by $F_h(f)=\int_a^b f\,dh$, is a linear continuous functional on $C([a,b])$, with $\|F_h\|\leq V(h)$. In the case that, for every $t\in]a,b[$, $h(t)$ belongs to the closed interval with end-points $h(t-)$ and $h(t+)$, then $\|F_h\|=V(h)$, and we will say that h is *norm-preserving*. Let us remark also that, if $f:[a,b]\longrightarrow \mathbb{K}$ is of bounded variation, if $c>0$ and if $g:[a/c,b/c]\longrightarrow \mathbb{K}$ is defined by $g(t)=f(ct)$, then $V_{a/c}^{b/c}(g)=V_a^b(f)$.

Corollary 1.4 Let $t_o\in]a,b[$. The following 5 subspaces of $C([a,b])$ are not barrelled:

$X_1=\{f\in C([a,b]):D^+f(t_o)$ and $D_+f(t_o)$ are finite$\}$

$X_2=\{f\in C([a,b]):$ there exists $f'(t_o+)\}$

$X_3=\{f\in C([a,b]):$ there exists $f'(t_o)\}$

$X_4=\{f\in C([a,b]):$ there exists $f'(t)$ for every $t\in[a,b]\}$

$X_5=\{f\in C([a,b]):$ there exists $f'\in C([a,b])\}=C^{(1)}([a,b])$.

Proof. Since $X_5\subset X_j\subset X_1$ for every j and X_5 is dense in $C([a,b])$, it is enough, by Remark 1.0, to prove that X_1 is not barrelled. Let n_o be an integer greater than $1/(b-t_o)$ and for $n\geq n_o$ consider the functions $g_n:[a,b]\to\mathbb{R}$ which are equal to $-n$ if $t\in[t_o,t_o+n^{-1}[$, and equal to 0 otherwise. Then, for $n\geq n_o$, $V(g_n)=2n$ and g_n are norm-preserving, so that, if $G_n=F_{g_n}$, we have $\sup\{\|G_n\|:n\geq n_o\}=\sup\{V(g_n):n\geq n_o\}=+\infty$. Now, for $n\geq n_o$ one has $G_n(f)=\int_a^b f\,dg_n=(f(t_o+n^{-1})-f(t_o))/(n^{-1})$, so that $\sup\{|G_n(f)|:n\geq n_o\}<\infty$ for every $f\in X_1$. By Corol.1.3, X_1 is not barrelled. ∎

Remark 1.5 The analogous result for $t_o=a$ also holds. Analogous results for derivatives at the left also hold. One only needs to consider convenient functions g_n.

Corollary 1.6 Let $t_o\in[a,b[$. The following 4 subspaces of $C([a,b])$ are not barrelled:

$Y_1=\{f\in C([a,b])$: there is $\delta>0$ so that f is of bounded var. on $[t_0,t_0+\delta]\}$.
(the δ depends on f and is less than $b-t_0$)

$Y_2=\{f\in C([a,b])$: f is of bounded variation in some nghb of $t_0\}$

$Y_3=BVC([a,b])$, $Y_4=AC([a,b])$.

Proof. Since $Y_4\subset Y_j\subset Y_1$ for every j and Y_4 is dense in $C([a,b])$, it is enough to prove that Y_1 is not barrelled. For simplicity of notations, we shall do so only in the case $t_0=0$. The sup-norm will be denoted by $\| \ \|$.

Let $g:\mathbb{R}\to\mathbb{R}$ be continuous, periodic with period 1, not identically zero, with $g(0)=0$ and $V_0^1(g)=M<\infty$ (hence $M>0$) . One may take, for instance, $g(t)=$ sen $2\pi t$. Then, $V_0^n(g)=nM$, for every positive integer n.

Let $n_0>b^{-1}$. For each $n\geq n_0$, let $g_n:[a,b]\to\mathbb{R}$ be defined by $g_n(t)=g(n^2t)$ if $t\in[0,n^{-1}]$ and $g_n(t)=0$ otherwise. Of course, $\|g_n\|=\|g\|$ for $n\geq n_0$ but, by the remark preceding Corol.1.4, with $c=n^2$, $a=0$ and $b=n$ we get $V_a^b(g_n)=V_0^{1/n}(g_n)=V_0^n(g)=nM$. Hence, if $G_n=F_{g_n}$ is the continuous linear functional on $C([a,b])$ defined by g_n for $n\geq n_0$, we have $\sup\{\|G_n\|:n\geq n_0\}=\sup\{V_a^b(g_n):n\geq n_0\}=+\infty$.

On the other hand, if $f\in Y_1$, let $\delta>0$ be such that f is of bounded variation on $[0,\delta]$. For $n\geq n_0$ we have $G_n(f)=\int_a^b f\,dg_n=$
$\int_a^0 f\,dg_n+\int_0^{1/n} f\,dg_n+\int_{1/n}^b f\,dg_n=\int_0^{1/n} f\,dg_n=f(1/n)g_n(1/n)-f(0)g_n(0)-\int_0^{1/n} g_n\,df=$
$-\int_0^{1/n} g_n\,df$.

Hence, for $n\geq\max\{n_0,\delta^{-1}\}$ we will have:
$|G_n(f)|=\left|\int_0^{1/n} g_n\,df\right|\leq\|g_n\|V_0^{1/n}(f)\leq\|g\|V_0^{\delta}(f)$, so that $\sup\{|G_n(f)| : n\geq n_0\}<\infty$, for each $f\in Y_1$. By Corol.1.3, Y_1 is not barrelled. ∎

Remark 1.7 An analogous result holds for functions which are of bounded variation at the left of t_0, if $t_0\in]a,b]$.

2. SOME CLASSES OF LTS AND LCS

The equivalence $c_1)\Leftrightarrow c_2)$ of Th.1.1, verified for lcs by Robertson ([13], Th.4), led her to define what she called *ultrabarrelled* spaces, here called *$\mathcal{L}$-barrelled* spaces, in accordance with [1].

Definition 2.1 Let E be a vector space. Then, a) a sequence A^n of subsets of E is a *basic sequence* if: i) each A^n is balanced and absorbent and ii) $A^{n+1}+A^{n+1}\subset A^n$ for each n; b) a subset A of E is *admissible in E* if there is a basic sequence A^n such that $A^1=A$; the sequence A^n is then called a <u>defining basic sequence</u> for A; c) a subset A of E is said to be <u>admissible</u> if it is admissible in sp(A); d) if A_m is an increasing sequence of admissible sets, we say that they <u>generate a table</u> if there are defining basic sequences $(A^n_m)_n$ for each A_m (in sp(A_m)) such that $A^n_m \subset A^n_{m+1}$ for every m and n.

<u>Definition 2.2</u> Let E be a lts. Then, a) a subset A of E is an <u>$\mathcal{L}$-barrel</u> (called <u>ultrabarrel</u> in [8]) if it is closed and admissible in E; b) E is <u>$\mathcal{L}$-barrelled</u> if every $\mathcal{L}$-barrel of E is a 0-nghb in E; c) E is <u>*-BL</u> if, given any increasing sequence A_m of closed admissible sets which covers E and generates a table, then there is an integer p such that A_p is a 0-nghb in E; d) E is <u>u-BL</u> if, given any sequence A_m of closed balanced sets such $A_m+A_m \subset A_{m+1}$ for every m and which covers E, then there is some integer p such that A_p is a 0-nghb in E.

Most properties of $\mathcal{L}$-barrelled spaces are included in the next theorem, which should be compared with Th.1.1. The letters of the conditions in Th.2.3 correspond to those in Th.1.1.

<u>Theorem 2.3</u> Let $E=(E,\mathcal{T}_0)$ be a lts, and consider the conditions:

a_1) E is *-BL

a_2) E is u-BL

b) if F is a metrizable complete lts and $f: E \to F$ a closed linear mapping, then f is continuous

c_1) E is $\mathcal{L}$-barrelled

c_2) if $\mathcal{T}_1$ is any linear topology on E with a base of $\mathcal{T}_0$-closed 0-nghb, $\mathcal{T}_1 \subset \mathcal{T}_0$

e) if F is a lts, if $f_\alpha: E \to F$, $\alpha \in A$, is a family of linear continuous mappings such that $\{f_\alpha(x): \alpha \in A\}$ is a bounded subset of F for every $x \in E$, then $\{f_\alpha: \alpha \in A\}$ is equicontinuous

f) the same as e), but for F metrizable lts

$\left.\begin{matrix}\tilde{e}) \\ \tilde{f})\end{matrix}\right\}$ the same as the corresponding ones without "~", but for sequences instead of families of functions

h) if F is a lts, if $f_n: E \to F$ is a sequence of linear continuous mappings such that there exists $\lim_{n\to\infty} f_n(x)$, called f(x), for every $x \in E$, then f is continuous.

Then: 1) conditions b), c_1), c_2), e) and f) are equivalent; 2) $e) \Rightarrow \tilde{e}) \left.\begin{matrix}\Rightarrow \tilde{f}) \\ \Rightarrow h)\end{matrix}\right\}$ and $a_1) \Rightarrow c_2)$; 3) in the case E is metrizable, all the conditions different from a_2) and h) are equivalent, and $c_2) \Rightarrow a_2)$.

<u>Remark-proof</u>(reference to). Definition 2.2b) coincides with the one considered in [1], while in [13] (resp. [19]) condition c_2) (resp. e)) was taken as definition of $\mathcal{L}$-barrelled spaces. The implications e)$\Rightarrow \bar{g}) \Rightarrow \bar{f}$), e)$\Rightarrow$ f)$\Rightarrow \bar{f}$) and $a_1) \Rightarrow c_1$) are evident. The implications $c_2) \Rightarrow \bar{g}$), e)$\Rightarrow$ h) and $c_2) \Rightarrow$ b) are respectively Th.5, the remark following its proof and Prop.15ii) of [13], while $c_1) \Leftrightarrow c_2$) and $c_2) \Leftrightarrow$ b) are resp. Th.3.1 and 3.2 of [8]. The implication e)$\Rightarrow c_2$) is Prop.5(pg 10) of [19]. The equivalences $c_2) \Leftrightarrow$ e)$\Leftrightarrow$ f) may be seen in §7(3), while the equivalence $c_2) \Leftrightarrow \bar{f}$) for metrizable lts in §7(4) of [1]. The name ultra-Baire-like was introduced in [9], but such a concept was already considered, without a name, in [8], where it is found the proof that, for metrizable lts, $c_1) \Rightarrow a_2$). The name *-BL was introduced in [2], although in [4] already appears the proof that, for metrizable lts, $c_1) \Rightarrow a_1$). ∎

In [14] a class of normed spaces was considered, called β-spaces. However, the definition makes sense for arbitrary lts, and amounts to:

<u>Definition 2.4</u> A lts E is a <u>β-space</u> if, whatever the sequence A_m of closed subsets of E, such that: i) $0 \in A_1$, ii) $A_m + A_m \subset A_{m+1}$ and $A_m - A_m \subset A_{m+1}$ for every m, iii) $\bigcup A_m = E$; then there is an integer p such that A_p has a non-empty interior.

<u>Proposition 2.5</u> Let E be a lts. Then, E is a β-space $\Leftrightarrow$ E is u-BL.

<u>Proof</u>. Let E be a β-space, and A_m a sequence of closed balanced sets such that $A_m + A_m \subset A_{m+1}$ for every m and which covers E. Since A_m is balanced, we have $0 \in A_1$ and $-A_m = A_m$, hence $A_m - A_m \subset A_{m+1}$. Since E is a β-space, there are $p \in N$, $x \in E$ and a 0-nghb V in E such that $x + V \subset A_p$. Hence, $V = (x+V) - x \subset A_p - A_p \subset A_{p+1}$, so that A_{p+1} is a 0-nghb in E. Therefore, E is u-BL.

Let E be a u-BL space and A_m a sequence of closed sets satisfying conditions i), ii) and iii) of Def.2.4. Let R_m be the balanced kernel of A_m (i e, $x \in R_m \Leftrightarrow \lambda x \in A_m$ for every λ with $|\lambda| \leq 1$: R_m is the biggest balanced subset of A_m). Since A_m is closed and the closure of balanced sets are balanced, it follows that R_m are closed. By ii), $R_m + R_m \subset A_{m+1}$, and since $R_m + R_m$ is balanced, we have $R_m+R_m\subset R_{m+1}$. for every m. Suppose that we have proved that $\bigcup R_m=E$: then, since E is u-BL, there will be some p for which R_p will be a 0-nghb in E, hence A_p has a non-empty interior, and E will be a β-space. Hence, it is enough to show that $\bigcup R_m=E$. Let $x\in E$, $x\neq 0$, $F=sp(x)$ with the topology induced by E and $S_m=A_m\cap F$. Then, S_m also satisfies i), ii) and iii) of Def.2.4 for F. Since F is a Baire space, there is p such that S_p has non-empty interior in F, hence S_{p+1} is a 0-nghb in F (proof like in part a)), which means that there is $\delta>0$ such that $\lambda x\in S_{p+1}$ for every λ with $|\lambda|\leqslant\delta$. It follows by induction that $\lambda x\in S_{p+m}$ for every λ with $|\lambda|\leqslant 2^{m-1}\delta$. .If we take m such that $2^{m-1}\delta\geqslant 1$, then we will have $\lambda x\in S_{p+m}$ for every λ with $|\lambda|\leqslant 1$, hence $x\in R_{p+m}$. ∎

We give for for reference next definition, usually considered for lcs.

Definition 2.6 Let E be a lts. Then,

a) E is suprabarrelled (shortly SB) if, given any increasing sequence E_n of subspaces of E which covers E, there is some p for which E_p is both barrelled and dense in E;

b) E is totally barrelled (shortly TB) if, given any sequence E_n of subspaces of E which covers E, there is some p for which E_p is barrelled and its closure in E is finite-codimensional;

c) E is unordered Baire-like (shortly UBL) if, given any sequence A_n of ac closed subsets of E which covers E, there is some p for which A_p is a 0-nghb in E.

Remark 2.7 a) It is proved in [3] (chap.9) that for lcs the following holds: Baire $\Rightarrow$ UBL $\Rightarrow$ TB $\Rightarrow$ SB $\Rightarrow$ BL $\Rightarrow$ barrelled and one or more counterexamples for each reverse implication is given. These implications are true also for lts, as can be easily verified. It is evident that u-BL $=$ BL, *-BL $\Rightarrow$ BL and $\mathcal{L}$-barrelled $\Rightarrow$ barrelled. b) If (E,τ) is a lts, let $\mathcal{V}$ be the set of all ac subsets V of E which are 0-nghb in E: then, $\mathcal{V}$ is the set of o-nghb in E for a lc topology τ_{oo} on E, coarser than τ (in fact, τ_{oo} is the finest lc topology on E, coarser than τ), but τ_{oo} is in general not Hausdorff, even if τ is Hausdorff. We will denote (E,τ_{oo}) by E_{oo}. It is easily verified that, if (E,τ) is a lts for which τ_{oo} is Hausdorff, and if (E,τ) is barrelled (resp. BL, UBL) then E_{oo} is also barrelled (resp. BL, UBL). c) Suppose that $(E,\mathcal{T})$ is a metrisable complete lts which is not lc, such that $\mathcal{T}_{oo}$ is Hausdorff. Then, $(E,\mathcal{T})$ is Baire, hence UBL; therefore, by b) E_{oo} is also UBL. On the other hand, E_{oo} is not $\mathcal{L}$-barrelled (see [13], Prop.20 and the example following it). Hence, none of the conditions UBL, TB, SB, BL or barrelled imply $\mathcal{L}$-barrelled, even for lcs. An example is given by ℓ^p, with $0<p<1$, and with the linear topology $\mathcal{T}$ given by the p-norm $\|(\lambda_n)\|_p=\sum_{n=1}^{\infty}|\lambda_n|^p$. Denote by $(\ell^p)^1$ the space ℓ^p endowed with the norm topology of ℓ^1. It is well known that $(\ell^p)_{oo}=(\ell^p)^1$. Moreover, if B_M is the set of $(\lambda_n)\in\ell^p$ with p-norm $\leq M$, then each B_M is balanced and closed in $(\ell^p)^1$, $B_M+B_M\subset B_{2M}$, but none of the B_M is a 0-nghb in $(\ell^p)^1$, hence $(\ell^p)_{oo}$ is not u-BL, either. d) $K^{(N)}$ is an $\mathcal{L}$-barrelled lcs which is not BL, hence not *-BL, not u-BL, not SB, not TB and not UBL. e) If E and F are infinite-dimensional Banach spaces, let H be the projective tensor product of E and F, and H_n be the subset of H of all tensor products of rank at most n. It is clear that H_n are balanced, that $H_n+H_n\subset H_{2n}$ and that they cover H. As already stated in [7] (Chap.III,§6,exerc.1), each H_n is closed (hence, H is a normed space which is not u-BL). In the same exercise it is stated that H is barrelled. In fact, it was proved in [15] that H is UBL. Moreover, H is ultrabornological (see Cor.11.3.18 in [3] ; according to 11.10 of [3], this result is based on or due to Floret). Hence, H is a normed ultrabornological UBL space which is not u-BL (therefore, not $\mathcal{L}$-barrelled). f) As shown by a comparison between Th.1.1 and 2.3, the $\mathcal{L}$-barrelled spaces constitute the class of lts analogous to the barrelled lcs. On the other hand, there are 2 different definitions for spaces analogous to the BL: the *-BL and the u-BL. We don't know if they coincide or if one of them implies the other (although for metrizable spaces *-BL$\Rightarrow$u-BL, since *-BL$\Rightarrow$ $\mathcal{L}$-barrelled and, for metrizable spaces, $\mathcal{L}$-barrelled$\Rightarrow$u-BL). Perhaps none of these is the "right" generalization of BL. g) Since we will not need then, we will not consider here the classes of lts "analogous" to the UBL, TB or SB, which have been studied by some authors.

Let us now take a look on the ultrabornological spaces.

If E is a lts, B an ac bounded subset of E and $E_B=sp(B)$, then the seminorm determined by B is a norm on E_B, and the inclusion $i_B:E_B\to E$ is continuous. We will always consider E_B as a normed space, with the norm given by B.

Definition 2.8 Let E be a lts. We say that B is a Banach disk of E if B is an ac bounded subset of E, such that E_B is a Banach space. We say that a sequence x_n is fast convergent to x_o in E if there is a Banach disk B of E such that x_o and the sequence x_n lie in E_B, and x_n converges to x_o in the Banach space E_B. In the case $x_o=0$, we say that the sequence x_n is null fast convergent in E.

One should remark that, if x_n is fast convergent in E, then there exists an ac compact subset K of E, to which every x_n belongs.

Proposition 2.9 a) Let X be a Banach space with unit ball B and D_n a sequence of closed balanced subsets of X which covers X, such that $D_n+D_n\subset D_{n+1}$ for every n. Then, there is some p such that D_p absorbs B. b) Let B be a Banach disk of a lts F and D_n a sequence of closed balanced subsets of F which covers F, such that $D_n+D_n\subset D_{n+1}$ for every n. Then, there is some p such that D_p absorbs B. c) Let E be a lts and B a closed ac bounded and sequentially complete subset of E. Then, B is a Banach disk of E (hence, every ac compact subset of E is a Banach disk).

Proof. a) Since X is Baire, there is p such that D_p has a non-empty interior hence (as in the proof of Prop.2.5) D_{p+1} is a 0-nghb in X, hence it absorbs B. b) follows from a) and the facts that E_B is a Banach space and $i_B:E_B\to E$ is continuous. c) The same proof given in §20.11(2) of [10] for lcs works also for lts. ∎

Definition 2.10 If E_α, $\alpha\in A$, is a family of lcs (resp. lts), E is a vector space and $u_\alpha:E_\alpha\to E$, $\alpha\in A$, a family of linear mappings, there exists the finest lc topology τ (resp. linear topology ξ) on E for which every u_α is continuous (of course, in the case E_α are lcs, both topologies may be defined and $\tau\subset\xi$; in general, these topologies may be non-Hausdorff). We call τ (resp. ξ) the final toplogy (resp. $\mathcal{L}$-final topology) on E determined by the family u_α, $\alpha\in A$. In the case A is a directed set, E_α is an inductive system of lcs (lts) and E the inductive limit of the *vector* spaces E_α (without consideration of topologies), then $(E,\mathcal{T})$ (resp. (E,ξ)) will be called the *inductive limit* (resp. *$\mathcal{L}$-inductive limit*) of the inductive system E_α, provided $\mathcal{T}$ (resp. ξ) is Hausdorff.

Next theorem includes most properties of ultrabornological spaces and is well known, at least for lcs.

Theorem 2.11 Let $(E,\mathcal{T})$ be a lts. Consider the conditions:

a_1) if V is an ac subset of E which absorbs every Banach disk of E, then V is a 0-nghb in E

b_1) if F is a lcs and $u: E \rightarrow F$ a linear mapping such that $u(B)$ is bounded in F for every Banach disk B of E, then u is continuous

c_1) the same as b_1), but for normed spaces F

d_1) if $\mathcal{T}_1$ is a lc topology (not necessarily Hausdorff) on E, such that every Banach disk of $(E,\mathcal{T})$ is bounded in $(E,\mathcal{T}_1)$, then $\mathcal{T}_1 \subset \mathcal{T}$.

a_2), b_2), c_2) and d_2): the same as the corresponding ones with sub-index "1" replacing "Banach disk" by "ac compact subset"

a_3), b_3), c_3) and d_3): the same as the corresponding ones with sub-index "1" replacing "Banach disk" by "null fast convergent sequence"

e) E is the inductive limit of an inductive system of Banach spaces.

Then, all the conditions different from e) are equivalent, e) implies the others and, in the case E is lc, is equivalent to the others.

Definition 2.12 Let $(E,\mathcal{T})$ be a lts. Then,

a) E is *ultrabornological* (shortly *ultrab.*) if it satisfies any of the conditions od Th.2.11 different from e). Usually a_1) is taken as definition.

b) E is *ultrabornologically Baire-like* (shortly *ultrab.BL*) if, for every increasing sequence W_n of ac sets which covers E and such that, for each Banach disk B of E there is some p (depending on B) such that W_p absorbs B, then there is some k such that W_k is a 0-nghb in E.

c) E is *u-ultrabornologically Baire-like* (shortly *u-ultrab.BL*) if, for every sequence W_n of balanced sets such that: i) $W_n + W_n \subset W_{n+1}$ for every n; ii) the sets W_n cover E; iii) for each Banach disk B of E, there is some p (depending on B) such that W_p absorbs B; then there is some k such that W_k is a 0-nghb E.

Remark 2.13 a) It is easy to verify that u-ultrab.BL $\Rightarrow$ ultrab.BL $\Rightarrow$

$\Rightarrow$ ultrab., and that, if (E,τ) is a ultrab.BL lts (resp. ultrab. lts) and τ_{oo} is Hausdorff, then E_{oo} is a ultrab.BL lcs (resp. ultrab. lcs). b) By Prop. 2.9b), u-ultrab.BL$\Rightarrow$u-BL and analogously ultrab.BL$\Rightarrow$BL, ultrab.$\Rightarrow$barrelled. c) One could define, similarly, the concepts of *$\mathcal{L}$-ultrabornological*, **-ultrab.BL*, *ultrab.UBL*, etc. but we will not need them.

Definition 2.14 We will say that a sequence x_n of a lts E *has property ℓ^∞ in E* if, for every $(\lambda_n)\in\ell^\infty$, the series $\sum_{n=1}^{\infty}\lambda_n x_n$ is convergent in E. In this case, we will denote by $B(\ell^\infty,x_n)$ the set of the sums of all such series, for $\|(\lambda_n)\|\leq 1$. (Remark that $x_k\in B(\ell^\infty,x_n)$ for every k).

Proposition 2.15 Let E be a lts and x_n a sequence in E. The sequence x_n has property ℓ^∞ in E and $B(\ell^\infty,x_n)$ is an ac compact subset of E (hence a Banach disk of E) in each of the following cases: a) E is metrizable complete, V_n is a base of balanced 0-nghb in E such that $V_n+V_n\subset V_{n-1}$ for every $n\geq 2$, and $x_n\in V_n$ for every n. b) E is the topological product of a family E_α, $\alpha\in A$, of lts and $x_n=(x_n^\alpha)_{\alpha\in A}$ is such that the set $J_\alpha=\{n\in\mathbb{N}:\ x_n^\alpha\neq 0\}$ is finite for every $\alpha\in A$.

Proof. a) Remark first that $\sum_{n=k+1}^{r}V_n\subset V_k$ if $r\geq k+1$. In fact, this is trivial if $r=k+1$ or $k+2$, and if assumed already proved for every pair (k,r) with $r=k+p$, then it also holds when $r=k+p+1$, because $\sum_{n=k+1}^{r}V_n=V_{k+1}+$ $+\sum_{n=k+2}^{r}V_n\subset V_{k+1}+V_{k+1}\subset V_k$.

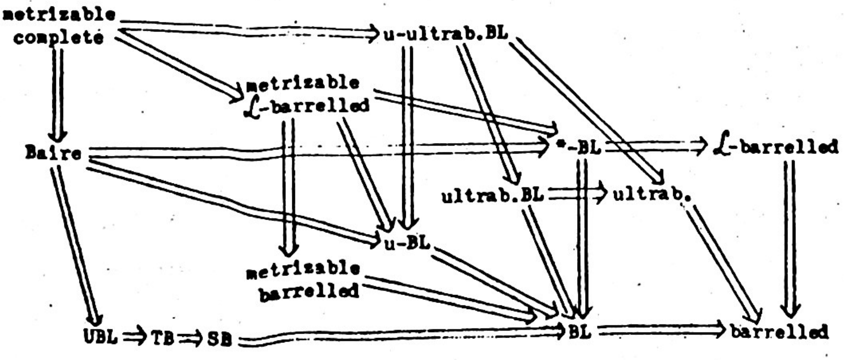
If $\sup\{|\lambda_n|:n\in\mathbb{N}\}\leq 1$, then $\sum_{n=k+1}^{r}\lambda_n x_n\in\sum_{n=k+1}^{r}V_n\subset V_k$, hence $s_m=s_m((\lambda_n))=\sum_{n=1}^{m}\lambda_n x_n$ is a Cauchy sequence in E (for each (λ_n)), therefore convergent in E, so that x_n has property ℓ^∞ in E. Since for every $r>k+1$ we have $\sum_{n=k+1}^{r}\lambda_n x_n\in V_k$, it follows that $\sum_{n=k+1}^{\infty}\lambda_n x_n\in\overline{V_k}\subset V_k+V_k\subset V_{k-1}$. Let $u:\ell^\infty\longrightarrow E$ be defined by $u((\lambda_n))=\sum_{n=1}^{\infty}\lambda_n x_n$. If B is the closed unit ball of ℓ^∞, then $u(B)=B(\ell^\infty,x_n)$ so that $B(\ell^\infty,x_n)$ is ac and we have shown that the sequence s_m is uniformly convergent to u on B. Consider on B the topology τ induced by $\sigma(\ell^\infty,\ell^1)$, which makes B compact (and topologically isomorphic to a countable product of spaces equal to $[-1,1]$). Since the mappings s_m are continuous on (B,τ), it follows that the uniform limit u is also continuous on (B,τ), hence its image $B(\ell^\infty,x_n)$ is compact.

b) Since each J_α is finite, the series $\sum_{n=1}^{\infty}\lambda_n x_n^\alpha$ is convergent for every α, hence the series $\sum_{n=1}^{\infty}\lambda_n x_n$ is convergent in E, so that x_n has property ℓ^∞ in E. Let $u:\ell^\infty\to E$ and B be as in part a) and let $\Pi_\alpha: E\to E_\alpha$, $\alpha\in A$, be the canonical projections. In order to show that $u:(\ell^\infty,\sigma(\ell^\infty,\ell^1))\to E$ is continuous, it is enough to show that $u_\alpha=\Pi_\alpha\circ u$ is continuous for every α. Since $u_\alpha((\lambda_n))=\sum_{n=1}^{\infty}\lambda_n x_n=\sum_{n\in J_\alpha}\lambda_n x_n$ and J_α is finite, it is clear that u_α is continuous. Hence, $u(B)=B(\ell^\infty,x_n)$ is an ac compact subset of E. ∎

<u>Corollary 2.16</u> If E is a metrizable complete lts, then E is u-ultrab.BL.

<u>Proof</u>. Let W_n be a sequence of balanced subsets of E satisfying conditions i), ii) and iii) of Def.2.12c). Let V_n be a basis of balanced 0-nghb in E such that $V_n+V_n\subset V_{n-1}$ for every $n\geq 2$. If none of the W_n were a 0-nghb in E, one could find $x_n\in V_n$ such that $x_n\notin nW_n$. By Prop.2.15a), $B(\ell^\infty,x_n)$ is a Banach disk of E, hence by condition iii) there exists some p such that W_p absorbs $B(\ell^\infty,x_n)$. But that is impossible since, for $n\geq p$, one has $W_p\subset W_n$ and therefore $x_n\notin nW_p$, $\forall\, n\geq p$. ∎

<u>Notation-Remark 2.17</u> If (E,τ) is a lcs, we denote by τ^u the lc topology on E defined by the family of all ac sets which absorbs all Banach disks B of (E,τ) as a basis of 0-nghb. Then, it is easily seen that: i) $\tau\subset\tau^u$, ii) (E,τ^u) is an ultrab. lcs and iii) τ^u is the weakest lc topology with properties i) and ii). The space $E^u=(E,\tau^u)$ is called the <u>ultrabornological space associated</u> with (E,τ). It is easy to see that E and E^u have the same Banach disks.



3. ULTRABORNOLOGICALNESS, u-ULTRAB.BL-NESS AND u-BL-NESS OF SUBSPACES OF SOME LTS

If E is a vector space, <u>projection of</u> E is a linear mapping $P:E\rightarrow E$ such that $P^2=P$. If F is a vector subspace of E such that $P(F)\subset F$, then the restriction $P|_F:F\rightarrow F$ is a projection of F.

<u>Proposition 3.1</u> Let E be a lts. a) If P and Q are projections of E such that $PQ=QP=P$, then $Q-P$ is also a projection of E. b) If F is a linear subspace of E and P a continuous projection of E such that $P(F)\subset F$, then: i) F is topologically isomorphic to the product of the subspaces $P(F)$ and $(I-P)(F)$ of E; ii) in the case F is dense in E, then $P(F)$ is dense in $P(E)$. c) If $Q_1, Q_2,\dots,Q_n$ $(n\geqslant 3)$ are continuous projections of E such that $Q_j(F)\subset F$ and $Q_iQ_j=Q_jQ_i=Q_{\min\{i,j\}}$, for every i and j, then $(Q_n-Q_1)(F)$ is topologically isomorphic to the product of the subspaces $(Q_{j+1}-Q_j)(F)$, $j=1,\dots n-1$, of E.

<u>Proof</u>. a) Compute $(Q-P)^2$. b) i) is well known; ii) $P(E)=P(\overline{F})\subset\overline{P(F)}$. c) It is enough to prove for $n=3$ since the general result follows from this by induction. Call $F=(Q_3-Q_1)(E)$, $P=Q_2$ and apply b)i), verifying that $P(F)=(Q_2-Q_1)(E)$, $(I-P)(F)=(Q_3-Q_2)(E)$ and that $P(F)\subset F$. For this last inclusion it is enough to show that $(Q_3-Q_1)P(Q_3-Q_1)=P(Q_3-Q_1)$, since $F=(Q_3-Q_1)(E)$. ∎

<u>Proposition 3.2</u> Let E and F be lts and V a subset of ExF. a) If V+V is not a 0-nghb in ExF, then at least one of the following conditions holds: i) $V\cap E$ is not a 0-nghb in E (more precisely, $V\cap(E\times\{0\})$ is not a 0-nghb in $E\times\{0\}$) or ii) $V\cap F$ is not a 0-nghb in F. b) If D_m is a sequence of subsets of ExF containing 0 such that $D_m+D_m\subset D_{m+1}$ for every m and if none of the D_m is a 0-nghb in ExF, then at lest one of the following conditions holds: i) none of the $D_m\cap E$ is a 0-nghb in E or ii) none of the $D_m\cap F$ is a 0-nghb in F. c) If V is ac but not a 0-nghb in ExF, then $V\cap E$ is not a 0-nghb in E or $V\cap F$ is not a 0-nghb in F.

<u>Proof</u>. a) If i) and ii) were both false, then $(V\cap E)\times(V\cap F)=(V\cap E)+(V\cap F)\subset V+V$ would be a 0-nghb in ExF. b) Remark first that for each m both sets $D_m\cap E$ and $D_m\cap F$ cannot be 0-nghb in the respective spaces, otherwise by

a) D_m+D_m would be a 0-nghb in $E\times F$ and a fortiori also D_{m+1}, against the hypothesis. Suppose that i) does not hold, i e, that for some p the set $D_p\cap E$ is a 0-nghb in E. Then, $D_m\cap E$ would be 0-nghb in E for every $m\geq p$ hence, by the initial remark, $D_m\cap F$ is not a 0-nghb in F for $m\geq p$, and a fortiori also for $m<p$ so that ii) holds. c) This follows from a), remarking that $V+V=2V$, since V is ac, and that V is a 0-nghb in $E\times F$ if and only if 2V is a 0-nghb in $E\times F$. ∎

Notation-Remark 3.3 a) Let E be a lts and for each $\lambda\in[a,b]$, $P_\lambda : E\rightarrow E$ be a continuous linear mapping such that

(3.4) $P_\lambda P_\mu = P_\mu P_\lambda = P_{\min\{\lambda,\mu\}}$, for every $\lambda,\mu\in[a,b]$.

Then each P_λ is a projection of E (take $\mu=\lambda$ in (3.4)) and, by Prop. 3.1a), P_d-P_c is a projection of E if $a\leq c<d\leq b$, which we will denote by $P^{(c,d)}$. Suppose that G is a linear subspace of E such that

(3.5) $P_\lambda(G)\subset G$ for every $\lambda\in[a,b]$.

Then, we will denote $G^{(c,d)}=P^{(c,d)}(G)$. By Prop.3.1c), if $a\leq c<e<d\leq b$, then $G^{(c,d)}=G^{(c,e)}\times G^{(e,d)}$. If one also assumes that

(3.4') $P_a=0$ and $P_b=I$,

then, $G^{(a,b)}=G$.

b) Suppose now that (3.4) holds and also:

$$(3.6)\begin{cases}\text{i) for each } \lambda\in\,]a,b] \text{ and each } x\in E, \text{ there exists } \lim\limits_{\mu\rightarrow\lambda-} P_\mu(x), \text{ which will} \\ \quad\text{be denoted } P_{\lambda-} \\ \text{i') } P_{\lambda-} : E\rightarrow E \text{ is continuous for every } \lambda\in\,]a,b] \\ \text{ii) for each } \lambda\in[a,b[\text{ and each } x\in E, \text{ there exists } \lim\limits_{\mu\rightarrow\lambda+} P_\mu(x), \text{ which will} \\ \quad\text{be denoted } P_{\lambda+} \\ \text{ii') } P_{\lambda+} : E\rightarrow E \text{ is continuous for every } \lambda\in[a,b[.\end{cases}$$

We shall always make the convention:

(3.6') $P_{a-}=0$ and $P_{b+}=I$.

Then, one can verify the following formulas:

$$P_\mu P_{\lambda+} = P_{\lambda+}P_\mu = \begin{cases}P_\mu \text{ if } \mu\leq\lambda \\ P_{\lambda+} \text{ if } \mu>\lambda\end{cases} ;\ P_\mu P_{\lambda-} = P_{\lambda-}P_\mu = \begin{cases}P_\mu \text{ if } \mu<\lambda \\ P_{\lambda-} \text{ if } \mu\geq\lambda\end{cases} ;\ P_{\mu-}P_{\lambda+} = P_{\lambda+}P_{\mu-} = \begin{cases}P_{\mu-} \text{ if } \mu\leq\lambda \\ P_{\lambda+} \text{ if } \mu>\lambda\end{cases}$$

$$P_{\mu+}P_{\lambda+} = P_{\lambda+}P_{\mu+} = P_{(\min\{\lambda,\mu\})+} \quad ; \quad P_{\mu-}P_{\lambda-} = P_{\lambda-}P_{\mu-} = P_{(\min\{\lambda,\mu\})-} \,,$$

from which it follows, in particular, that $P_{\lambda+}$ and $P_{\lambda-}$ are projections of E for every $\lambda\in[a,b]$. By Prop.3.1a) and the formulas above, the following mappings (with the corresponding notations) are projections of E:

$P_{(c,d)}=P_{d-}-P_{c+}$, if $a\leq c<d\leq b$; $P_{\{\lambda\}}=P_{\lambda+}-P_{\lambda-}$, $P_{\{\lambda+\}}=P_{\lambda+}-P_\lambda$, $P_{\{\lambda-\}}=P_\lambda-P_{\lambda-}$, if $\lambda\in[a,b]$.

Suppose that G is a linear subspace of E such that, besides (3.5), it also satisfies

(3.5') $P_{\lambda-}(G)\subset G$ and $P_{\lambda+}(G)\subset G$, for every $\lambda\in[a,b]$.

Then, we will denote $G_{(c,d)}=P_{(c,d)}(G)$ if $a\leq c<d\leq b$ and $G_{\{\lambda\}}=P_{\{\lambda\}}(G)$, $G_{\{\lambda+\}}=P_{\{\lambda+\}}(G)$, $G_{\{\lambda-\}}=P_{\{\lambda-\}}(G)$ if $\lambda\in[a,b]$. By Prop.3.1c), if $\lambda\in[a,b]$ we have $G_{\{\lambda\}}=G_{\{\lambda-\}}\times G_{\{\lambda+\}}$, and $G^{(c,d)}=G_{\{c+\}}\times G_{(c,d)}\times G_{\{d-\}}$ if $a\leq c<d\leq b$, without assuming that (3.4') holds.

c) If E is an $\mathcal{L}$-barrelled lts or a barrelled lcs and $P_\lambda:E\to E$ are linear continuous mappings for $\lambda\in[a,b]$, satisfying (3.4), (3.6) i) and ii), then (3.6) i') and ii') follows from the implication $c_1)\Rightarrow h)$ of Th.1.1 or 2.3, while from the implication $c_1)\Rightarrow e)$ of the same theorems and a little thought, one can show that the set of all P_λ, all $P_{\lambda+}$ and all $P_{\lambda-}$, with $\lambda\in[a,b]$, is equicontinuous. We will use these results only in Theorem 3.11''b).

Proposition 3.7 Let E be a lts, G a linear subspace of E, $P_\lambda:E\to E$ linear continuous mappings for $\lambda\in[a,b]$, and D_m a sequence of balanced (non-empty) subsets of G such that $D_m+D_m\subset D_{m+1}$ for every m, but such that none of the D_m is a 0-nghb in G. a) If (3.4), (3.4') and (3.5) holds, then there is $c\in[a,b]$ and a sequence $d_n\in[a,b]$ such that at least one of the following conditions holds:

i) $d_n\uparrow c$ (i e, d_n is strictly increasing with limit c) and $D_m\cap G^{(d_n,c)}$ is not a 0-nghb in $G^{(d_n,c)}$, for every m and n; or

ii) $d_n\downarrow c$ (i e, d_n is strictly decreasing with limit c) and $D_m\cap G^{(c,d_n)}$ is not a 0-nghb in $G^{(c,d_n)}$, for every m and n.

b) If (3.4), (3.6), (3.6'), (3.5) and (3.5') hold and if, for each $\lambda\in[a,b]$, there is some p (depending on λ) such that $D_p\cap G_{\{\lambda\}}$ is a 0-nghb in $G_{\{\lambda\}}$, then there is $c\in[a,b]$ and a sequence $d_n\in[a,b]$ such that at least one of the following conditions hold:

i) $d_n\uparrow c$ and $D_m\cap G_{(d_n,c)}$ is not a 0-nghb in $G_{(d_n,c)}$, for every m and n; or

ii) $d_n\downarrow c$ and $D_m\cap G_{(c,d_n)}$ is not a 0-nghb in $G_{(c,d_n)}$, for every m and n.

Proof. a) Let e be the middle point of $[a,b]$. Then, $G=G^{(a,b)}=G^{(a,e)}\times G^{(e,b)}$, the first equality due to (3.4'). By Prop.3.2b), one of the following conditions holds: for each m, $D_m\cap G^{(a,e)}$ is not a 0-nghb in $G^{(a,e)}$, or, for each m, $D_m\cap G^{(e,b)}$ is not a 0-nghb in $G^{(e,b)}$. By induction and subsequent subdivision of each interval in two equal parts. one gets a decreasing sequence of intervals $[a_n,b_n]$ contained in $[a,b]$, whose intersection reduces to a single point (call it c), such that $D_m\cap G^{(a_n,b_n)}$ is not a 0-nghb in $G^{(a_n,b_n)}$ for any m and n. There are 3 possible cases:

i) there is some p for which $a_p=c$. Then, $c=a_n$ for all $n\geqslant p$ and $D_m\cap G^{(c,b_n)}$ is not a 0-nghb in $G^{(c,b_n)}$ for every m and for every $n\geqslant p$. It is enough now to relabel the points b_n.

ii) there is some p for which $b_p=c$. Then, $c=b_n$ for every $n\geqslant p$ and $D_m\cap G^{(a_n,c)}$ is not a 0-nghb in $G^{(a_n,c)}$ for every m and for every $n\geqslant p$.

iii) $a_n<c<b_n$ for every n. Let J_1 be the set of integers n such that $D_m\cap G^{(a_n,c)}$ is not a 0-nghb in $G^{(a_n,c)}$, for any m, and let J_2 be the set of n such that $D_m\cap G^{(c,b_n)}$ is not a 0-nghb in $G^{(c,b_n)}$ for any m. Since $G^{(a_n,b_n)}=G^{(a_n,c)}\times G^{(c,b_n)}$, it follows from Prop.3.2b) that $J_1\cup J_2=\mathbb{N}$, hence at least one of the sets J_1, J_2 is infinite, and it is enough to relabel the indices of that infinite set, throwing out indices which give the same point.

b) Since $G_{\{\lambda\}}=G_{\{\lambda-\}}\times G_{\{\lambda+\}}$ and $D_p\cap G_{\{\lambda\}}$ is a 0-nghb in $G_{\{\lambda\}}$ for some p, it follows that $D_p\cap G_{\{\lambda-\}}$ and $D_p\cap G_{\{\lambda+\}}$ are 0-nghb in the respective spaces, for some p. Apply part a) of this Prop., redefining $P_a=0$ and $P_b=I$, if needed. In case a)i) one may assume $d_n\neq a$ for every n and in case a)ii) one may assume $d_n\neq b$ for every n.

If a)i) holds with $c\neq b$ or a)ii) with $c\neq a$, we may see that these statements do not depend on the definitions of P_a and P_b, hence they hold for their initial definitions, too. In these cases, since $G^{(d_n,c)}=G_{\{d_n+\}}\times G_{(d_n,c)}\times G_{\{c-\}}$, and $G^{(c,d_n)}$ has a similar formula, it follows from the initial remark and two applications of Prop.3.2b), that one of the conditions b)i) or b)ii) holds.

If a)i) holds with c=b, it is enough to remark that, with the redefinition $P_b=I$, one has $G^{(d_n,b)}=G_{\{d_n+\}}\times G_{(d_n,b)}\times G_{\{b\}}$, and to apply then Prop.3.2b), to get b)i). The case in which a)ii) holds with c=a is proved similarly. ■

Proposition 3.8 Let E be a lts, G a linear subspace of E, $P_\lambda : E \to E$ continuous projections for $\lambda\in[a,b]$, satisfying (3.4), (3.6), (3.6'), (3.5) and (3.5'), and let F be a linear subspace of E such that $G\subset F$. Suppose also

$$(3.9)\left\{\begin{array}{l}\text{for every } x\in E\text{: i) if } c\in\,]a,b] \text{ and } P_\mu(x)\in G \text{ for all } \mu\in[a,c[\text{, then } P_{c-}(x)\in F\\ \text{and} \quad \text{ii) if } c\in[a,b[\text{ and } (I-P_\mu)(x)\in G \text{ for all } \mu\in\,]c,b]\text{, then } (I-P_{c+})(x)\in F.\end{array}\right.$$

If x_n is a sequence with property ℓ^∞ in E and if there exists $c\in[a,b]$ and a sequence $d_n\in[a,b]$ such that: a) $d_n\uparrow c$ and $x_n\in G_{(d_n,c)}$ for every n; or b) $d_n\downarrow c$ and $x_n\in G_{(c,d_n)}$ for every n; then $B(\ell^\infty,x_n)\subset F$.

Proof. of case a). Since G satisfies (3.5), we have $P_\mu(x_n)\in G$ for all $\mu\in[a,b]$ and all n. Let $x\in B(\ell^\infty,x_n)$; then $x=\sum_{n=1}^{\infty}\lambda_n x_n$. If $\mu\in[a,c[$, then there is p such that $\mu<d_p$ hence $\mu<d_n$ for all $n\geqslant p$, so that $P_\mu(x_n)\in P_\mu G_{(d_n,c)}=P_\mu(P_{c-}-P_{d_n+})(G)=\{0\}$ for all $n\geqslant p$, hence $P_\mu(x)=\sum_{n=1}^{p-1}\lambda_n P_\mu(x_n)\in G$. By (3.9) i), it follows that $P_{c-}(x)\in F$. Since $x_n\in G_{(d_n,c)}$, we have $x_n=(P_{c-}-P_{d_n+})(x_n)$, hence $P_{c-}(x_n)=x_n$, therefore $P_{c-}(x)=x$, so that $x\in F$.

The proof of case b) is similar. ■

Proposition 3.8' Let E, F, G and P_λ be like in last proposition, but with (3.4), (3.4'), (3.5) and

$$(3.9')\left\{\begin{array}{l}\text{for all } x\in E\text{: i) if } c\in\,]a,b] \text{ and } P_\mu(x)\in G \text{ for all } \mu\in[a,c[\text{, then } P_c(x)\in F\\ \text{and} \quad \text{ii) if } c\in[a,b[\text{ and } (I-P_\mu)(x)\in G \text{ for all } \mu\in\,]c,b]\text{, then } (I-P_c)(x)\in F.\end{array}\right.$$

If x_n is a sequence with property ℓ^∞ in E and if there exists $c\in[a,b]$ and a sequence $d_n\in[a,b]$ such that: a) $d_n\uparrow c$ and $x_n\in G^{(d_n,c)}$ for all n; or b) $d_n\downarrow c$ and $x_n\in G^{(c,d_n)}$ for all n; then $B(\ell^\infty,x_n)\subset F$.

Proof. Analogous to that of Prop.3.8. ■

Remark 3.10 Let E_α, $\alpha\in[a,b]$, be lts and $E=\prod\{E_\alpha;\ \alpha\in[a,b]\}$. If $x=(x_\alpha)\in E$ and $\lambda\in[a,b]$, we define $P_\lambda(x)=y=(y_\alpha)$, where $y_\alpha=x_\alpha$ if $\alpha\leqslant\lambda$ and $y_\alpha=0$ otherwise. It is clear then that $P_\lambda : E\to E$ is a continuous projection for every $\lambda\in[a,b]$, that (3.4), (3.6) hold, $P_b=I$, and that with the convention (3.6'), we have $P_{\lambda+}=P_\lambda$ for every $\lambda\in[a,b]$. Whenever E is the product of a family E_α, $\alpha\in[a,b]$, we shall consider the projections P_λ defined above, which we will call the *canonical* P_λ. In this case, $E_{\{\lambda\}}=P_{\{\lambda\}}(E)$ is canonically isomorphic to E_λ, for every $\lambda\in[a,b]$.

<u>Theorem 3.11</u> a) Suppose that either: 1) E is a metrizable complete lts and $P_\lambda : E\to E$, $\lambda\in[a,b]$ continuous projections satisfying (3.4), (3.6) and (3.6'); or 2) $E=\prod\{E_\alpha : \alpha\in[a,b]\}$, where the E_α are lts and the P_λ, $\lambda\in[a,b]$ are the canonical P_λ (see Remark 3.10 : (3.4), (3.6) and (3.6') automatically hold). Suppose further that G and F are linear subspaces of E, with $G\subset F$, such that G satisfies (3.5), (3.5'), (3.9) and also

(3.12) G is dense in F,

(3.13) $G_{\{\lambda\}}$ is u-BL (resp. BL, resp. barrelled) for every $\lambda\in[a,b]$.

Then, F is u-BL (resp. BL, resp. barrelled).

b) If, furthermore, $G=F$ and

(3.13') $G_{\{\lambda\}}$ is u-ultrab.BL (resp. ultrab.BL, resp. ultrab.) for all $\lambda\in[a,b]$,

then, F is u-ultrab.BL (resp. ultrab.BL, resp. ultrab.). (See also the Remark 3.25, for further information in this case).

<u>Proof</u>. a) I) Let us prove the u-BL-ness.

If F were not u-BL, there would exist a sequence A_m of closed balanced subsets of F such that $A_m+A_m\subset A_{m+1}$, which covers F, but none of the A_m would be a 0-nghb in F. By (3.12), none of the $A_m\cap G$ would be a 0-nghb in G (otherwise, the closure of $A_m\cap G$ in F, which is contained in A_m, would be a 0-nghb in F). Call $D_m=A_m\cap G$. By (3.13), for each $\lambda\in[a,b]$ there exists some p such that $D_p\cap G_{\{\lambda\}}$ is a 0-nghb in $G_{\{\lambda\}}$, hence by Prop.3.7b) there exists $c\in[a,b]$ and a sequence $d_n\in[a,b]$ such that one of the conditions holds:

$$(3.14)\begin{cases} \text{i) } d_n\uparrow c \text{ and } D_m\cap G_{(d_n,c)} \text{ is not a 0-nghb in } G_{(d_n,c)}, \text{ for all } m \text{ and } n; \text{ or} \\ \text{or} \\ \text{ii) } d_n\downarrow c \text{ and } D_m\cap G_{(c,d_n)} \text{ is not a 0-nghb in } G_{(c,d_n)}, \text{ for all } m \text{ and } n. \end{cases}$$

<u>Claim</u>: In case (3.14) i), there is a sequence $x_n\in G_{(d_n,c)}$ which has property ℓ^∞ in E, such that $B(\ell^\infty,x_n)$ is a Banach disk of E and such that $x_n\notin nA_n$ for every n. In case (3.14) ii), there is a sequence $x_n\in G_{(c,d_n)}$ which has property ℓ^∞ in E, such that $B(\ell^\infty,x_n)$ is a Banach disk of E and such that $x_n\notin nA_n$.

Then, by Prop.3.8, we have $B(\ell^\infty,x_n)\subset F$, so that $B(\ell^\infty,x_n)$ is a Banach disk of F, containing all the x_n. By Prop.2.9b), there is some p such that A_p absorbs $B(\ell^\infty,x_n)$. Since $x_n \notin nA_n$ for every n, we have $x_n \notin nA_p$ for every $n \geq p$, which is a contradiction. Hence, F is u-BL, assumed the claim.

Proof of the claim. The proof is a little different for each of the cases 1) and 2). 1): E metrizable complete. Let V_k be a fundamental system of balanced 0-nghb in E, such that $V_k+V_k \subset V_{k-1}$ for every $k \geq 2$. In the case (3.14) i), since $D_n \cap G_{(d_n,c)} = A_n \cap G_{(d_n,c)}$ is not a 0-nghb in $G_{(d_n,c)}$ for each n, there is $x_n \in V_n \cap G_{(d_n,c)}$ such that $x_n \notin nA_n$. By Prop.2.15a), the sequence x_n has property ℓ^∞ in E and $B(\ell^\infty,x_n)$ is a Banach disk of E. The case (3.14) ii) has a similar proof. 2): $E = \prod\{E_\alpha : \alpha \in [a,b]\}$. In the case (3.14) i), since $A_n \cap G_{(d_n,c)}$ is not a 0-nghb in $G_{(d_n,c)}$ for each n, there is $x_n \in G_{(d_n,c)}$ such that $x_n \notin nA_n$ for every n. Since $d_n \uparrow c$ and $x_n \in G_{(d_n,c)}$, it follows that, for every $\alpha \in [a,b]$, $J_\alpha = \{m : x_m^\alpha \neq 0\}$ is finite. By Prop.2.15b), the sequence x_n has property ℓ^∞ in E and $B(\ell^\infty,x_n)$ is a Banach disk of E. The case (3.14) ii) has a similar proof.

II) Proof of the BL-ness.

If F were not BL, there would exist a sequence C_m of closed ac subsets of F which covers F, such that $C_m \subset C_{m+1}$ for every m, but none of the C_m being a 0-nghb in F. If we take $A_m = 2^{m-1}C_m$, then $A_m+A_m = 2A_m = 2^m C_m \subset A_{m+1}$ and A_p is a 0-nghb if and only if C_p is a 0-nghb. Hence, a similar proof may be given, utilizing the corresponding condition (3.13).

III) Proof of the barrelledness. It is exactly the same as that of I), utilizing the corresponding condition of (3.13) and the remark that F is barrelled if and only if, for each barrel T of F, if we call $A_m = 2^m T$, then some A_p is a 0-nghb in F.

b) I) Let us prove the u-ultrab.BL-ness.

Let A_m be a sequence of balanced sets of F which covers F, such that $A_m+A_m \subset A_{m+1}$ for every m and, for each Banach disk B of F, there is p such that A_p absorbs B. We shall prove that there is q such that $A_q \cap G$ is a 0-nghb in G (without assuming that $G = F$), in almost the same way as in a)I).

Suppose that none of the $A_m \cap G$ were a 0-nghb in G. By (3.13'), for each $\lambda \in [a,b]$ there is some p such that $A_p \cap G_{\{\lambda\}}$ is a 0-nghb in $G_{\{\lambda\}}$ hence, by Prop.3.7b), there exists $c\in[a,b]$ and a sequence $d_n\in[a,b]$ such that one of the conditions i) or ii) of (3.14) holds. Then, the same claim of a)I) is proved in exactly the same way. Then, by Prop.3.8, $B(\ell^\infty,x_n)\subset F$ so that $B=B(\ell^\infty,x_n)$ is a Banach disk of F. By hypothesis, there is p such that A_p absorbs B, which is a contradiction, since $x_n\notin nA_p$ for every $n\geq p$.

Hence, if $G=F$, we conclude that A_q is a 0-nghb in F.

II) and III) The proofs of the ultrab.BL-ness and of ultrabornologicalness are related to that of u-ultrab.BL-ness in the same way as those of BL-ness and barrelledness are to that of u-BL-ness.

<u>Theorem 3.11'</u> Let E be a metrizable complete lts, $P_\lambda:E\to E$, $\lambda\in[a,b]$ continuous projections, G and F linear subspaces of E with $G\subset F$, satisfying (3.4), (3.4'), (3.5), (3.9') and (3.12). Then, a) F is u-BL; b) if, moreover, $G=F$, then F is u-ultrab.BL.

<u>Proof</u>. a) u-BL-ness: Analogous to that of Th.3.11, with the following differences: one applies Prop.3.7a) instead of Prop.3.7b), concluding that there is $c\in[a,b]$ and a sequence d_n such that

$$(3.14')\begin{cases}\text{i) } d_n\uparrow c \text{ and } D_m\cap G^{(d_n,c)} \text{ is not a 0-nghb in } G^{(d_n,c)} \text{ for any } m \text{ and } n\\ \text{or}\\ \text{ii) } d_n\downarrow c \text{ and } D_m\cap G^{(c,d_n)} \text{ is not a 0-nghb in } G^{(c,d_n)} \text{ for any } m \text{ and } n.\end{cases}$$

One proves then a similar claim, with x_n belonging to $G^{(d_n,c)}$ or $G^{(c,d_n)}$ instead of $G_{(d_n,c)}$ or $G_{(c,d_n)}$, in exactly the same way. Applying Prop.3.8' instead of Prop.3.8, one concludes that $B(\ell^\infty,x_n)\subset F$, and continues as before.

b) u-ultrab.BL-ness: it is analogous.

<u>Theorem 3.11''</u> Let E be a metrizable lts, P_λ, G and F be as in Th.3.11 and $\hat{E}$ the completion of E (the projections $P_\lambda:E\to E$ have then a unique continuous extension $\hat{P}_\lambda:\hat{E}\to\hat{E}$.)

a) Suppose that (3.4), (3.4'), (3.5), (3.12) and the following hold:

$$(3.9'')\begin{cases}\text{for all } x\in\hat{E}\text{: i) if } c\in\,]a,b] \text{ and } \hat{P}_\mu(x)\in G \text{ for all } \mu\in[a,c[\text{, then } \hat{P}_c(x)\in F\\ \text{and ii) if } c\in[a,b[\text{ and } (I-\hat{P}_\mu)(x)\in G \text{ for all } \mu\in\,]c,b] \text{ then } (I-\hat{P}_c)(x)\in F.\end{cases}$$

Then, F is u-BL. If, moreover, $G=F$, then F is u-ultrab.BL.

b) Suppose that (3.4), (3.5), (3.5'), (3.6), (3.6'), (3.12), (3.13) and the following condition hold:

$$(3.9''')\begin{cases}\text{for all } x\in\hat{E}\text{: i) if } c\in\,]a,b] \text{ and } \hat{P}_\mu(x)\in G \text{ for all } \mu\in[a,c[\text{, then } \hat{P}_{c-}(x)\in F\\ \text{and ii) if } c\in[a,b[\text{ and } (I-\hat{P}_\mu)(x)\in G \text{ for all } \mu\in\,]c,b]\text{, then } (I-\hat{P}_{c+})(x)\in F\end{cases}$$

(where $\hat{P}_{\lambda-}$,and $\hat{P}_{\lambda+}$ denote the continuous extensions of $P_{\lambda-}$ and $P_{\lambda+}$)

Suppose also that $\{P_\lambda : \lambda\in[a,b]\}$ is equicontinuous (which, by Remark 3.3c) is automatically verified if one assumes that E is an $\mathcal{L}$-barrelled lts or a barrelled lcs).

Then, a) F is u-BL (resp. BL, resp. barrelled); b) if, moreover, $G=F$ and (3.13') holds, then F is u-ultrab.BL (resp. ultrab.BL, resp. ultrab.).

<u>Proof</u>. a) From (3.4) one gets immediately $\hat{P}_\lambda\hat{P}_\mu=\hat{P}_\mu\hat{P}_\lambda=\hat{P}_{\min\{\lambda,\mu\}}$ for all $\lambda,\mu\in[a,b]$, which is (3.4) for $\hat{E}$ and $\hat{P}$; (3.9") is (3.9') for $\hat{E}$ and $\hat{P}$; (3.5) and (3.12) are the same for E and P, and for $\hat{E}$ and $\hat{P}$. Apply then Th.3.11' to $\hat{E}$ and $\hat{P}$.

b) As in a), from (3.4) one gets the corresponding (3.4) for $\hat{E}$ and $\hat{P}$. Since the set of all P_λ is equicontinuous, the same happens to the set of all $\hat{P}_\lambda$, and since E is dense in $\hat{E}$, it follows from (3.6) that the corresponding (3.6) for $\hat{E}$ and $\hat{P}$ also holds. It is clear that (3.5), (3.5'), (3.12), (3.13) and (3.13') for E and P are the same for $\hat{E}$ and $\hat{P}$, and that (3.9''') is (3.9) for $\hat{E}$ and $\hat{P}$. Apply then Th.3.11 to $\hat{E}$ and $\hat{P}$. ∎

<u>Remark 3.15</u> If F is a linear subspace of a lts E such that F is ultrabornological and dense in E, it does not follow in general that E is ultrabornological (compare with Remark 1.0; see also Prop.5.7, under another axiomatic). Suppose that Th.3.11b) were true without the hypothesis that $G=F$. Since, if conditions (3.4),(3.5),(3.5'),(3.6),(3.6'),(3.9),(3.12), (3.13') are true for G,F and E, they are also true for G,$\tilde{F}$and E, provided $F\subset\tilde{F}$ and F is dense in $\tilde{F}$, it would follow that for every such $\tilde{F}$, $\tilde{F}$ would be ultrabornological, which is a rather strong property, without appearance of being possible in general. This explains why the proof of Th.3.11b) didn't work without the hypothesis $G=F$. However, one can circumvent such a nuisance describing precisely what kind of space F one will deal with. What we do in the sequel is to describe the smallest linear subspace of E which contains

G and satisfies (3.9).

Definition 3.16 Let E be a lts, G a linear subspace of E and $P_\lambda : E \to E$, $\lambda \in [a,b]$ continuous linear mappings satisfying (3.4), (3.5), (3.5'), (3.6) and (3.6'). If $x \in E$ and $a \leq c < d \leq b$, we say that x is of *type G_1* in $[c,d]$ if $(P_\mu - P_c)(x) \in G$ for all $\mu \in [c,d[$, and of *type G_2* in $[c,d]$ if $(P_d - P_\mu)(x) \in G$ for all $\mu \in]c,d]$. We denote by F(G) the *$\mathcal{P}$-closure* of G in E (where $\mathcal{P}$ is the set of all P_λ), which is defined to be the set of all $x \in E$ for which there is some division $a = t_0 < t_1 \dots < t_n = b$ of $[a,b]$ (depending on x) such that $P_{\{t_j\}}(x) \in G$ for $j = 0, 1, \dots, n$, and such that in each interval $[t_{j-1}, t_j]$, $j = 1, \dots, n$, x is of type G_1 or of type G_2 (the type may vary from one interval to the other).

Proposition 3.17 Assume the conditions of Def.3.16 hold, and that F(G) is the $\mathcal{P}$-closure of G in E. Then:

i) F(G) is a linear subspace of E, such that $G \subset F(G)$

ii) G is dense in F(G)

iii) F(G) satisfies (3.9)

iv) F(G) is the smallest linear subspace of E containing G, for which (3.9) holds.

Proof (sketch of)i)In order to show that $x+y \in F(G)$ when x and y belong to F(G), one should first verify that: a) if x is of type G_1 or G_2 in $[c,d]$, then it is of the same type in every subinterval of $[c,d]$; b) if $x \in F(G)$, then $P_{\{\lambda\}}(x) \in G$ for all $\lambda \in [a,b]$; c) if x and y are of the same type in $[c,d]$, then x+y is of that same type in $[c,d]$; d) if x and y are of different types in $[c,d]$ and $c < e < d$, then x and y are of the same type in $[c,e]$ and of the same type (the other one) in $[e,d]$.

ii) If $x \in F(G)$ and $a = t_0 < \dots < t_n = b$ is some associated division, then by Prop.3.1c) one has

$$(3.18) \quad x = \sum_{j=0}^{n-1} (P_{t_{j+1}^-} - P_{t_j^+})(x) + \sum_{j=0}^{n} P_{\{t_j\}}(x) \, ,$$

where the second sum belongs to G. If x is of type G_1 in $[t_j, t_{j+1}]$, one shows first that $(P_\mu - P_{t_j^+})(x) \in G$, for all $\mu \in]t_j, t_{j+1}[$. Since $P_{t_{j+1}^-}(x) = \lim_{\mu \to t_{j+1}^-} P_\mu(x)$, it follows that $(P_{t_{j+1}^-} - P_{t_j^+})(x) \in \overline{G}$, and the first sum belongs to $\overline{G}$.

iii) If $x \in E$ and $P_\mu(x) \in G$ for all $\mu < c$, consider the division $a < c < b$.

Since $(P_\mu-P_a)P_{c-}(x)=P_\mu(x)-P_a(x)\in G$ for all $\mu<c$, $P_{c-}(x)$ is of type G_1 in $[a,c]$. Since $(P_b-P_\mu)P_{c-}(x)=0\in G$ for all $\mu>c$, $P_{c-}(x)$ is of type G_2 in $[c,b]$. Moreover, $P_{\{c\}}P_{c-}(x)=P_{\{b\}}P_{c-}(x)=0\in G$ and, if one takes some $\mu\in]a,c[$, we have $P_{\{a\}}P_{c-}(x)=P_{a+}(x)=P_{a+}(P_\mu(x))\in G$, hence $P_{c-}(x)\in F(G)$.

iv) Let F be a linear subspace of E containing G for which (3.9) holds, and let $x\in F(G)$. One has to show that $x\in F$. By (3.18), it is enough to show that $y=(P_{t_{j+1}-}-P_{t_j+})(x)\in F$. Suppose x is of type G_1 in $[t_j,t_{j+1}]$. We have $P_\mu(y)=0\in G$ if $\mu\leq t_j$ and $P_\mu(y)=(P_\mu-P_{t_j+})(x)\in G$ if $t_j<\mu<t_{j+1}$. Since F satisfies (3.9) we have $P_{t_{j+1}-}(y)\in F$. But $P_{t_{j+1}-}(y)=y$, hence $y\in F$. ■

<u>Lemma 3.19</u> Let τ_1 and τ_2 be two linear Hausdorff topologies on a vector space F, with $\tau_1\subset\tau_2$, G a linear subspace of F, such that G is dense in (F,τ_2), and that the restrictions of τ_1 and τ_2 to G coincide. Then, $\tau_1=\tau_2$.

<u>Proof</u>. Let us call $F_1=(F,\tau_1)$ and $F_2=(F,\tau_2)$. Since G is dense in F_2, it is also dense in F_1, hence the canonical inclusion $i_1:F_1\supset G\rightarrow F_2$, which is continuous by hypothesis, has a (unique) continuous extension $v_1:F_1\rightarrow\hat{F}_2$. Consider also the canonical mappings $i_2:F_2\rightarrow F_1$ and $i:F_2\rightarrow\hat{F}_2$, which are continuous. Since $v_1\circ i_2$ and i are both linear continuous mappings from F_2 into $\hat{F}_2$ which leave each element of G fixed and G is dense in F_2, we must have $v_1\circ i_2=i$. Since i_2 is a bijective mapping and i the canonical inclusion, we conclude that the image of v_1 is F_2. Then, calling $w:F_1\rightarrow F_2$ the mapping defined by $w(x)=v_1(x)$ for all $x\in F_1$, one has $w=i_2^{-1}$, therefore i_2^{-1} is continuous and i_2 is a topological isomorphism. ■

<u>Corollary 3.20</u> Let F be a <u>lcs</u>, F^u the ultrabornological space associated to F and G a linear subspace of F. If F and F^u induce the same topology on G and G is dense in F^u, then $F=F^u$, hence F is ultrabornological.

<u>Theorem 3.21</u> Let E be a lcs, $P_\lambda:E\rightarrow E$, $\lambda\in[a,b]$, linear continuous mappings, G a linear subspace of E satisfying (3.4), (3.5), (3.5'), (3.6), (3.6'), $F(G)$ the $\mathcal{P}$-closure of G in E, and suppose that

(3.13") $G_{\{\lambda\}}$ is ultrabornological for every $\lambda\in[a,b]$.

Then, $F(G)$ is ultrabornological in each of the following cases: 1) E is metrizable complete (and lcs); 2) $E=\prod\{E_\alpha:\alpha\in[a,b]\}$, where the E_α are les and the P_λ are the canonical P_λ.

Proof. Remark first that, by Prop.3.17, $F(G)$ satisfies (3.9) and (3.12). Let $F(G)^u$ be the ultrab. lcs associated to $F(G)$ and V an ac 0-nghb in $F(G)^u$, that is, V is an ac set which absorbs all Banach disks of $F(G)$. Since all the hypothesis of Th.3.11b) hold, with the exception of the condition $G=F(G)$, one concludes from the initial part of the proof of Th.3.11b) (which doesn't need such hypothesis), that $V\cap G$ is a 0-nghb in G (with the topology induced by $F(G)$). This means that $F(G)$ and $F(G)^u$ induce the same topology on G. By Cor.3.20, it remains only to prove that G is dense in $F(G)^u$ (we only know, by Prop.3.17, that G is dense in $F(G)$).

Let $x\in F(G)$ and $a=t_0<\ldots<t_n=b$ an associated division. By (3.18), it is enough to show that for each j, the element $y=(P_{t_{j+1}-}-P_{t_j+})(x)$ belongs to the closure of G in $F(G)^u$. We will prove that only in the case x is of type G_1 in $[t_j,t_{j+1}]$, leaving to the reader the case of type G_2. Call for simplicity $c=t_j$, $d=t_{j+1}$, so that $y=(P_{d-}-P_{c+})(x)$. Since x is of type G_1 in $[c,d]$, we have $(P_\mu-P_c)(x)\in G$ for all $\mu\in]c,d[$, and using the fact $P_{\{c\}}(x)\in G$ one can show also that $y_\mu=(P_\mu-P_{c+})(x)$ belongs to G, hence $y=\lim\limits_{\mu\to d-} y_\mu$, with $y_\mu\in G$ for $\mu\in]c,d[$.

Claim: There is $w\in G$, a sequence $a_n\in]c,d[$ such that $a_n\uparrow d$ and a sequence $x_n\in G_{(a_n,d)}$ such that: i) $y=w+\sum\limits_{n=1}^{\infty}x_n$, ii) the sequence $z_n=2^n x_n$ has property ℓ^∞ in E, and iii) $B(\ell^\infty,z_n)$ is a Banach disk of E.

Then, by Prop.3.8 we have $B=B(\ell^\infty,z_n)\subset F(G)$, hence $F(G)_B$ is a Banach space, whose norm will be denoted by $\|\ \|$. Since $z_n\in B$, we have $\|z_n\|\leqslant 1$ for all n, hence $\|x_n\|\leqslant 2^{-n}$, so that the series $\sum\limits_{n=1}^{\infty}x_n$ is convergent to some element θ in $F(G)_B$. Since the inclusion $i:F(G)_B\to F(G)^u$ is continuous, it follows that such series converges to θ also in $F(G)^u$, and a fortiori in $F(G)$, so by condition i) of the claim, $\theta=y-w$. Since each partial sum of that series belongs to G, this shows that $y-w$ belongs to the closure of G in $F(G)^u$, hence also y belongs to that closure (because $w\in G$).

Proof of the claim. 1) E metrizable complete (and lcs). Let V_k be a basis of closed ac 0-nghb in E, with $2V_k\subset V_{k-1}$ for all $k\geqslant 2$, and call

$W_n = 2^{-n}V_n$; then, W_n is also a basis of 0-nghb in E, and $4W_n = 2^{-n+2}V_n \subset W_{n-1}$ for $n \geq 2$. Since $y = \lim_{\mu \to d-} y_\mu$, there exists a sequence $d_n \in]c,d[$ such that $d_n \uparrow d$ and $y_\mu - y_{d_n} \in W_n$ for all $\mu \in [d_n, d[$. Take $x_n = y_{d_{n+1}} - y_{d_n}$ and $z_n = 2^n x_n$; then, $x_n \in W_n \cap G$ and $z_n \in 2^n W_n - V_n$. By Prop. 2.15a), z_n has property ℓ^∞ in E and $B(\ell^\infty, z_n)$ is a Banach disk of E. Now, $\sum_{n=1}^{k} x_n = \sum_{n=1}^{k} (y_{d_{n+1}} - y_{d_n}) = y_{d_{k+1}} - y_{d_1}$, hence the series $\sum_{n=1}^{\infty} x_n$ is convergent (in E) to $(\lim_{\mu \to d-} y_\mu) - y_{d_1} = y - y_{d_1}$, so if we take $w = y_{d_1}$, we will have $w \in G$ and the item 1) of the claim. On the other hand, $x_n = y_{d_{n+1}} - y_{d_n} = (P_{d_{n+1}} - P_{c+})(x) - (P_{d_n} - P_{c+})(x) = (P_{d_{n+1}} - P_{d_n})(x)$, hence if we take $a_n \in]d_{n-1}, d_n[$ for $n \geq 2$ and $a_1 \in]c, d_1[$, then $a_n \uparrow d$ and $(P_{d-} - P_{a_n+})(x_n) = (P_{d-} - P_{a_n+})(P_{d_{n+1}} - P_{d_n})(x) = (P_{d_{n+1}} - P_{d_n})(x) = x_n$, hence $x_n \in G_{(a_n, d)}$.

2) $E = \prod\{E_\alpha : \alpha \in [a,b]\}$, ($E_\alpha$ being lcs). Take $d_n \in]c,d[$ such that $d_n \uparrow d$, and put $x_n = y_{d_{n+1}} - y_{d_n}$ and $w = y_{d_1} \in G$. As in case 1) one shows that the item 1) of the claim holds and that, if we take $a_n \in]d_{n-1}, d_n[$ for $n \geq 2$ and $a_1 \in]c, d_1[$, then $a_n \uparrow d$ and $x_n \in G_{(a_n, d)}$. Call now $z_n = 2^n x_n$. Then $z_n \in G_{(a_n, d)}$ and, since $a_n \uparrow d$, we have $J_\alpha = \{n \in N : z_n^\alpha \neq 0\}$ finite for every $\alpha \in [a,b]$. By Prop. 2.15b), z_n has property ℓ^∞ in E and $B(\ell^\infty, z_n)$ is a Banach disk of E. ∎

Remark-Definition 3.22 Let E and G be as in Def.3.16. Then, it is easy to verify that $P_\lambda(F(G)) \subset F(G)$, $P_{\lambda-}(F(G)) \subset F(G)$ and $P_{\lambda+}(F(G)) \subset F(G)$ for all $\lambda \in [a,b]$. Moreover, $F(G)_{\{\lambda\}} = G_{\{\lambda\}}$, for all $\lambda \in [a,b]$. Hence, one may consider also $F(F(G))$ (the $\mathcal{P}$-closure of F(G) in E), which is in general different from F(G). More generally, if we denote $F^1(G) = F(G)$ and α is any ordinal number, we may denote $F^\alpha(G) = F(F^{\alpha-1}(G))$ if α has antecedent $\alpha-1$, and $F^\alpha(G) = F(\bigcup_{\beta<\alpha} F^\beta(G))$, if α has no antecedent. Of course, there exists the first ordinal γ such that $F^{\gamma+1}(G) = F^\gamma(G)$; this set we will call the total $\mathcal{P}$-closure of G in E. One has $(F^\alpha(G))_{\{\lambda\}} = G_{\{\lambda\}}$ for every $\lambda \in [a,b]$ and every ordinal α.

Corollary 3.23 If all the hypothesis of Th.3.21 are satisfied, then $F^\alpha(G)$ is ultrabornological for every ordinal number α. In particular, the total $\mathcal{P}$-closure of G in E is ultrabornological (and also, by Th.3.11, u-ultrab. BL, provided $G_{\{\lambda\}}$ are u-ultrab.BL, even when $E = \prod E_\alpha$).

We give now a definition and some results only for reference (without proof) which are analogous to Th.3.21 (or, rather, to Cor.3.23) when condition

(3.9') is considered, instead of (3.9).

Definition 3.16' Let E be a lts, G a linear subspace of E and $P_\lambda : E \rightarrow E$, $\lambda \in [a,b]$, continuous projections satisfying (3.4), (3.4') and (3.5). We denote by $F((G))$ the $\mathcal{P}'$-closure of G in E, which is defined to be the set of all $x \in E$ for which there is some division $a=t_0<\ldots<t_n=b$ such that in each interval $[t_{j-1},t_j]$, $j=1,\ldots,n$, x is of type G_1 or of type G_2. By (3.5) it follows that $P_\lambda(\overline{G}) \subset \overline{G}$ for all $\lambda \in [a,b]$, hence (3.4), (3.4') and (3.5) also holds for G instead of E. Hence we may also consider the $\mathcal{P}'$-closure of G in $\overline{G}$, which will be denoted by $F[G]$.

Proposition 3.17' Assume the conditions of Def.3.16' hold. Then:

i) $F((G))$ is a linear subspace of E such that $G \subset F((G))$

iii) $F((G))$ satisfies (3.9')

iv) $F((G))$ is the smallest linear subspace of E containing G, satisfying (3.9')

In general, however, G is not dense in $F((G))$, but $F[G]$ satisfies i), iii), iv) with E replaced by $\overline{G}$, and of course also:

ii) G is dense in $F[G]$.

As in (3.22), one may define $F^{\alpha}((G))$ and $F^{\alpha}[G]$ for every ordinal number α, as well as the total $\mathcal{P}'$-closures of G in E and of G in $\overline{G}$.

It doesn't seem to be possible to get an adaptation of Th.3.21 in order to conclude that $F[G]$ is ultrabornological when E is a metrizable complete lcs and G a subspace satisfying (3.4), (3.4') and (3.5). However, as a simple consequence of Th.3.11' one gets:

Corollary 3.23' Let E be a metrizable complete lts, G a linear subspace, $P_\lambda : E \rightarrow E$ linear continuous, satisfying (3.4), (3.4') and (3.5). Then, the total $\mathcal{P}'$-closures of G in E and in $\overline{G}$ are u-ultrab.BL.

Remark 3.24 When E is only a metrizable lcs, satisfying all the other conditions of Th.3.21 and also with $\{P_\lambda : \lambda \in [a,b]\}$ equicontinuous, then we may conclude that the $\mathcal{P}$-closure of G in $\hat{E}$ is ultrabornological, as well as the iterated α-$\mathcal{P}$-closure and total $\mathcal{P}$-closure of G in $\hat{E}$.(which, in general, may be not contained in E). A similar remark, corresponding to Cor.3.23' when E is not complete, may also be formulated.

Remark 3.25 In all this remark, we assume that the hypothesis of Th.3.11b) hold (with G=F, and $G_{\{\lambda\}}$ ultrabornological for every $\lambda\in[a,b]$).

a) By Th.2.11, if F is an ultrab. lts and V is an ac subset of F, then a_1), a_2) and a_3) are equivalent, that is: V is a 0-nghb in F $\Longleftrightarrow$ V absorbs all Banach disks of F $\Longleftrightarrow$ V absorbs all ac compact subsets of F $\Longleftrightarrow$ V absorbs all null fast convergent sequences of F. Assuming the hypothesis of Th.3.11b), let us say that a sequence x_n is null $\mathcal{P}$-fast convergent if there is $c\in[a,b]$ and either a sequence $d_n\uparrow c$ with $x_n\in F_{(d_n,c)}$ or $d_n\downarrow c$ with $x_n\in F_{(c,d_n)}$ and if, moreover, in the case E is metrizable complete and is given a basis of 0-nghb V_n in E such that $V_n+V_n\subset V_{n-1}$ for all $n\geq 2$, one also has $x_n\in V_n$ for all n. Of course, every null $\mathcal{P}$-fast convergent sequence is null fast convergent. If one looks carefully at the proof of Th.3.11b) in the ultrab. case, one sees that, if V is an ac subset of F, then V is a 0-nghb in F $\Longleftrightarrow$ V absorbs all null $\mathcal{P}$-fast convergent sequences.

b) As in the proof of the equivalence b_1)$\Leftrightarrow$$b_2$)$\Leftrightarrow$$b_3$) of Th.2.11, one may also see that, if H is a lcs and $u:F\to H$ is a linear mapping, then u is continuous $\Longleftrightarrow$ u maps null $\mathcal{P}$-fast convergent sequences in bounded sequences.

c) If F is an ultrab. lcs, then F has the final limit topology given by the linear mappings $i_B:F_B\to F$, where we may choose B to run through either of the sets $\mathcal{B}_1$, $\mathcal{B}_2$ or $\mathcal{B}_3$, $\mathcal{B}_1$ being the set of all Banach disks of F, $\mathcal{B}_2$ the set of all ac compact subsets of F and $\mathcal{B}_3$ the set of ac closed hulls of all null fast convergent sequences of F. If besides the hypothesis of Th.3.11b) one also assumes that F is a lcs, then the same happens when B runs through $\mathcal{B}_4$, the set of ac closed hulls of all null $\mathcal{P}$-fast convergent sequences of F.

As a final observation, which may be useful in some applications, we have:

Proposition 3.26 Let E be a lcs, $P_\lambda:E\to E$ continuous projections satisfying (3.4), with convention (3.6'). Let A,B,C be subsets of $[a,b]$, and let E_1 be the set of all $x\in E$ such that: i) there exists $\lim\limits_{\mu\to\lambda^-} P_\mu(x)$ for all $\lambda\in A\cup C$ (where we convention the existence and equality to 0 of such limit if $\lambda=a$), ii) there exists $\lim\limits_{\mu\to\lambda+} P_\mu(x)$ for all $\lambda\in B\cup C$ (with a similar remark), and iii) $\lim\limits_{\mu\to\lambda-} P_\mu(x) = \lim\limits_{\mu\to\lambda+} P_\mu(x) = P_\lambda(x)$ for all $\lambda\in C$. Then, a) E_1 is a linear subspace of E, such that $P_\lambda(E_1)\subset E_1$ for all $\lambda\in[a,b]$; b) in the case E is sequentially complete and the set of all P_λ is equicontinuous, then E_1 is a closed subspace of E.

Proof. Straightforward, but boring. ∎

Questions 3.27 a) Under the hypothesis of Th.3.11a) (or of Th.3.11b) if needed), with the convenient (3.13) or (3.13'), can we conclude that F is ℓ-barrelled? *-BL? SB? TB? UBL?

b) Same as a), but for Th.3.11'.

c) Under the hypothesis of Th.3.21, with the convenient (3.13"), can one conclude that F(G) is ultrab.BL? u-ultrab.BL? With the same (3.13") but without assuming that E is a lcs, can one conclude that F(G) is ultrabornological? (the proof of Th.3.21 does not seem to be adaptable to these cases).

d) Under the hypothesis of Cor.3.23', can one conclude that F[G] (or F((G))) is ultrab.? at least if E is assumed to be lcs?

4. EXAMPLES OF APPLICATION OF THE RESULTS OF §3

1) E metrizable complete

A) Subspaces of $\ell^\infty([a,b],X)$

Let X be a Banach space and $E=\ell^\infty([a,b],X)$ the space of functions $f:[a,b]\to X$ which are bounded, with $\|f\|=\sup\{\|f(t)\|: t\in[a,b]\}$, and define $P_\lambda: E\to E$ by $(P_\lambda f)(t)=f(t)$ if $t\in[a,\lambda]$, $(P_\lambda f)(t)=f(\lambda)$ if $t\in[\lambda,b]$, for each $\lambda\in[a,b]$. Then, E is a Banach space and $\{P_\lambda: \lambda\in[a,b]\}$ is an equicontinuous (because all with norm 1) set of linear projections, which satisfies (3.4), and $P_b=I$.

If we make the convention (3.6'), then the set E_1 of the elements f of E for which there exists $\lim\limits_{\mu\to\lambda-} P_\mu f$ and $\lim\limits_{\mu\to\lambda+} P_\mu f$ for all $\lambda\in[a,b]$, which by Prop.3.26 is a closed subspace of E, is the space $G([a,b],X)$ of regulated functions defined on $[a,b]$ with values in X, while the set E_2 of the elements f of E_1 for which the above limits are equal to $P_\lambda f$ if $\lambda\in]a,b]$ and such that $\lim\limits_{\mu\to a+} P_\mu f = P_a f$ is simply $C([a,b],X)$, and E_3, the set of the elements of E_1 for which the above limits are equal to $P_\lambda f$ for all $\lambda\in[a,b]$ is $C_o(\ a,b\ ,X)$, the space of the continuous functions which vanish at the point a.

Remark that, if $f\in E_1$, then $(P_{\lambda-}f)(t)=f(t)$ if $t\in[a,\lambda[$, $(P_{\lambda-}f)(t)=f(\lambda-)=\lim\limits_{t\to\lambda-} f(t)$ if $t\in[\lambda,b]$, for every $\lambda\in]a,b]$, while $(P_{\lambda+}f)(t)=f(t)$ if $t\in[a,\lambda]$, $(P_{\lambda+}f)(t)=f(\lambda+)=\lim\limits_{t\to\lambda+} f(t)$ if $t\in]\lambda,b]$, for every $\lambda\in[a,b]$. Hence, $P_{\{\lambda\}}(E_1)$ is isomorphic to $X\times X$ for $\lambda\in[a,b[$ and to X for $\lambda=b$, while $P_{\{\lambda\}}(E_2)$ is isomorphic to $\{0\}$ for $\lambda\in]a,b]$ and to X for $\lambda=a$, and $P_{\{\lambda\}}(E_3)$ is isomorphic to $\{0\}$ for $\lambda\in[a,b]$. Of course, E_1, E_2 and E_3 satisfies (3.6).

The only results applicable to E are Th.3.11' and Cor.3.23', while to E_1, E_2 and E_3 the Th.3.11, Th.3.21 and Cor.3.23 are also available. Moreover, by the remark above, if X is finite-dimensional and G is any linear subspace of E_i $(i=1,2,3)$ satisfying (3.5) and (3.5'), then $G_{\{\lambda\}}\subset P_{\{\lambda\}}(E_i)$, hence G automatically satisfies (3.13), (3.13') and (3.13"). The same happens without restriction on X, if G is a subspace of E_3. We state below the formulation of Th.3.11 and Th.3.21 for these spaces.

<u>Theorem 4.1</u> Let H be one of the spaces $E_2=C([a,b],X)$ or $E_3=C_o([a,b],X)$, G and F linear subspaces of H such that $G\subset F$, G is dense in F and:

(3.5) $P_\lambda(G)\subset G$ for all $\lambda\in[a,b]$

(3.9) $\begin{cases}\text{for all } f\in H\text{: i) if } c\in]a,b] \text{ and } P_\mu f\in G \text{ for all } \mu\in[a,c[\text{, then } P_c f\in F\\ \text{and ii) if } c\in[a,b[\text{ and } (I-P_\mu)f\in G \text{ for all } \mu\in]c,b]\text{, then } (I-P_c)f\in F\end{cases}$

Then, F is u-BL in each of the following cases: a) $H=E_3$; b) $H=E_2$ and X is finite-dimensional; c) $H=E_2$ and $\{f(a):f\in G\}$ is a u-BL subspace of X. If G=F, then F is u-ultrab.BL in cases a), b) as before, and c') $H=E_2$ and $\{f(a):f\in G\}$ is a u-ultrab.BL subspace of X. (similar results hold for BL-ness, barrelledness and for ultrab.BL-ness and ultrabornologicalness, when $H=E_2$ and X is infinite-dimensional, with the appropriate requirement on $\{f(a):f\in G\}$).

<u>Theorem 4.2</u> Let $H=C([a,b],X)$, G and F linear subspaces of H such that $G\subset F$, G is dense in F and:

(3.5) and (3.5') $P_\lambda(G)\subset G$, $P_{\lambda-}(G)\subset G$, $P_{\lambda+}(G)\subset G$ for all $\lambda\in[a,b]$

(3.9) $\begin{cases} \text{for all } f\in H\text{: i) if } c\in]a,b] \text{ and } P_\mu f\in G \text{ for all } \mu\in[a,c[\text{, then } P_{c-}f\in F \\ \text{and ii) if } c\in[a,b[\text{ and } (I-P_\mu)f\in G \text{ for all } \mu\in]c,b]\text{, then } (I-P_{c+})f\in F \end{cases}$

(3.13) $\{f(\lambda+)-f(\lambda)\colon f\in G\}$ and $\{f(\lambda)-f(\lambda-)\colon f\in G\}$ are u-BL (resp. BL, resp. barrelled) subspaces of X for all $\lambda\in[a,b]$.

Then, F is u-BL (resp. BL, resp. barrelled). Moreover, if G=F and

(3.13') the subspaces of X mentioned in (3.13) are u-ultrab.BL (resp. ultrab. BL, resp. ultrab.) for all $\lambda\in[a,b]$.

then, F is u-ultrab.BL (resp. ultrab.BL, resp. ultrab.).

Theorem 4.3 Let H be one of the spaces E_2 or E_3 (as in Th.4.1), G a linear subspace of H, F(G) the $\mathcal{P}$-closure of G in H (see Def.3.16) and suppose that:

(3.5) $P_\lambda(G)\subset G$ for all $\lambda\in[a,b]$.

Then, F(G) is ultrabornological in each of the following cases: a) $H=E_3$, b) $H=E_2$ and X is finite dimensional, c) $H=E_2$ and $\{f(a)\colon f\in G\}$ is an ultrabornological subspace of X.

Theorem 4.4 Let $H=G([a,b],X)$, G a linear subspace of H, F(G) the $\mathcal{P}$-closure of G in H and suppose that

(3.5) and (3.5') $P_\lambda(G)\subset G$, $P_{\lambda-}(G)\subset G$, $P_{\lambda+}(G)\subset G$ for all $\lambda\in[a,b]$

(3.13") $\{f(\lambda+)-f(\lambda)\colon f\in G\}$ and $\{f(\lambda)-f(\lambda-)\colon f\in G\}$ are ultrabornological subspaces of X for all $\lambda\in[a,b]$.

Then, F(G) is ultrabornological.

We will give now a few concrete examples of ultrabornological subspaces of $C([a,b],X)$. The reader may provide some which are subspaces of $G([a,b],X)$ but not of $C([a,b],X)$.

Example 4.5 a) As immediate consequences of Th.4.1, the following spaces are u-ultrab.BL, where $H=C([a,b],X)$ and the sets mentioned in each case depend on the function f:

$F_1=\{f\in H\colon$ f has derivative except at most on a countable set$\}$

$F_2=\{f\in H\colon$ f has deriv. except at most on a set of null measure$\}$

$F_3=\{f\in H\colon$ f has deriv. exc. at most on a set contained in a closed count. set$\}$

$F_4=\{f\in H\colon$ f has deriv. of all orders except on a set like those of $F_3\}$

$F_5=\{f\in F_4\colon f''(t)=0$ on the points where f has deriv. of all orders$\}$
(recall that, if $f''=0$ on an open interval J, then f is linear on J)

b) As immediate consequences of Th.4.3 (and Th.4.1), the following subspaces of H-$C([a,b],X)$ are ultrabornological (and u-BL):

$F_6 = \left\{ f \in H : \text{f has deriv. exc. at most on a set with a finite number of accumulation points} \right\}$
(because F_6 is the β-closure of $G_6 = \left\{ f \in H : \text{f has der. exc. at most on a finite set} \right\}$)

$F_7 = \left\{ f \in H : \text{f has der. of all orders exc. on a set like those of } F_6 \right\}$
(because F_7 is the β-closure of $G_7 = \left\{ f \in H : \text{f has der. of all orders exc. at most on a finite set} \right\}$)

$F_8 = \left\{ f \in F_7 : f''(t) = 0 \text{ on the points where f has deriv. of all orders} \right\}$
(because F_8 is the β-closure of $G_8 = \left\{ f \in H : \text{f is piecewise linear} \right\}$).

It is interesting to compare these examples with X_3 and X_4 of Cor.1.4, which are not even barrelled.

The next kind of examples is related to integrable functions.

Suppose that for each pair (a,b) of real numbers with $a \leq b$, $\mathcal{M}([a,b])$ is a vector space of X-valued functions defined on $[a,b]$ and $I_{(a,b)} : \mathcal{M}([a,b]) \longrightarrow X$ is a linear mapping ($I_{(a,b)}(f)$ will be called the <u>integral</u> of f on $[a,b]$ and also denoted by $\int_a^b f$, while the elements f of $\mathcal{M}([a,b])$ will be called <u>I-integrable</u> functions on $[a,b]$; we make the convention that every function is integrable on $[a,a]$, and that $\int_a^a f = 0$), such that :

i) if $f : [a,b] \longrightarrow X$ and $c \in [a,b]$, then $f \in \mathcal{M}([a,b])$ if and only if $f|_{[a,c]} \in \mathcal{M}([a,c])$ and $f|_{[c,b]} \in \mathcal{M}([c,b])$ and in this case $\int_a^b f = \int_a^c f|_{[a,c]} + \int_c^b f|_{[c,b]}$.

ii) if $f : [a,b] \longrightarrow X$ is zero except at most on a finite set, then $f \in \mathcal{M}([a,b])$ and $\int_a^b f = 0$.

iii) if $f \in \mathcal{M}([a,b])$ and $\tilde{f}(t) = \int_a^t f|_{[a,t]}$ for all $t \in [a,b]$, then $\tilde{f} \in C_o([a,b],X)$.

iv) if $f \in \mathcal{M}([a,b])$ and $a \leq c < d \leq b$, then $\chi_{[c,d]} f \in \mathcal{M}([a,b])$.
(where $\chi_{[c,d]}$ denotes the characteristic function of $[c,d]$).

Let then (with the notation of iii)), $\mathcal{J}_{[a,b]} : \mathcal{M}([a,b]) \longrightarrow C_o([a,b],X)$ be defined by $\mathcal{J}_{[a,b]}(f) = \tilde{f}$. Of course, $\mathcal{J}_{[a,b]}$ is a linear mapping and, if $I([a,b],X)$ denotes the quotient of $\mathcal{M}([a,b])$ by $\ker \mathcal{J}_{[a,b]}$, we may consider the quotient mapping $J : I([a,b],X) \rightarrow C_o([a,b],X)$. If, for each $f \in \mathcal{M}([a,b])$ we put $|||f||| = \|\tilde{f}\|$, then $|||\ |||$ is a seminorm on $\mathcal{M}([a,b])$ and its quotient norm, still denoted by $|||\ |||$, makes $I([a,b],X)$ isometric, through J , to a linear subspace G of $C_0([a,b],X)$. We denote by $\tilde{f}$ the equivalence class determined by f.

It is clear that, if $f \in \mathcal{M}([a,b])$, then $P_\lambda(\tilde{f}) = P_\lambda(\int_{[a,b]} f) = \int_{[a,b]}(\chi_{[a,\lambda]} f)$, hence $P_\lambda(G) \subset G$ for all $\lambda \in [a,b]$.

Given such an integral I, we say that $f: [a,b] \to X$ is <u>I-integrable in the improper sense</u> (on $[a,b]$) in one of the following cases:

a) $f|_{[a,\mu]} \in \mathcal{M}([a,\mu])$ for all $\mu \in [a,b[$, and there exists $\lim\limits_{\mu \to b^-} \int_a^\mu f|_{[a,\mu]}$ (denoted $I^1_{(a,b)}(f)$);

b) $f|_{[\mu,b]} \in \mathcal{M}([\mu,b])$ for all $\mu \in]a,b]$, and there exists $\lim\limits_{\mu \to a^+} \int_\mu^b f|_{[\mu,b]}$ (also denoted $I^1_{(a,b)}(f)$).

Let then $\mathcal{M}^1([a,b])$ be the set of X-valued functions defined on $[a,b]$ such that there exists a division $a=t_0<\ldots<t_n=b$ of $[a,b]$ (depending on f) such that $f|_{[t_{i-1},t_i]}$ is I-integrable in the improper sense for $i=1,\ldots,n$, and define $I^1_{(a,b)}(f) = \sum_{i=1}^n I^1_{(t_{i-1},t_i)}(f|_{[t_{i-1},t_i]})$.

Then, $\mathcal{M}^1([a,b])$ and $I^1_{(a,b)}$, as a and b vary, satisfy the same conditions i), ii), iii), iv), being I^1 an extension of I, and the corresponding mapping $\int^1_{[a,b]}$ is an extension of $\int_{[a,b]}$. Let us call G^1 the image of $\int^1_{[a,b]}$, which is a subspace of $C_0([a,b],X)$ isometric to $I^1([a,b],X)$. It is easy to verify that G^1 is the ρ-closure of G in $C_0([a,b],X)$. We call $\mathcal{M}^1([a,b])$ the space of functions which are <u>Cauchy-I-integrable of the first order</u> on $[a,b]$. Of course, this procedure may be iterated, so that for each ordinal number α we get the space $\mathcal{M}^\alpha([a,b])$ of the functions which are <u>Cauchy-I-integrable of order α</u> on $[a,b]$, which will correspond to the ρ-closure of order α of G.

The three most important examples of integrals are the Riemann, the Bochner (which reduces to the Lebesgue, when X is R or C) and the Kurzweil (which reduces to the Denjoy-Perron, when X is R or C) integrals, with the corresponding spaces of integrable functions. All of them satisfy conditions i), ii), iii) and iv). Let $\mathcal{M}_0$ be the space of functions $f: [a,b] \to X$ which vanish outside a set of null measure. In the case of the Bochner integral one has $\mathcal{M}_0 = \mathrm{Ker} \int_{[a,b]}$, but in the case of Riemann or Kurzweil integrals and for some infinite-dimensional spaces X, the Ker $\mathcal{I}_{[a,b]}$ may be not contained in $\mathcal{M}_0$. What one always has is: $f\in \operatorname{Ker} \mathcal{I}_{[a,b]} \Leftrightarrow \int_a^t f|_{[a,t]} = 0$ for all $t\in[a,b]$. The norm $|||\ |||$ on $I([a,b],X)$ in the case of the Kurzweil integral is called the <u>Alexiewicz</u> norm.

We may then consider the Cauchy-Riemann, Cauchy-Bochner and Cauchy-Kurzweil integrable functions of order α. The class of Kurzweil integrable functions has the special property that it coincides with the class of Cauchy-Kurzweil integrable functions of order 1 (hence also of every order α).

As an immediate consequence of Th.3.11 and Th.3.21 we have:

<u>Theorem 4.6</u> a) If X is a Banach space, $\mathcal{M}([a,b])$ and $I_{(a,b)}$ defined for $a\leq b$ satisfying i), ii), iii) and iv), and α is any ordinal number ≥ 1, then the normed space $I^{\alpha}([a,b],X)$ of Cauchy-I-integrable functions of order α on $[a,b]$ is ultrabornological and u-BL. b) If, furthermore, $\mathcal{M}^1([a,b]) = \mathcal{M}([a,b])$, then $I([a,b],X)$ is even u-ultrab.BL.

<u>Example 4.7</u> a) The following spaces are u-ultrab.BL:

$F_9 = \{\dot f : f:[a,b]\to X$ is Kurzweil-integrable$\}$
(when $X=R$ or C, F_9 is the space of (classes) Denjoy-Perron int.func.)

$F_{10} = \{\dot f : f:[a,b]\to R$ is Denjoy-Khintchin integrable$\}$

b) The following spaces are ultrabornological and u-BL (for all $\alpha\geq 1$):

$F_{11}^{\alpha} = \{\dot f : f:[a,b]\to X$ is Cauchy-Riemann integrable of order $\alpha\}$

$F_{12}^{\alpha} = \{\dot f : f:[a,b]\to X$ is Cauchy-Bochner integrable of order $\alpha\}$
(when $X=R$ or C, F_{12}^{α} is the space of Cauchy-Lebesgue int.func. ord. α)

It is interesting to compare F_{12}^1 (and also F_9), in the case $X=R$ or C with the example Y_4 of Cor.1.6. The space $\{\dot f : f:[a,b]\to R$ is Lebesgue integ.$\}$ is isometric to the space of absolutely continuous functions on $[a,b]$ which vanish at the point a, which is a closed hyperplane of Y_4, hence it is not even barrelled.

<u>Remark 4.8</u> In the case of F_9 and F_{10} (or, more generally, of Th.4.6b), a sequence $\dot f_n$ will be null $\mathcal{P}$-fast convergent if and only if $|||\dot f_n||| < 2^{-n}$ for all n and if there is $c\in[a,b]$ and a sequence d_n such that either $d_n\uparrow c$ and $\dot f_n = 0$ outside $[d_n,c]$ (that is, the integral of f_n on every subinterval of $[a,b]$ which is disjoint of $]d_n,c[$ is 0) or $d_n\downarrow c$ and $\dot f_n = 0$ outside $[c,d_n]$.

Then, one may apply Remark 3.25.

<u>Remark 4.9</u> a) The example given in Remark 2.7c) of a normed lcs which is UBL but not u-BL ($(\ell^p)^1$, with $0<p<1$) is also, by Cor.2.16 and Remark 2.13a), ultrabornological.

b) In Remark 2.7e) examples were given (the projective tensor products of two infinite-dimensional Banach spaces) of non-complete normed ultrabornological spaces which are UBL but not u-BL (hence, not u-ultrab.BL).

c) Examples a) and b) of this Remark were probably the simplest known ones of non-complete ultrabornological normed spaces (but they are not u-BL). We believe that the most natural examples of such spaces are those given in Example 4.5 and 4.7, which have the further advantage of being u-BL (as a compensation, we don't know wether they are UBL).

B) Subspaces of a Hilbert space

We only mention that, if a resolution of the identity $\{E(\lambda) : \lambda\in R\}$ on a Hilbert space H is given and if one takes a homeomorphism $\varphi: R \to]0,1[$ which is increasing, then it is enough to call $P_\lambda = E(\varphi^{-1}(\lambda))$ if $\lambda\in]0,1[$, $P_0=0$ and $P_1=I$, in order to have a family $P_\lambda : H\to H$, $\lambda\in[0,1]$, of continuous projections satisfying (3.4) and (3.6), and with the convention (3.6') one has $P_{\lambda+}=P_\lambda$ for all $\lambda\in[0,1]$, and $P_{1-}=I$. Thus, the stage is ready for the application of the theorems of §3.

2) Subspaces of $E=\prod\{E_\alpha : \alpha\in[a,b]\}$, E_α lts

If $x=(x_\alpha)$, we put $\Pi_\lambda(x)=x_\lambda$, so $\Pi_\lambda : E\to E_\lambda$ is a linear continuous mapping, and supp $x = \{\alpha\in[a,b] : x_\alpha\neq 0\}$. Here, we take the canonical P_λ (see Remark 3.10) hence (3.4), (3.6) and (3.6') always hold. Moreover, $P_\lambda=P_{\lambda+}$, for all $\lambda\in[a,b]$, hence, if G satisfies (3.5) and (3.5') we have $G_{\{\lambda\}}=G_{\{\lambda-\}}$, which is topologically isomorphic to the subspace of E_λ given by $\Pi_\lambda(G)$. The statements of Th.3.11 and Th.3.21 cannot be much simplified in the general case, but it can when we assume that $\Pi_\lambda(G)=E_\lambda$ for every $\lambda\in[a,b]$. Then, G is dense in E and for $x\in E$ one has $P_\lambda(x)\in G \Leftrightarrow P_{\lambda-}(x)\in G$, therefore we get:

<u>Theorem 4.10</u> Let $E=\prod\{E_\alpha : \alpha\in[a,b]\}$, where E_α are lts, with the canonical P_λ. Let G and F be linear subspaces of E with $G\subset F$ and suppose that:

(3.5) $P_\lambda(G)\subset G$ for all $\lambda\in[a,b]$

(4.11) $\pi_\lambda(G)=E_\lambda$, for all $\lambda\in[a,b]$ (hence, by (3.5), also (3.5') holds)

$$(3.9)\left\{\begin{array}{l}\forall\, x\in E,\ \text{i) if } c\in\,]a,b] \text{ and } P_\mu(x)\in G \text{ for all } \mu\in[a,c[\text{, then } P_{c-}(x)\in F\\ \qquad\qquad \text{(or, equivalently, } P_c(x)\in F)\\ \quad\text{and}\\ \quad\text{ii) if } c\in[a,b[\text{ and } (I-P_\mu)(x)\in G \text{ for all } \mu\in\,]c,b]\text{, then } (I-P_{c+})(x)\in F\\ \qquad\quad \text{(or, equivalently, } (I-P_c)(x)\in F).\end{array}\right.$$

Then: a) F is u-BL (resp. BL, resp. barrelled), provided the E_α are u-BL (resp. BL, resp. barrelled) for all $\alpha\in[a,b]$. b) If we assume further that $G=F$, then F is u-ultrab.BL (resp. ultrab.BL, resp. ultrab.) provided the E_α are of the respective type for every $\alpha\in[a,b]$.

<u>Theorem 4.12</u> Let $E=\prod\{E_\alpha : \alpha\in[a,b]\}$, where the E_α are ultrabornological lcs, with the canonical P_λ, and G be a linear subspace of E satisfying (3.5) and (4.11). If F is the ρ-closure of G in E, then F is ultrabornological.

<u>Example 4.13</u> a) The following subspaces of $E=\prod\{E_\alpha : \alpha\in[a,b]\}$ are u-ultrab.BL (resp. ultrab.BL, resp. ultrab., resp. u-BL, resp. BL, resp. barrelled) provided every E_α is of the respective type:

$H_1=\{x\in E$: supp x is at most countable$\}$

$H_2=\{x\in E$: supp x is a set of null measure$\}$

$H_3=\{x\in E$: supp x is contained in a closed countable set$\}$

b) The following subspaces of E are ultrabornological, provided every E_α is an ultrabornological lcs:

$H_4=\{x\in E$: supp x is at most a countable set with a finite number of accumulation points$\}$
(that is, $x\in E \Leftrightarrow$ the derived set of supp x is finite)
(because H_4 is the ρ-closure of $G_4=\{x\in E$: supp x is finite$\}$ in E)

$H_5=\{x\in E$: the derived set of second-order of supp x is finite$\}$
(because H_5 is the ρ-closure of H_4)

etc

<u>Remark 4.14</u> It is well known that the space H_1 is: $\mathcal{L}$-ultrabornological, ultrab., $\mathcal{L}$-barrelled, barrelled, provided the E_α are of the respective type (these follows from Prop.2 of [5]) and that G_4 is bornological

when E_α are bornological, but G_4 is not barrelled even when $E_\alpha = \mathbb{R}$ for all α. It could be somewhat surprising that H_3, and mainly H_4, are ultrabornological (hence barrelled), due to the "smalness" of these spaces.

<u>Remark 4.15</u> In the case of Th.4.8b), a sequence x_n in F will be null ρ-fast convergent if and only if there is $c \in [a,b]$ and a sequence d_n such that either $d_n \uparrow c$ and supp $x_n \subset]d_n, c[$ or $d_n \downarrow c$ and supp $x_n \subset]c, d_n[$. Remark 3.25 can then be applied.

5. SOLOVAY'S AXIOM AND THE ULTRABORNOLOGICAL LTS

Solovay's axiom asserts that: "each subset of $\mathbb{R}$ is Lebesgue-measurable". Of course this axiom is inconsistent with the (uncountable) axiom of choice but, as shown by Solovay [16], that statement cannot be disproved in the system ZF+DC, where DC is a weakened form of the axiom of choice, which implies the countable axiom of choice. Hence, only by means of the uncountable axiom of choice it is possible to construct a non-measurable subset of $\mathbb{R}$.

Let us denote Solovay's axiom by S. The lcs in which every seminorm is continuous were called <u>good</u> by Garnir [6], and he proved that in the system ZF+DC+S every Frechet (or, even more generally, every sequentially complete bornological) space is a good space. He also remarked that, if E is a good space, F a lcs and $u: E \to F$ a linear mapping, then u is continuous, thus trivializing the closed graph theorem, since it supresses any closed graph assumption when the departure space is a good space. In particular, every linear functional on a good space is continuous. From the point of view of those who like to work with the system ZFC (that is, with the (even uncountable) axiom of choice), that result means that it is impossible only within the system ZF+DC, without appeal to the uncountable axiom of choice, to construct a discontinuous linear functional on a good space (thus, for instance, on a Frechet space). As a compensation, not every vector space has an algebraic basis in the system ZF+DC+S.

Remark that the definition of good spaces makes sense also for lts.

One of the main purposes of this § is to characterise the good spaces in the system ZF+DC+S (within the class of lts) as the ultrabornological lts. The proof is, in fact, so simple, that it is difficult to understand how Garnir could have missed it, after having done all the hard work. Thus, the ultrabornological lcs rise in importance, since not only there is a very general closed graph theorem for ultrabornological lcs as departure spaces in the system ZFC (it is enough to take the $\mathcal{C}$-webbed spaces of DeWilde, for range spaces, which is a large class with several stability properties, including every space of interest for practical analysts), but the closed graph theorem becomes trivial for them in the system ZF+DC+S.

From now on, we are assuming the system ZF+DC+S, unless the contrary is explicitly stated.

Remark 5.1 a) By the correspondence between seminorms and certain ac absorbent subsets of a vector space, it is clear that a lts is a good space if and only if every ac absorbent set is a 0-nghb. Thus, the good lcs are those which have the finest lc topology!

b) Let us be precise about the meaning of ultrabornological space: they are those which satisfy condition a_1) of Th.2.11 (they may be only lts).

c) By [6], every Banach space is a good space.

d) In §3 we never used the uncountable axiom of choice, hence those results remain valid in the system ZF+DC. Therefore, the examples of §4 are ultrabornological in the system ZF+DC, and a fortiori also in ZF+DC+S.

Proposition 5.2 A lts is good if and only if it is ultrabornological.

Proof. a) Let E be an ultrab. lts and V an ac absorbent subset of E. Let B be a Banach disk of E. Then, $i_B : E_B \to E$ is linear and $i_B^{-1}(V)=V\cap E_B$ is an ac absorbent subset of E_B. Since E_B is Banach, hence a good space (Remark 5.1c)), $V\cap E_B$ is a 0-nghb in E_B (Remark 5.1a)) hence V absorbs B. Since E is ultrab., V is a 0-nghb (Remark 5.1b)).

b) Let E be a good space and V an ac subset of E which absorbs every Banach disk. Then, V is in particular absorbent, hence (Remark 5.1a)) a 0-nghb in E. Thus, E is ultrabornological. ■

<u>Corollary 5.3</u> Let E be a lts. Then, E is ultrabornological if and only if, for every lcs F, every linear mapping $u: E \to F$ is continuous.

Since Hahn-Banach's theorem does not hold in every lcs, one has to be careful with the definition of Mackey spaces.

<u>Definition 5.4</u> a) A lcs (E,τ) is a <u>Mackey space</u> if there is not any lc topology on E strictly finer than τ, which admits exactly the same linear continuous functionals as (E,τ). b) Given a lcs (E,τ_1) and a lc topology τ_2 on E, finer than τ_1, we say that (E,τ_2) is the <u>Mackey space associated to (E,τ_1)</u> if: i) (E,τ_2) is a Mackey space; ii) (E,τ_1) and (E,τ_2) have the same continuous linear functionals.

It is not clear if every lcs admits a Mackey space associated to it. However, every ultrab. lcs (E,τ) is a Mackey space, since τ is the finest lc topology on E.

<u>Corollary 5.5</u> Let (E,τ_1) be a lcs. The following conditions are equivalent:

a) every linear functional on (E,τ_1) is continuous,

b) there exists the Mackey space (E,τ_2) associated to (E,τ_1), and (E,τ_2) is ultrabornological.

<u>Proof</u>. a)$\Rightarrow$b): By hypothesis, $(E,\tau_1)' = E^*$. Let τ_2 be the lc topology on E which has as a basis of 0-nghb the set of all ac absorbent subsets; then, (E,τ_2) is a good space, hence a Mackey space. Moreover, $E^* \supset (E,\tau_2)' \supset (E,\tau_1)' = E^*$, therefore $(E,\tau_2)' = E^* = (E,\tau_1)'$, so that (E,τ_2) is the Mackey space associated to (E,τ_1).

b)$\Rightarrow$a): is trivial, by Cor.5.3. ∎

In the next remark we point out some facts (in ZF+DC+S) which are false (hence unexpected) in ZFC. Part a)i) of it was already made in [6], for good spaces.

<u>Remark 5.6</u> a) i) Let E be an ultrab. lcs, F and G linear subspaces of E which are algebraic complements. Then, F and G are topological complements, hence closed in E, and ultrab. (as quotients of E). In fact, let $P: E \to F$ be the linear projection of E onto F with kernel G. Then, by Cor.5.3

P has to be continuous, therefore E and F are topological complements in E. ii) It follows, in particular, that every finite or infinite-countable codimensional subspace of an ultrab. lcs E is closed in E and is ultrab.; in the infinite-countable case, moreover, every algebraic complement is closed in E, is a topological complement, and is isomorphic to $K^{(N)}$ (since every countable dimensional ultrab. space is isomorphic to $K^{(N)}$). iii) As a consequence, a metrizable ultrab. lcs has no infinite countable codimensional subspace. iv) Therefore, an infinite-countable dimensional subspace of a metrizable ultrab. lcs has no algebraic complement.

b) If E and F are ultrab. lcs and $u: E \rightarrow F$ is a linear injective mapping such that u(E) is not topologically isomorphic to E, then u(E) has no algebraic complement in F. Otherwise, by a)i), u(E) would be ultrab. and both $u: E \rightarrow u(E)$ and $u^{-1}: u(E) \rightarrow E$ should be continuous, hence E isomorphic to u(E). Thus, for instance, ℓ^1 (= $\ell^1(N)$) has no algebraic complement in ℓ^2, since ℓ^1 with the topology induced by ℓ^2 is not complete and a fortiori not isomorphic to ℓ^2.

Analogously, if E and F are ultrab. lcs and $u: E \rightarrow F$ a linear mapping such that u(E) is not topologically isomorphic to a quotient of E (by a closed subspace), then u(E) has no algebraic complement in F.

c) If E is an ultrab. lcs and is a proper, dense, finite or infinite-countable codimensional subspace of a lcs F, then F is not ultrab.. Otherwise, since E has an algebraic complement in F, E would be closed in F by a)i), against the hypothesis. By the importance of this result, we state a particular case of it as a proposition:

<u>Proposition 5.7</u> Let H be an ultrab. and dense proper subspace of a lcs E. Let $x \in E \setminus H$, and $F = H + sp(x)$. Then, F is not ultrabornological.

Since in ZF+DC+S the class of lcs with the finest lc topology collapses into the class of ultrab. lcs, one could suppose that this later one could collapse into the class of barrelled lcs, at least when intersected with the class of normed spaces. Next Corollary shows that this is not the case.

Corollary 5.8 There exists a normed barrelled space which is not ultrabornological (in ZF+DC+S).

Proof. Take $E=C([a,b])$ and $H=F_1$ of Example 4.5a) (with $X=R$), which is an ultrab. dense subspace of $C([a,b])$ by Remark 5.1d). By Baire's theorem or by Weierstrass' example of a continuous function without derivative at any point, H is a proper subspace of E. So, if $x \in E\setminus H$ and $F=H+sp(x)$, F cannot be ultrab. by Prop.5.7. But F is barrelled by Remark 1.0, which holds also in ZF+DC+S. ∎

Remark 5.9 Many non-ultrab. spaces constructed in the system ZFC cannot be constructed in the system ZF+DC+S. For instance, assuming ZFC, as a particular case of a result of Valdivia [18], every infinite dimensional Frechet space has a dense hyperplane which is not ultrab. (his proof relies heavily on the uncountable axiom of choice). By Remark a)ii) such construction is impossible in ZF+DC+S. However, the same question in the positive case is not so clear:

Question 5.10 a) Is every lcs E which can be constructed and proved to be ultrab. in ZFC also constructible and ultrab. in ZF+DC+S? or at least: b) If a lcs can be constructed in both systems and proved to be ultrab. in ZFC, is it also ultrab. in ZF+DC+S?

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