LEFT AND RIGHT TYPES OF TILTED ALGEBRAS

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Along these notes all algebras are basic, connected, associative, finite dimensional algebras with identities over a fixed algebraically closed field k, and all modules are finitely generated right modules. For an algebra A, let mod$A$ denote the category of modules over A and ind$A$ denote the full subcategory of mod$A$ with one representative of each isoclass of indecomposable modules.

The main objective in representation theory is the study of mod$A$, for a given algebra A. A possible general strategy is to consider a class of well-known algebras and from there to construct another class, where it is possible to transport informations from the old class to the new one

In general, if A is an algebra and $M \in$ mod$A$, then $B = (\mathrm{End}_A M)^{op}$ is also an algebra. Depending on the hypothesis imposed on A and on M, we will have some control over the algebra B constructed as above (or equivalently, over the category mod$B$). For instance, if M is a projective progenerator of mod$A$, then $B = (\mathrm{End}_A M)^{op} \cong A$. Moreover, the functor $\mathrm{Hom}_A(M, -)\colon \mathrm{mod}A \longrightarrow \mathrm{mod}B$ gives an equivalence of the categories mod$A$ and mod$B$ in this case.

Another interesting situation is the following. Let A be a representation-finite algebra, that is an algebra such that ind$A$ has only finitely many objects. Let M be the sum of one copy of each module of ind$A$. The algebra $B = (\mathrm{End}_A M)$ is called the *Auslander algebra* of A and it has well-known properties (see [4]).

However, we shall concentrate our attention in the so-called *tilted algebra*, construct from hereditary algebras. In section 1, we will recall the definition and basic properties of this class of algebras introduced by Happel-Ringel [9]. Section 2 is devoted to discuss the notions of left and right types of tilted algebras as introduced in a joint work with I. Assem [2] (also [3]). The main results concern the homological properties of modules with respect to their position in the Auslander-Reiten quiver of the algebra (see definitions in section 1). These notes are based in a conference given in the *XIII Escola de Algebra*, held in Campinas, Brazil, in July, 1994.

The author was supported in part by CNPq

1. Tilted algebras

From now on, let A be a hereditary algebra, that is, an algebra such that its radical is a projective module. An A-module T_A is called *tilting* provided:

(i) $\mathrm{Ext}^1_A(T,T) = 0$; and

(ii) There exists a short exact sequence

$$0 \longrightarrow A_A \longrightarrow T'_A \longrightarrow T''_A \longrightarrow 0$$

with T' and T'' direct sums of direct summands of T, that is, they belong to add$T$.

If T_A is a tilting module, the algebra $B = \mathrm{End}_A T$ is called *tilted.*

Examples. Let A be a hereditary algebra and $A_A = P_1^{n_1} \oplus \cdots \oplus P_t^{n_t}$ be its decomposition into indecomposable modules. Since we are assuming that A is basic, then $n_i = 1$, for each $i = 1, \cdots, t$.

(1) The module $T = P_1 \oplus \cdots \oplus P_t$ is clearly a tilting module.

(2) Let $T = DA = \mathrm{Hom}_k(A,k)$. Then, T is the sum of the indecomposable injective A-modules. Therefore, $\mathrm{Ext}^1_A(T,T) = 0$. Consider now the following short exact sequence

$$0 \longrightarrow A_A \xrightarrow{\iota} T_1 \longrightarrow Coker(\iota) \longrightarrow 0$$

where ι is the injective envelope of A. Then $T_1 \in$ add$T$, and since Cokerι is a quotient of T_1, it also belongs to add$T$. Therefore, T is a tilting module.

(3) Let A be the matrix algebra

$$\begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

The indecomposable A-modules are $P_1 = e_{11}A$, $P_2 = e_{22}A$, $P_3 = e_{33}A = S_3$, $S_1 = P_1/\mathrm{rad}P_1$, $S_2 = P_2/\mathrm{rad}P_2$ and P_1/S_3, where e_{ii} is the matrix with 1 in the coordinate (i,i) and 0 in the other coordinates. Observe that P_i is the indecomposable projective associated to the row i and S_i is the simple module associated to P_i. It is not difficult to see that $T = P_1 \oplus S_1 \oplus S_3$ is a tilting module. Moreover,

$$\mathrm{B} = (\mathrm{End}_A T)^{op} = \begin{pmatrix} k & k & 0 \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

The notion of tilted algebras was introduced by Happel-Ringel [9], based in the work of Brenner-Butler [5] (see also []). In order to give a characterization of tilted algebras we shall recall the notion of the Auslander-Reiten quiver of an algebra A.

Given two indecomposable A-modules X and Y, and $i \geq 1$, we denote by $\mathrm{rad}^i(X,Y)$ the vector space generated by the morphisms from X to Y which are composite of i non-invertible morphisms and by $\mathrm{rad}^\infty(X,Y)$ the intersection of all $\mathrm{rad}^i(X,Y)$, $i \geq 1$.

Finally, denote by $\mathrm{Irr}(M,N) = \mathrm{rad}(M,N)/\mathrm{rad}^2(M,N)$, which is clearly a k-vector space. A morphism in $\mathrm{Irr}(M,N)$ is usually called *irreducible morphism.*

For a given algebra A, the Auslander-Reiten quiver Γ_A of A is defined as follows. For each indecomposable A-module M we assign a vertex $[M]$ in Γ_A, and for a given pair of modules M and N in ind$A$, the number of arrows from $[M]$ to $[N]$ is defined to be the dimension $\dim_k \mathrm{Irr}(M,N)$. We shall denote by $(\Gamma_A)_0$ and $(\Gamma_A)_1$ the sets of vertices and arrows of Γ_A, respectively.

The Auslander-Reiten quiver Γ_A of an algebra A satisfies:

(i) Γ_A is locally finite, that is, for each vertex $[M]$ in Γ_A, there is at most finitely many vertices linked by an arrow to $[M]$.

(ii) Γ_A has no loops, that is, there is no arrows starting and ending at the same vertex.

(iii) There is a bijection $\tau\colon (\Gamma_A)'_0 \longrightarrow (\Gamma_A)''_0$, where $(\Gamma_A)'_0$ and $(\Gamma_A)''_0$ are the subsets of $(\Gamma_A)_0$ corresponding to the non-projective and non-injective modules, respectively, such that for each $[M] \in (\Gamma_A)'_0$, and $[N] \in (\Gamma_A)_0$, the number of arrows from $[N]$ to $[M]$ equals the number of arrows from $[\tau M]$ to $[N]$. Moreover, for representation-infinite algebras the quiver Γ_A is usually not connected. We shall refer to the connected components of Γ_A simply by components. From now on we shall identify the vertices of Γ_A with the corresponding indecomposable modules.

Let Γ be a component of Γ_A. Recall that a *section* Σ in Γ is a connected full subquiver of Γ such that:

(1) Σ contains no oriented cycles;
(2) Σ meets each τ-orbit in Γ exactly once;
(3) Σ is convex in Γ, that is, any path in Γ with endpoints in Σ lies entirely in Σ; and
(4) for each arrow $M \to N$ in Γ, if M is in Σ, either N or τN is in Σ and, if N is in Σ, either M or $\tau^{-1}M$ is in Σ.

The next result, due to Liu [11] and Skowroński [12], characterizes the tilted algebras in terms of the so-called complete slices.

Theorem. *An algebra A is tilted if and only if Γ_A has a component Γ with a faithful section Σ such that $\mathrm{Hom}_A(M,\tau N) = 0$ for all M, N in Σ. In this situation, Σ is called a complete slice and Γ is called a connecting component of Γ_A.*

For further details in tilting theory, we refer the reader to [1].

2. Left and right types

Let $B=\mathrm{End}_A(T)$ be a tilted algebra. In [9], Happel-Ringel have shown that the global dimension of B is at most two. Moreover, if $M \in \mathcal{X} = \{N_B |\ N \otimes_B T) = 0\}$, then the injective dimension of M (id$M$) is at most one and if $M \in \mathcal{Y} = \{N_B |\ \mathrm{Tor}_1^B(N,T) = 0\}$ then, the projective dimension of M (pd$M$) is at most one.

Consequently, we have the following proposition. We say that a property holds for almost all modules if it fails only for a finite number of non-isomorphic indecomposable modules.

Proposition 3. *Let B be a tilted algebra. Then B is representation-finite (that is, there are only finitely many non-isomorphic indecomposable B-modules) if and only if $pdM = idM = 2$ for almost all indecomposable modules.*

For a representation-infinte tilted algebra B, it is reasonable to expect that the structure of its Auslander-Reiten quiver can be used to predict the homological dimensions of a B-module, from its position with respect to a complete slice in Γ_B. We have studied this question in a joint work with I. Assem [2, 3].

Let now A be a hereditary algebra and T be a tilting A-module. In case T satisfies $\mathrm{rad}^{\infty}(-,T) = 0$, then the algebra $B = \mathrm{End}T$ is called *concealed.* It is well-known that a tilted algebra B has at most two connecting components and it has two if and only if B is concealed. In many aspects, the concealed algebras are those which are closer to hereditary algebras. We have proven in [2] the following result.

Theorem 4. *Let A be a representation-infinite algebra. The following are equivalent:*
(i) A is concealed;
(ii) $rad^{\infty}(-,A) = 0$ and $rad^{\infty}(DA,-) = 0$;
(iii) $pdM = 1$ and $idM = 1$ for almost all indecomposable modules M.

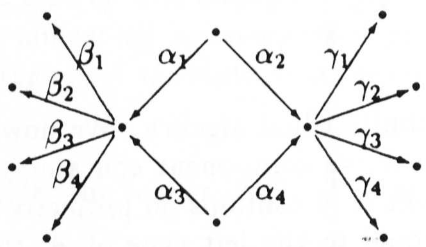
The next step is to consider those tilted algebras which satisfy one of the following properties: (a) $\mathrm{pd}M = 2$ and $\mathrm{id}M = 1$ for almost all indecomposable modules; and (b) $\mathrm{pd}M = 1$ and $\mathrm{id}M = 2$ for almost all indecomposable modules. In order to do so, we shall introduce the notions of left and right types of a tilted algebra.

Let now A be a tilted algebra. We define the *left type* of A as follows. If A has a complete slice in a postprojective component, the left type of A is defined to be the empty graph. Otherwise, A has a unique connecting component Γ which is not postprojective. If Γ contains no projective module (so that every module in Γ is left stable), we define the left type of A to be the type of the tilted algebra A, as defined above. Suppose Γ contains a projective module. Let Σ be the subsection of Γ consisting of the left stable modules M in Γ such that there exists a path in Γ of length at least one from M to some projective, and any such path is sectional. Since Σ is generally not connected, we can write it as $\Sigma_1 \cup \cdots \cup \Sigma_t$, where each Σ_i is a connected component of Σ. Then $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$ will be called *the left extremal subsection of A*, and its underlying graph $\overline{\Sigma}$ will be called the left type of A. Observe that, since Γ contains a complete slice, no injective module is a predecessor of Σ.

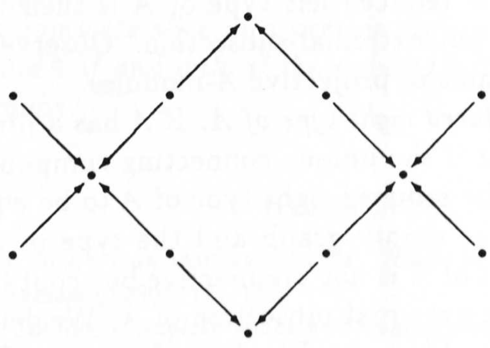
Dually, we define the *right type* of A as follows. If A has a complete slice in a preinjective component, the right type of A is defined to be the empty graph.

Otherwise, A has a unique connecting component Γ which is not preinjective. If Γ contains no injective module (so that every module in Γ is right stable), we define the right type of A to be the type of the tilted algebra A. If Γ contains an injective module, let Σ be the subsection of Γ consisting of the right stable modules M in Γ such that there exists a path in Γ of length at least one from some injective to M, and any such path is sectional. Let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$, where each Σ_i is a connected component of Σ. Then $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$ will be called *the right extremal subsection of* A, and its underlying graph $\overline{\Sigma}$ will be called the right type of A. Observe that, since Γ contains a complete slice, no projective module is a successor of Σ.

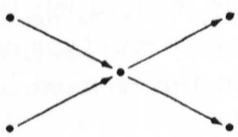
Example. We borrow this example from [10](5). Let A be given by the quiver



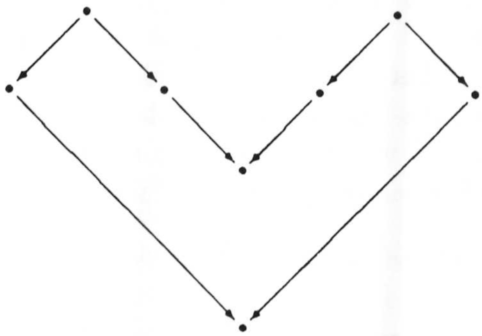
bound by $\alpha_1\beta_i = 0 = \alpha_2\gamma_i$ for $i = 1,2,3$ and $\alpha_3\beta_j = 0 = \alpha_4\gamma_j$ for $j = 2,3,4$. Then A is tilted, and its type is the underlying graph of the following representing slice



The left type of A equals the disjoint union of two copies of $\tilde{\mathbb{D}}_4$, and the left extremal subsection equals the disjoint union of two copies of the quiver



The right type of A is equal to $\tilde{A}_7$, and the right extremal subsection is given by the quiver



Let A be a representation-infinite tilted algebra. We now define the *reduced left type of A*. If A has a postprojective component containing a complete slice or if the unique connecting component of A contains no projective module, we define the reduced left type of A to be equal to the left type of A, that is, respectively, the empty graph and the type of A. Assume that the unique connecting component of A is not postprojective but contains projectives, and let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$ be the left extremal subsection of A. We define the reduced left extremal subsection of A to be $\Sigma'_1 \cup \cdots \cup \Sigma'_t$ where, for each i, Σ'_i is the full (convex) subquiver of Σ_i obtained by deleting all the sinks. The reduced left type of A is then the underlying graph $\overline{\Sigma'_1} \cup \cdots \cup \overline{\Sigma'_t}$ of the reduced left extremal subsection. Observe that the sinks of Σ_i correspond to radical summands of projective A-modules.

Likewise, we define the *reduced right type of A*. If A has a preinjective component containing a complete slice or if the unique connecting component of A contains no injective module, we define the reduced right type of A to be equal to the right type of A, that is, respectively, the empty graph and the type of A. Assume that the unique connecting component of A is not preinjective but contains injectives, and let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$ be the right extremal subsection of A. We define the reduced right extremal subsection of A to be $\Sigma'_1 \cup \cdots \cup \Sigma'_t$ where, for each i, Σ'_i is the full (convex) subquiver of Σ_i obtained by deleting all the sources. The reduced right type of A is then the underlying graph $\overline{\Sigma'_1} \cup \cdots \cup \overline{\Sigma'_t}$ of the reduced right extremal subsection . Observe that the sources of Σ_i correspond to socle factors of injective A-modules.

Since A is representation-infinite, the reduced left (or right) type of A is empty if and only if so is the left (or right, respectively) type of A.

For instance, in the above example, the reduced left extremal subsection is the disjoint union of two copies of the quiver

while the reduced right extremal subsection is the disjoint union of two copies of the quiver

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

In particular, both the reduced right and left types of A are disjoint unions of Dynkin graphs, so that A satisfies the conditions of our main theorem below.

Theorem. *Let A be a representation-infinite algebra which is tilted but not concealed, and Σ be a complete slice in Γ_A.*

(a) *$pdM = 2$ for almost all indecomposable successors M of Σ if and only if the reduced right type of A is empty or a disjoint union of Dynkin graphs.*

(b) *$idM = 2$ for almost all indecomposable predecessors M of Σ if and only if the reduced left type of A is empty or a disjoint union of Dynkin graphs.*

For a proof, we refer the reader to [2]. The next result is a direct consequence of the above theorem.

Corollary. *Let A be a representation-infinite tilted algebra.*

(a) *If A has a complete slice in a postprojective component then $pdM = 2$ for almost all M in $indA$ if and only if the reduced right type of A is a disjoint union of Dynkin graphs.*

(b) *If A has a complete slice in a preinjective component then $idM = 2$ for almost all M in $indA$ if and only if the reduced left type of A is a disjoint union of Dynkin graphs.*

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