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MORE ON THE UNIT LOOP OF AN ALTERNATIVE LOOP RING

EDGAR G. GOODAIRE AND CÉSAR POLCINO MILIES

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ABSTRACT. Let L be a loop and R a commutative, associative ring of characteristic different from 2 such that the loop ring RL is alternative. It is known that the (necessarily Moufang) loop L is solvable and "torsion of bounded exponent over the centre" in the sense that, for some integer n, ℓ^n is central for all $\ell \in L$. In this paper, we describe work which explores the extent to which these two properties are shared by the unit loop of RL in the cases that R is the ring of rational integers or a field.

RÉSUMÉ. Soit L une boucle et R un anneau commutatif et associative de caracteristique $\neq 2$ tel que l'anneau de boucle RL est alternative. Il est bien connu que la boucle L (necessairement Moufang) est resoluble et "torsion d'exposant borné", c'est-à-dire, il existe un entier n tel que ℓ^n soit centrale pour tout *ell* $\in L$. Cet article etudie comment la boucle d'unitées de RL a ces deux propriètés aux cas ou R soit l'anneau des entiers rationels ou un corps.

1. Introduction. An *alternative* ring is one which satisfies both the left and right alternative laws

$$x(xy) = x^2y \quad \text{and} \quad (yx)x = yx^2.$$

A *Moufang* loop is a loop which satisfies any of the three equivalent identities

$$((xy)x)z = x(y(xz)), \quad ((xy)z)y = x(y(zy)) \quad \text{and} \quad (xy)(zx) = (x(yz))x.$$

Given a Moufang loop L and a commutative, associative ring R, one forms the loop ring RL precisely as if L were a group. Unlike the associative situation, however, it is rarely the case that RL inherits the Moufang identity from L. When it does, and if char $R \neq 2$, L is called an *RA loop* because RL is an alternative ring. A suitable single source of information about Moufang loops, RA loops, alternative rings and alternative loop rings is the monograph [GJM96].

If L is an RA loop, the set $\mathcal{U}(\text{RL})$ of *units* (or invertible elements) in RL is a Moufang loop (containing L) and it is interesting to compare the properties

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of L and $\mathcal{U}$(RL). Since the square of any element in L is central (hence $L/\mathcal{Z}(L)$ is an elementary abelian 2-group), for instance, L is nilpotent. Necessary and sufficient conditions for the nilpotence of $\mathcal{U}$(RL), for various R, were determined a few years ago [GM95], [GM97], so we have recently turned our attention to solvability, with results in the cases that $R = \mathbb{Z}$ is the ring of rational integers or a field of arbitrary characteristic. A Moufang loop $\mathcal{U}$ is *torsion over the centre* if, for each $\mu \in \mathcal{U}$, μ^n is central for some integer $n = n(\mu)$. If n can be chosen independent of μ, then $\mathcal{U}$ is torsion *of bounded exponent* n over the centre. We have already said that an RA loop is torsion of bounded exponent 2 over its centre. It is natural then to see if the unit loop can exhibit similar behaviour.

The questions with which we are concerned in this paper have been considered for groups. Group rings whose unit groups are torsion over their centres have been investigated by Sehgal [Seh78, Section II.2], Cliff and Sehgal [CS80], Coelho [Coe82] and Bist [Bis94]. The first paper on solvability was written by Bateman [Bat71], who considered the group algebra of a finite group over a field. Solvability of the unit group of infinite groups and over other rings of coefficients has been investigated by a number of people, including Motose, Tominaga and Ninomiya [MT71], [MN72], Sehgal [Seh75], Passman [Pas77], Gonçalves [Gon86] and Bovdi [Bov92].

2. **Integral loop rings.** An element of finite order in a loop is a *torsion* element. In an RA loop, the set of all torsion elements forms a subloop called the *torsion* subloop of L. In [GM95] (see also [GJM96, Chapter XII]), the authors showed that the nilpotence of $\mathcal{U}(\mathbb{Z}L)$, the unit loop of an integral alternative loop ring, is equivalent to a number of conditions, one of which is:

* The torsion subloop T of L is an abelian group or a hamiltonian Moufang 2-loop and $x^{-1}tx = t^{\pm 1}$ for any $t \in T$ and any $x \in L$. Moreover, if T is an abelian group and $x \in L$ is an element that does not centralize T, then $x^{-1}tx = t^{-1}$ for all $t \in T$.

Recently, we have shown that solvability is equivalent to a suitable weakening of this statement [GMa].

THEOREM 1. *Let L be an RA loop with torsion subloop T. Then $\mathcal{U}(\mathbb{Z}L)$ is solvable if and only if T is either an abelian group or a hamiltonian Moufang 2-loop and every subloop of T is normal in L.*

It follows that solvability and nilpotence of $\mathcal{U}(\mathbb{Z}L)$ are equivalent for torsion RA loops. (There are examples which show that this is not always the case.) Also, and this was quite unexpected, we have shown that (2) is a condition also equivalent to $\mathcal{U}(\mathbb{Z}L)$ being torsion over its centre. Specifically, we have this result [GMb].

THEOREM 2. *Let L be an RA loop with torsion subloop T. Then the following conditions are equivalent:*

(i) $\mathcal{U}(\mathbb{Z}L)$ *is torsion of bounded exponent over the centre.*
(ii) $\mathcal{U}(\mathbb{Z}L)$ *is torsion of bounded exponent* 2 *over the centre.*
(iii) $\mathcal{U}(\mathbb{Z}L)$ *is nilpotent.*
(iv) Condition (2) holds.

We do not know when $\mathcal{U}(\mathbb{Z}L)$ is torsion of unbounded exponent over its centre.

3. Unit loops of loop algebras. Questions about loop algebras over fields require different approaches and usually have very different answers from analogous questions about integral loop rings and such was our experience with these investigations. In certain cases, for instance, we were able to establish that torsion over the centre was equivalent to torsion of bounded exponent. Our results depend on the characteristic of K and are presented in four theorems below, proofs of which can be found in [GMb]. Those of Theorems 3 and 5, which consider the case of unbounded exponent, make fundamental use of the observation that every idempotent of KL is central whenever $\mathcal{U}(KL)$ is torsion over its centre, but not torsion. At this point, we have very explicit information about the nature of units: specifically, every unit μ can be written as $\mu = \sum \mu_q q$, the q in L and the μ_q in KT, where T is the torsion subloop of L [GJM96, Lemma XII.1.1], [GM95].

THEOREM 3. *Let L be an RA loop with torsion subloop T and let K be a field of characteristic* 0. *Suppose $\mathcal{U}(KL)$ is not torsion. Then the following conditions are equivalent.*
(i) $\mathcal{U}(KL)$ *is torsion over its centre.*
(ii) T is central.
(iii) $\mathcal{U}(KL)$ *is torsion of bounded exponent* 2 *over its centre.*
(iv) $\mathcal{U}(KL)$ *is torsion of bounded exponent over its centre.*

Since the prime 2 plays a very important role in the theory of RA loops, when K has positive characteristic p, the case $p = 2$ is usually special and easy to handle.

THEOREM 4. *Let L be an RA loop and let K be a field of characteristic* 2. *Then $\mathcal{U}(KL)$ is torsion of bounded exponent* 2 *over its centre.*

THEOREM 5. *Let K be a field of positive characteristic $p \neq 2$ and let L be an RA loop with torsion subloop T. Suppose $\mathcal{U}(KL)$ is not torsion. Then $\mathcal{U}(KL)$ is torsion over its centre if and only if T is an abelian group, every idempotent of KT is central in KL and, if T is not central, then K is algebraic over its prime field.*

THEOREM 6. *Let K be a field of positive characteristic $p \neq 2$ and let L be an RA loop with torsion subloop T. Let P and $A = T_{p'}$ denote, respectively, the sets of p- and p'-elements in T. Suppose $\mathcal{U}(KL)$ is not torsion. Then the following statements are equivalent.*
(i) $\mathcal{U}(KL)$ *is torsion of bounded exponent over its centre.*

(ii) T *is abelian, there exists an integer* k *such that* x^{p^k} *is central for all* $x \in \Delta(L, P)$ *and either*

- T *is central and* $\mathcal{U}(KL)$ *has exponent* $2p^k$ *over its centre, or*
- K *is finite,* $A^m = \{1\}$ *for some* m *and* $\mathcal{U}(KL)$ *has exponent* $2rp^k$ *over its centre, where* $r = |K(\zeta)| - 1$, ζ *a primitive* m*-th root of unity.*

We return to questions of solvability of the unit loop of a loop algebra. Over a field K, when the RA loop L is torsion, we can show that $\mathcal{U}(KL)$ is solvable if and only if char $K = 2$. When L is not torsion, however, the question is considerably more complicated and not completely settled. Our results rely heavily on the ability to find *Zorn's vector matrix algebra* in loop algebras. Given a field K, Zorn's algebra $\mathfrak{Z}(K)$ is the set of 2×2 matrices of the form $\begin{bmatrix} a & \mathsf{x} \\ \mathsf{y} & b \end{bmatrix}$ where $a, b \in K$, $\mathsf{x}, \mathsf{y} \in K^3$, with obvious addition and the following multiplication:

$$\begin{bmatrix} a_1 & \mathsf{x}_1 \\ \mathsf{y}_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & \mathsf{x}_2 \\ \mathsf{y}_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + \mathsf{x}_1 \cdot \mathsf{y}_2 & a_1\mathsf{x}_2 + b_2\mathsf{x}_1 - \mathsf{y}_1 \times \mathsf{y}_2 \\ a_2\mathsf{y}_1 + b_1\mathsf{y}_2 + \mathsf{x}_1 \times \mathsf{x}_2 & b_1b_2 + \mathsf{y}_1 \cdot \mathsf{x}_2 \end{bmatrix}.$$

Here, $\cdot$ and $\times$ denote the dot and cross products respectively in K^3. Lowell Paige showed that the unit loop of any Zorn's algebra is simple modulo its centre [Pai56]. Of particular relevance here is the consequence that the unit loop of $\mathfrak{Z}(K)$ is never solvable.

Suppose K is a field of characteristic different from 2, L is an RA loop with torsion subloop T and the subloop generated by some $t \in T$ is not normal in L. We are able to construct quite explicitly a copy of $\mathfrak{Z}(K)$ inside KL. Similarly, if L contains the quaternion group of order 8 and K has characteristic 3, again we can exhibit $\mathfrak{Z}(K)$ within KL. These facts were the key in obtaining the following result.

THEOREM 7. *Let* K *be a field of characteristic* $p \geq 0$ *and suppose* L *is an RA loop which contains no elements of order* p *(in the case* $p > 0$*). Then* $\mathcal{U}(KL)$ *is solvable if and only if (i)* $p = 2$ *or (ii)* T *is an abelian group and every idempotent of* KT *is central in* KL.

Comparison with Theorem 4 gives the following.

COROLLARY 8. *With the hypotheses on* K *and* L *as in Theorem 7, if* $\mathcal{U}(KL)$ *is torsion over its centre but not torsion, then* $\mathcal{U}(KL)$ *is solvable.*

When L contains elements of order p, we have results only in positive characteristic. The corresponding problem for group rings is also unsettled, although A. Bovdi has made progress for nilpotent nontorsion groups [Bov92].

THEOREM 9. *Let* K *be a field of characteristic* $p > 0$. *Let* L *be an RA loop which is not torsion and which contains elements of order* p. *The* $\mathcal{U}(KL)$ *is solvable if and only if (i)* $p = 2$ *or (ii) the set* P *of* p*-elements in* L *is a finite central group and* $\mathcal{U}(K[L/P])$ *is solvable.*

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Memorial University of Newfoundland
St. John's, Newfoundland
A1C 5S7
email: edgar@math.mun.ca

Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 66.281
CEP 05315-970, São Paulo SP
Brasil
email: polcino@ime.usp.br