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Abstract

We construct geometrically finite free Kleinian groups acting on S^3 whose limit sets are wild Cantor sets.

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Introduction

In [5], M Freedman and R. Scora have constructed exotic examples of co–compact topological group actions on the 3-dimensional sphere S^3 with wild Cantor sets as their limit sets.

Their groups have interesting features: each element of a group is individually conjugate to a conformal (hyperbolic) transformation of S^3, but the whole group is not topologically conjugate to a conformal group; so the wildness of the limit set arises from the interplay of the generators and not from the dynamics of any element alone.

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They conjectured in [6] that if a group acts conformally on S^3 with limit set homeomorphic to a Cantor set and with compact quotient of the domain of discontinuity, then that Cantor set is tame.

The purpose of this paper is to exhibit explicit examples of conformal groups - Kleinian groups - acting on S^3, whose limit sets are wild Cantor sets. As opposed to the Freedman-Scora examples, the groups we constructed have non-compact quotients, they contain lots of parabolic elements. We will call those groups *Fake Schottky type groups* (or FST-groups).

We will present three different constructions of FST-groups. It is interesting to note that we obtain in this way examples of non-equivalent wild Cantor sets in S^3.

In the first example we use Klein's Combination Theorem to build FST-groups from Schottky type groups. The key point in this part is in constructing the Schottky type group (acting on S^3) with non-standard isometric fundamental domain.

It should be noted here that the first attempt to construct FST-groups was made by M. Bestvina and D. Cooper [1], but unfortunately their paper contains a gap (see comments below). Nevertheless their idea is beautiful and fruitful. Our second example is in fact a realization of their idea.

The third construction is a generalisation of the first one. Using this construction we obtain the following result. For any positive integer N there are at least N free FST-groups acting on S^3 with the same rank $k(N)$ which uniformize N non-homeomorphic manifolds. Moreover, the limit sets of these groups are non-equivalent wild Cantor sets.

The organization of the paper runs as follows. In section 1 we review Kleinian groups and discuss some examples including M. Bestvina and D. Cooper's one. The first and the second constructions are given in sections 2 and 3 respectively. Section 4 contains the topological part of the proofs. In section 5 we present the third construction. In section 6 we prove that the extensions of the FST-groups we constructed to the action on 4-dimensional sphere are Schottky type groups.

1 Preliminaries

1.1 We denote the Euclidean n-space by $\mathbf{E}^n$. We will write a point $x \in \mathbf{E}^n$ as $x = (x_1, \ldots, x_n)$. The unit sphere in $\mathbf{E}^n$ is $\mathbf{S} = \{x \in \mathbf{E}^n : |x| = 1\}$, the open unit ball is $\mathbf{B} = \{x \in \mathbf{E}^n : |x| < 1\}$, and the upper half space is $\mathbf{H}^n = \{x = (x_1, \ldots, x_n) \in \mathbf{E}^n : x_n > 0\}$. The one point compactification of $\mathbf{E}^n$ is denoted by $\bar{\mathbf{E}}^n$ or $\mathbf{S}^n$. The natural inclusion of $\mathbf{E}^{n-1}$ into $\mathbf{E}^n$ is given by $\mathbf{E}^{n-1} = \{x = (x_1, \ldots, x_n) \in \mathbf{E}^n : x_n = 0\}$ and extends to the one point compactifications, so that we have $\bar{\mathbf{E}}^{n-1} = \partial \mathbf{H}^n$.

1.1.1 The differential metric on $\mathbf{H}^n$ is given by $ds^2 = (dx_1^2 + \ldots + dx_n^2)/x_n^2$. With this metric $\mathbf{H}^n$ is a model of hyperbolic space.

1.1.2 Let $\mathrm{M}(n)$ be the group of all orientation preserving Möbius transformations of $\bar{\mathbf{E}}^n$, that is, each element of $\mathrm{M}(n)$ is a composition of a finite (even) number of inversions in spheres in $\bar{\mathbf{E}}^n$. This group is isomorphic to the connected component of the unity of the Lorentz group $SO(n+1,1)$. In dimension $n=2$ there is also a canonical identification of $M(2)$ with $PSL(2,C)$.

1.1.3 It is well known that there is a natural embedding of $M(n)$ into $M(n+1)$, that is, for each $g \in M(n)$ there is a $g^* \in M(n+1)$ such that $g^*|_{E^n} = g$ and $g^*(H^{n+1}) = H^{n+1}$.

We remark also that $M(n)$ is both the full group of orientation-preserving isometries of H^{n+1} and also the full group of orientation-preserving conformal mappings of $\bar{E}^n$.

1.2 Classification of the elements of $M(n)$

1.2.1 Every element $g \in M(n)$ has at least one fixed point in the closure of H^{n+1}. If g has a fixed point in H^{n+1}, then it is *elliptic*; if g is not elliptic, and has exactly one fixed point on ∂H^{n+1}, then it is *parabolic*; otherwise, it is *loxodromic*. A loxodromic transformation which is conjugate to a dilation $x \mapsto \lambda x$, $\infty \mapsto \infty$, $\lambda > 0$ $\lambda \neq 1$, is called *hyperbolic*.

1.2.2 In dimension $n = 2$ we may identify $M(2)$ with $PSL(2,C)$.

Proposition 1.1 *Let $g \in PSL(2,C)$, and let $Tr^2(g)$ denote the square of the trace of a matrix in $SL(2,C)$ representing g. Then:*

1. *$Tr^2(g)$ is real with $0 \leq Tr^2(g) \leq 4$ if and only if g is elliptic;*
2. *$Tr^2(g) = 4$ if and only if g is either parabolic or the identity;*
3. *$Tr^2(g)$ is real with $Tr^2(g) > 4$ if and only if g is hyperbolic;*
4. *$Tr^2(g)$ is not in the interval $[0,\infty)$ if and only if g is loxodromic, but not hyperbolic.*

1.3 Isometric spheres

For a transformation $g \in M(n)$ with $g(\infty) \neq \infty$, the isometric sphere (isometric circle in dimension 2) $I(g)$ of g is defined by $I(g) = \{x \in E^n : \|D_x g\| = 1\}$.

Proposition 1.2 *A transformation $g \in M(n)$ such that $g(\infty) \neq \infty$ can be written in the form $g = O \circ q \circ p$, where p is the inversion in $I(g)$; q is the reflection in the bisector of the centers of $I(g)$ and $I(g^{-1})$ if $I(g) \neq I(g^{-1})$, or the reflection in an arbitrary hyperplane in E^n passing through the center of $I(g)$ if $I(g) = I(g^{-1})$; and O is a rotation around the center of $I(g^{-1})$.*

In particular, for dimension 2 we have:

Proposition 1.3 *The transformation $g \in M(2)$ with $g(\infty) \neq \infty$ has the following form*

$$g(z) = \beta - \frac{r^2 e^{i(\theta+2\lambda)}}{z-\alpha},$$

where $\alpha \in C$ is the center of $I(g)$, r is its radius, β is the center of $I(g^{-1})$, λ is the angle which the bisector of α and β (if distinct) makes with the imaginary axis (or the angle of a line passing through α if $\alpha = \beta$), and θ is the angle of rotation around β.

1.4 Kleinian groups

Let Γ be a subgroup of $M(n)$. We say that the action of Γ at a point $x \in \bar{E}^n$ is *discontinuous* if

1. The stabilizer $\Gamma_x = \{g \in \Gamma : gx = x\}$ is finite;
2. There is a neighborhood U of x such that $g(U) \cap U = \emptyset$ for all $g \in \Gamma \setminus \Gamma_x$.

The set of points at which the action of Γ is discontinuous is called the *regular set*, and is denoted by $R(\Gamma)$. Its complement $L(\Gamma) = \bar{E}^n \setminus R(\Gamma)$ is called the *limit set* of Γ. A group Γ is called *Kleinian* if $R(\Gamma) \neq \emptyset$. An *elementary* group is a Kleinian group whose limit set has a finite number of points.

1.5 Fundamental domains

A *fundamental domain* D for the Kleinian group Γ is an open subset of $R(\Gamma)$ such that:

1. $g(D) \cap D = \emptyset$, for all $g \in \Gamma \setminus \{id\}$;
2. For every $x \in R(\Gamma)$ there is a $g \in \Gamma$, with $g(x) \in \bar{D}$ ($\bar{D}$ is the closure of D).

A fundamental domain D for Γ is said to be *isometric* if D bounded by isometric spheres (isometric circles in dimension 2) of generators of Γ.

A Kleinian group Γ is *geometrically finite* if it has a finite sided fundamental hyperbolic polyhedron for its action on hyperbolic space H^{n+1}.

1.6 Klein's combination theorem

Let Γ_1 and Γ_2 be Kleinian groups. Suppose that there are fundamental domains D_i of Γ_i $(i = 1, 2)$, such that $D_1 \cup D_2 = \bar{E}^n$ and $D = D_1 \cap D_2 \neq \emptyset$. Then $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is a Kleinian group, and D is a fundamental domain for Γ and $\Gamma = \Gamma_1 * \Gamma_2$ (the free product of Γ_1 and Γ_2).

1.7 Poincaré's polyhedron theorem

This section is devoted to the exposition of a fundamental theorem of Poincaré. It will be given in the form we need for our purpose. A general treatment can be found in [9].

Let $\{(T_i, T_i') : 1 \le i \le m\}$ be a family of closed metric balls in $\hat{\mathbf{E}}^n$. Assume that any pair of them either intersect in a point or are disjoint. A point of intersection of two balls will be called a *point of contact*. Let C be the set of all points of contact of those balls. Let $S_i = \partial T_i \setminus \{\text{points of contact}\}$ and $S_i' = \partial T_i' \setminus \{\text{points of contact}\}$. Then either $S_i = \partial T_i$, ($S_i' = \partial T_i'$), or S_i, (respectively S_i') is a punctured $(n-1)$-sphere. Let $S = \{S_i, S_i'\}$. The complement of the union of all T_i, T_i' we denote by P. An element of S will be called a *side* of P.

Suppose that for each i there is a $g_i \in \mathbf{M}(n)$ such that $g(S_i) = S_i'$, $g_i^{-1}(S_i') = S_i$ and $g_i(P) \cap P = \varnothing$. Let $F = \{g_i, g_i^{-1}\} = \{f_1, \ldots, f_{2m}\}$. An element of F is called a *side pairing transformation*. A side pairing transformation, say f_i, sends a point of contact $e \in T_i$ to a point of contact $e' \in T_i'$. We say that e and e' are related. This relation gives an equivalence relation in C, partitioning C into equivalence classes, called *cycles of points of contact*.

Each cycle c can be cyclically ordered as $c = \{e_1, \ldots, e_{k-1}, e_k = e_0\}$, in such a way that for each i, $1 \le i \le k$, there is a $f_i \in F$ such that $f_i(e_{i-1}) = e_i$. Let $f_c = f_k \circ \ldots \circ f_1$. The element f_c is called the *cyclic transformation* related to the cycle c. Clearly $f_c(e_0) = e_0$, that is, e_0 is a fixed point of f_c.

Theorem 1.1 *Let P be a spherical polyhedron constructed above. Let $F = \{g_i, g_i' : 1 \le i \le m\}$ be a set of side pairing transformations related to P. Suppose that for each cycle of points of contact c we have that g_c is parabolic. Then Γ generated by F is a Kleinian subgroup of $\mathbf{M}(n)$, and P is a fundamental domain for Γ.*

This theorem is a particular case of general Poincaré's Polyhedron Theorem proved by Maskit in [9].

We will also need the following corollary of the proof of Poincaré's Polyhedron Theorem.

Theorem 1.2 *Let Γ be a Kleinian group as above. Then each parabolic element from Γ is conjugated in Γ to the element of the form g_c^k, where g_c is a cyclic transformation and $k \in \mathbf{Z}$.*

1.8 Schottky type groups

1.8.1 We say that a Kleinian group $\Gamma \subseteq \mathbf{M}(n)$ is an ST-group of type (r,s) (*Schottky Type group*) if Γ has generators $g_1, \ldots, g_r, h_1, \ldots, h_s$ and a fundamental domain D bounded Jordan surfaces (or curves in dimension 2) $S_1, S_1' \ldots, S_r, S_r', T_1, T_1', \ldots, T_s, T_s'$, and they satisfy the following conditions:

1. The surfaces (or curves) are disjoint, except that T_j and T_j' have a common point x_j;

2. $g_i(S_i) = S_i'$, $h_i(T_j) = T_j'$;

3. h_j is parabolic with fixed point x_j.

The elements g_i, h_j are called ***standard generators*** of Γ, and D is called a ***standard fundamental domain*** for Γ.

If $s = 0$ then Γ is called a *Schottky group*.

Figure 1.

One can see that every ST-group of type (r,s) is constructed from r cyclic loxodromic groups and s cyclic parabolic groups by Klein's Combination Theorem.

1.8.2 It is easy to verify that an ST-group Γ has the following properties:

1. Γ has a free product decomposition $\Gamma = F_1 * \ldots * F_r * H_1 * \ldots * H_s$, where F_i is cyclic loxodromic and H_j is cyclic parabolic;
2. The limit set $L(\Gamma)$ is totally disconnected;
3. In dimension 2 the regular set $R(\Gamma)$ is connected, and in dimension $n > 2$ $R(\Gamma)$ is simply connected.

1.9 Wild Cantor sets in $\bar{\mathbf{E}}^n$ and Fake Schottky type groups

1.9.1 Cantor sets imbedded in $\bar{\mathbf{E}}^n$ are of two types. A Cantor set $K \subseteq \bar{\mathbf{E}}^n$ is called *tame* if there is a homeomorphism $h: \bar{\mathbf{E}}^n \longrightarrow \bar{\mathbf{E}}^n$ such that $h(K)$ lies on a smoothly embedded arc. Otherwise, K is called *wild*.

Two Cantor sets $K_1, K_2 \subseteq \bar{\mathbf{E}}^n$ are *equivalent* is there is a homeomorphism $h: \bar{\mathbf{E}}^n \longrightarrow \bar{\mathbf{E}}^n$ such that $h(K_1) = K_2$. It is well known that any two Cantor sets in $\bar{\mathbf{E}}^2$ are equivalent, and any two tame Cantor sets in $\bar{\mathbf{E}}^n$ are equivalent.

1.9.2 It is not difficult to show that the limit set of an ST-group is either finite or a tame Cantor set, and up to topological conjugation two ST-groups of the same type are equivalent.

1.10 In [4] Freedman has considered a topological generalization of the ST-groups. He defines a group Γ of homeomorphisms of $\bar{\mathbf{E}}^n$ to be *admissible* if:

1. The limit set $L(\Gamma)$ is a Cantor set;
2. Γ acts discontinuously on $R(\Gamma)$;
3. The quotient $R(\Gamma)/\Gamma$ is compact.

Γ is called *weakly admissible* if the condition (3) above is dropped.

Schottky groups provide examples of admissible actions, and Schottky type groups provide examples of weakly admissible actions.

1.11 We say that a Kleinian group $\Gamma \subseteq M(n)$ is an FST-group of type (r,s) *(Fake Schottky Type group)* if:

1. Γ has a free product decomposition $\Gamma = F_1 * \ldots * F_r * H_1 * \ldots * H_s$, where F_i is cyclic loxodromic and H_j is cyclic parabolic;
2. The limit set $L(\Gamma)$ is a Cantor set;
3. Γ is not an ST-group.

Remark 1. We will show that in dimension 3, conditions (1), (2), (3) above imply that $L(\Gamma)$ is wild. The same is true in dimension greater than 4, but we do not know about it in dimension 4.

Remark 2. It is well-known (see, for instance, Chuckrow [3]) that when n=2 conditions 1 and 2 imply that Γ is an ST-group. Thus, there are no FST-groups in dimension 2.

1.12 Examples related to Poincaré's Polyhedron Theorem

In this section we present two examples which show that one should be careful in applying spherical polyhedra to construct fundamental domains for Kleinian groups. We also recall Bestvina- Cooper's example.

1.12.1 Consider the domain $D \subseteq \bar{E}^3$ bounded by the spheres T_1, T_1', T_2, T_2', where T_1, T_1' are spheres centered at the origin and with radii 1 and 3 respectively, and T_2, T_2' are spheres of radii 1 and centered at $a = (0,-2,0)$ and $b = (0,2,0)$ respectively. (See figure 2.)

Figure 2.

1.12.2 Example 1

Let $g_1(x) = 3x$, $g_2 = \jmath \circ \imath$, where $\imath$ is the inversion in T_2, and $\jmath$ is the reflection in the (x_1,x_3)-plane. We see that there are four points of contact p_1, p_2, p_3, p_4. Let $S_1 = T_1 \setminus \{p_2,p_3\}$, $S_1' = T_1' \setminus \{p_1,p_4\}$, $S_2 = T_2 \setminus \{p_1,p_2\}$, and $S_2' = T_2' \setminus \{p_3,p_4\}$. Then we have that $g(S_1) = S_1'$, $g_2(S_2) = S_2'$. In addition, $g_i(D) \cap D = \emptyset$, $i = 1,2$.

Let $\Gamma = \langle g_1, g_2 \rangle$. In order to prove that D is a fundamental domain for Γ, we need to verify whether all the cyclic transformations are parabolic.

It is easy to verify that we have only one cycle of points of contact $c = \{p_2,\ p_1 = g_1(p_2),\ p_4 = g_2 \circ g_1(p_2),\ p_3 = g_1^{-1} \circ g_2 \circ g_1(p_2)\}$. The cyclic element corresponding to this cycle of points of contact is $g_c = g_2^{-1} \circ g_1^{-1} \circ g_2 \circ g_1$.

Observe that Γ leaves invariant the (x_1,x_2)-plane which we identify with the complex plane C, and put $z = x_1 + ix_2$. Then the action of the elements g_1, g_2 on this C-plane is given by $g_1(z) = 3z$, and $g_2(z) = (2z+3)/(z+2)$ (see proposition 1.3).

We obtain that the action of g_c on the C-plane is given by $g_c(z) = (-5z-4)/(4z+3)$. It is obvious that g_c is parabolic if and only if the restriction of g_c to the C-plane is parabolic if we consider it as an element of $\mathrm{PSL}(2,\mathbf{C})$. Since $Tr^2(g_c) = 4$, we obtain that g_c is parabolic. Therefore, by Poincaré's Polyhedron Theorem, D is a fundamental domain for Γ.

Notice that the restriction of Γ to the C-plane is a Fuchsian group of the first kind. In particular, the limit set $L(\Gamma)$ is the real axis completed by ∞.

1.12.3 Example 2

Let now $\tilde{g}_1(x) = 3x$, $\tilde{g}_2 = \rho \circ \jmath \circ \imath$, where $\imath$ is the inversion in T_2, $\jmath$ is the reflection in the (x_1, x_3)-plane, and ρ is the rotation of π around the line $L = \{x_2 = 2,\ x_1 = 0\}$. We have again that $\tilde{g}_i(S_i) = S_i'$, and $\tilde{g}_i(D) \cap D = \emptyset$, $i = 1, 2$. The cycle of points of contact is $c = \{p_2, p_4 = \tilde{g}_2(p_2), p_3 = \tilde{g}_1^{-1}\tilde{g}_2(p_2), p_1 = \tilde{g}_2^{-1}\tilde{g}_1^{-1}\tilde{g}_2(p_2)\}$. The cyclic element is $\tilde{g}_c = \tilde{g}_1^{-1}\tilde{g}_2^{-1}\tilde{g}_1^{-1}\tilde{g}_2$.

Let $\tilde{\Gamma} = \langle \tilde{g}_1, \tilde{g}_2 \rangle$. Then $\tilde{\Gamma}$ leaves invariant the (x_1, x_2)-plane, which we again identify with the C-plane.

The action of the elements $\tilde{g}_i$ on this plane are given by: $\tilde{g}_1(z) = 3z$, and $\tilde{g}_2(z) = (2z+5)/(z+2)$. The action of $\tilde{g}_c$ is given by

$$\tilde{g}_c(z) = \frac{(11/3)z + (20/3)}{-4z - 7},$$

with $Tr^2(\tilde{g}_c) = 100/9 \neq 4$. We see that $\tilde{g}_c$ is loxodromic, and therefore the conditions of Poincaré's Polyhedron Theorem are not satisfied.

It follows from Maskit's result [10] that D is not a fundamental domain for $\tilde{\Gamma}$.

It is interesting to note that the limit set $L(\tilde{\Gamma})$ is a *Cantor set* lying on the x_2-axis completed by ∞ and $\tilde{\Gamma}$ does not contain parabolic elements. This also follows from Maskit [10]. In fact, $\tilde{\Gamma}$ is a Schottky group.

1.13 Bestvina-Coopers' example

In this section we outline Bestvina – Cooper's example [1].

Let K be the graph consisting of two disjoint simple closed curves K_1 and K_2 joined by an arc L, and embedded in $\mathbf{E}^3$ as in Figure 3. The arc L will be called a *bridge* of K.

Figure 3.

Consider a collection $S = \{T_1, T_2, \ldots\}$ of closed round balls placed along K so that adjacent balls touch in one point (see Figure 4.)

Figure 4.

Let $\phi: S \longrightarrow S$ be a fixed point-free involution such that:

- $\phi(T_1) = T_2$;

- along each circular part K_1, K_2 of K there are at least two balls T', T'' such that $\phi(T')$ and $\phi(T'')$ lie along L.

For each $T \in S$, choose a Möbius transformation $h_T: \mathbf{E}^3 \longrightarrow \mathbf{E}^3$ so that:

- $h_T(T) = \bar{\mathbf{E}}^3 \setminus \mathrm{int}(\phi(T))$;
- h_T maps the points of contact of T to the points of contact of $\phi(T)$;
- $h_{\phi(T)} = h_T^{-1}$.

Let G be the group generated by $\{h_T : T \in S\}$.
Then in [1] it has been concluded without proof that:

1. G is a free group of finite rank;
2. G acts freely and discontinuously in the complement of its limit set $L(G)$;
3. $D = S^3 \setminus (\bigcup_{T \in S} T)$ is a fundamental domain for G;
4. $L(G) = \bigcap_{n=0}^{\infty} S_n$, where $S_0 = S$, and $S_{n+1} = \bigcup_{T \in S} h_T(S_n)$.

But we have seen in section 1.12 that conclusions (3) and (4) are not true in general. Examples 1 and 2 show that it depends on the particular choice of a set $\{h_T : T \in S\}$ of side pairing transformations.

Observe that if T is the collection of the closed balls corresponding to the spheres in section 1.12.1 then in the first example $L(\Gamma) = \bigcap_{n=0}^{\infty} T_n$, while in the second one $L(\tilde{\Gamma})$ is a proper subset of $\bigcap_{n=0}^{\infty} T_n$, where the sets T_n are constructed by the same way as the sets S_n above.

Let us remark that (3) implies (4), so in order to construct a correct example, we need to find a set of side pairing transformations satisfying the conditions of Poincaré's Polyhedron theorem.

We will give a realization of Bestvina-Cooper's idea in section 3.

2 The first example

2.1 ST - groups with non - standard fundamental domain

2.1.1 In this section we construct an example of an ST-group acting on the plane with non-standard isometric fundamental domain.

We start with the description of the isometric circles of the generators. In what follows we identify the plane $\mathbf{E}^2$ with the complex plane $\mathbf{C}$ and write $z = x_1 + ix_2$.

The following table gives the centers of the isometric circles S_i and S_i'. All the circles S_j and S_j' have radius 1.

j	S_j	S'_j
1	-3	3
2	-5	-1
3	5	1
4	-7	7
5	-9	9
6	$-9+2i$	$9+2i$
7	$-9+4i$	$9+4i$

j	S_j	S'_j
8	$-9+6i$	$9+6i$
9	$-7+6i$	$7+6i$
10	$-5+6i$	$5+6i$
11	$-3+6i$	$3+6i$
12	$-3+4i$	$3+4i$
13	$-3+2i$	$3+2i$

Table 1: The centers of the isometric circles.

We define the Möbius transformations g_i as $g_i = p_i \circ q_i$, where q_i is the inversion in S_i, and p_i is the reflection in the bisector of the centers of S_i and S'_i. Then g_i is hyperbolic, and S_i and S'_i are the isometric circles of g_i and g_i^{-1} respectively.

Figure 5 shows all the isometric circles and the transformations g_i.

Figure 5.

Applying proposition 1.3 in section 1, we obtain that the matrices of the transformations are:

$$g_1 = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix}, \quad g_2 = \begin{pmatrix} -1 & -6 \\ 1 & 5 \end{pmatrix}, \quad g_3 = \begin{pmatrix} -1 & 6 \\ -1 & 5 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 7 & 48 \\ 1 & 7 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 9 & 80 \\ 1 & 9 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 9+2i & 84 \\ 1 & 9-2i \end{pmatrix},$$

$$g_7 = \begin{pmatrix} 9+4i & 96 \\ 1 & 9-4i \end{pmatrix}, \quad g_8 = \begin{pmatrix} 9+6i & 116 \\ 1 & 9-6i \end{pmatrix}, \quad g_9 = \begin{pmatrix} 7+6i & 84 \\ 1 & 7-6i \end{pmatrix},$$

$$g_{10} = \begin{pmatrix} 5+6i & 60 \\ 1 & 5-6i \end{pmatrix}, \quad g_{11} = \begin{pmatrix} 3+6i & 44 \\ 1 & 3-6i \end{pmatrix}, \quad g_{12} = \begin{pmatrix} 3+4i & 24 \\ 1 & 3-4i \end{pmatrix},$$

and

$$g_{13} = \begin{pmatrix} 3+2i & 12 \\ 1 & 3-2i \end{pmatrix}.$$

Let $\Gamma = \langle g_1, \ldots, g_{13} \rangle$ be the group generated by $g_1, \ldots, g_{13}$.

Let D be the complement of all closed discs bounded by the circles S_i and S'_i. Notice that D has three connected components D_1, D_2, D_3 (see Figure 6.)

Figure 6.

Our purpose now is to prove that D is a fundamental domain for Γ.

First we observe that $g_i(S_i) = S'_i$ and $g_i(D) \cap D = \emptyset$. Besides, g_i sends points of contact to points of contact. So the only hypothesis of Poincaré's Polyhedron Theorem we need to verify is that each cyclic transformation is parabolic.

Let us list all the cycles of points of contact. One easily sees that there are 12 cycles of points of contact as follows:

(1) The cycle $\{-6;6;0\}$ with the cyclic transformation

$$h_1 = g_2^{-1}g_3g_4 = \begin{pmatrix} -17 & -108 \\ 3 & 19 \end{pmatrix}.$$

(2) The cycle $\{-4;4;2;-2\}$ with the cyclic transformation

$$h_2 = g_2^{-1}g_1^{-1}g_3g_1 = \begin{pmatrix} -17 & -64 \\ 4 & 15 \end{pmatrix}.$$

(3) The cycles consisting of two points with the cyclic transformations $h_3 = g_{13}^{-1}g_1$, and $h_i = g_i^{-1}g_{i+1}$ for $4 \le i \le 12$.

We have two types (modulo changing coordinates):

(a) Type 1: S_i is centered at $-a$ and S_i' at a, and S_{i+1} at $-a-2$ and S_{i+1}' at $a+2$; $a > 0$. Then

$$g_i = \begin{pmatrix} a & a^2-1 \\ 1 & a \end{pmatrix}, \quad g_{i+1} = \begin{pmatrix} a+2 & (a+2)^2-1 \\ 1 & a+2 \end{pmatrix}.$$

and

$$h_i = g_i^{-1}g_{i+1} = \begin{pmatrix} 1+2a & -2(a+1)^2 \\ 2 & -3-2a \end{pmatrix}.$$

(b) Type 2: S_i is centered at $-a+i$, S_i' at $a+i$, and S_{i+1} at $-a-i$, and S_{i+1}' at $a-i$. Then

$$g_i = \begin{pmatrix} a+i & a^2 \\ 1 & a-i \end{pmatrix}, \quad g_{i+1} = \begin{pmatrix} a-i & a^2 \\ 1 & a+i \end{pmatrix}.$$

and

$$h_i = g_i^{-1}g_{i+1} = \begin{pmatrix} -1+2ai & -2a^2i \\ 2i & -1-2ai \end{pmatrix}.$$

We see that all the cyclic transformations h are parabolic. Therefore, we can conclude from Poincaré's Polyhedron Theorem that D is a fundamental polyhedron for Γ, and Γ is Kleinian . It is clear that Γ is free on these generators, i.e., $\Gamma = \langle g_1 \rangle * \ldots * \langle g_{13} \rangle$. In particular, the minimal number of generators is 13.

We now show that Γ constructed above is an ST-group.

Lemma 2.1 *The regular set $R(\Gamma)$ of Γ is connected.*

Proof: Consider the path $\alpha: [0,1] \longrightarrow \bar{D}$ connecting the points $p, q \in \partial \bar{D}_3$ as shown in Figure 6.

Let $\beta: [0,1] \longrightarrow \bar{\mathbf{C}}$ be the path $\beta = g_2^{-1} \circ \alpha$. Then it is clear that $\beta(t) \in R(\Gamma)$ for all $t \in [0,1]$, and $\beta(0) = p' \in \partial D_1$ and $\beta(1) = q' \in \partial D_3$. It follows that for any pair of points $x \in D_1$ and $y \in D_3$ there is a path in $R(\Gamma)$ connecting x and y. The same argument shows that for any pair of points $x' \in D_2$ and $y' \in D_3$ there is a path in $R(\Gamma)$ connecting x' and y'. Observing that

$$R(\Gamma) = \bigcup_{g \in \Gamma} g(\bar{D} \setminus \{\text{points of contact}\}).$$

we conclude that $R(\Gamma)$ is connected. □

Theorem 2.1 *Γ is an ST-group of type (1, 12).*

Proof: Observe that the natural extension of Γ to the action in $\mathbf{H}^3$, as follows from the construction, is a geometrically finite discrete subgroup of the isometry group of $\mathbf{H}^3$. Applying the lemma above and Theorem 6.2 in [8], we conclude that Γ has no totally degenerate groups as subgroups. Since $R(\Gamma)$ is connected, it is clear that Γ has no quasi-Fuchsian subgroups of the first kind. Then it follows from Proposition 5.8 in [8] that Γ is constructed by Klein's Combination Theorem from a finite number of elementary groups. Since Γ does not contain free abelian subgroups of rank 2, we obtain that Γ is an ST-group in the sense of our definition.

We also note that every maximal parabolic subgroup of Γ has rank 1, and that there are exactly 12 distinct conjugacy classes of such subgroups. Therefore, Γ is an ST-group of type (1, 12).

□

Corollary 2.1 *The limit set $L(\Gamma)$ of Γ is a Cantor set.*

The proof is contained in, e.g., [3].

Corollary 2.2 *For the group Γ constructed above, $S(\Gamma) = R(\Gamma)/\Gamma$ is a Riemannian surface of signature (1, 24), that is, $S(\Gamma)$ is compact Riemannian surface of genus 1 with 24 punctures.*

2.2 Let Γ be the group constructed in section 2.1. Consider the natural extension Γ^* of Γ to $\bar{\mathbf{E}}^3$. Let P be the spherical polyhedron in $\bar{\mathbf{E}}^3$ formed by the spheres spanning the circles S_i and S_i'. We will keep the same letters S_i and S_i' for denoting the sides of P. Using again Poincaré's Polyhedron Theorem, we obtain that P is a fundamental polyhedron for Γ^*. It is not difficult to show that Γ^* is an ST-group acting on $\bar{\mathbf{E}}^3$, and P is its non-standard isometric fundamental domain, but we will only need the fact that the limit set of Γ is a Cantor set.

2.2.1 Let K be the graph in the xy-plane, depicted in Figure 7. K has the centers of the spheres S_i and S_i' as its verteces; the edges of K are the straight segments connecting centers of adjacent spheres.

We will call the graph K a *spine* of the group Γ^*.

Figure 7.

2.3 Constructing FST–groups acting on E^3.

In this section we construct the first example of a Kleinian group acting on E^3 with the limit set a wild Cantor set.

2.3.1 Let Γ^* be the group built in section 2.2; P its fundamental polyhedron; and K its spine.

2.3.2 Take a Mobius transformation $h \in M(3)$. Let $\Gamma_h^* = h \circ \Gamma^* \circ h^{-1}$. Then $h(P)$ is a fundamental polyhedron for Γ_h^*.

One easily sees that one can find Mobius transformations h_1, h_2, h_3, h_4 from $M(3)$ such that the groups $\Gamma_i = h_i \circ \Gamma^* \circ h_i^{-1}$, $i = 1,2,3,4$ satisfy the following:

1). The polyhedra $P_i = h_i(P)$, $i = 1,2,3,4$, satisfy the conditions of Klein's Combination Theorem, that is, the complement of P_i in E^3 is contained in P_j, $i \neq j$;

2). The spines K_i of the groups Γ_i form the link as shown in Figure 8.

Figure 8.

Let $H = \langle \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \rangle$ be the group generated by Γ_i. Then it follows from Klein's Combination Theorem that H is a Kleinian group; $F = P_1 \cap P_2 \cap P_3 \cap P_4$ is a fundamental domain for H; $H = \Gamma_1 * \Gamma_2 * \Gamma_3 * \Gamma_4$. (See Figure 9.)

Figure 9.

Remark. One sees that the group H is of type (4, 48).

2.3.3 In section 4 we will prove that the limit set of the group H is a wild Cantor set, that is, H is a Fake Schottky type group.

3 The second example

Our second example of an FST-group acting on E^3 is closely related to the example of Bestvina and Cooper [1]. In fact, this is its correct version. The construction we offer is quite complicated and much harder than the first one. The main difficulties are in finding a suitable linear construction of the spine and a set of side pairing transformations in order to satisfy Poincaré's Polyhedron Theorem. It is reasonable to believe that this is not an ideal construction, and that different approaches could work better. On the other hand, many approaches that at first glance appear to be easy, do not work, and we believe that, in any case, the construction must be far from trivial.

3.1 We start with the description of the spine. Here we adopt the coordinates (x, y, z) for E^3.

Let K be the graph embedded in E^3 as shown in Figure 10. As in section 2.2.1, we call this graph the spine of the group.

Figure 10.

The coordinates of its vertices $p_i = (x_i, y_i, z_i)$, $p'_i = (x'_i, y'_i, z'_i)$ are given in Table 2 below. The length of each edge of K equals 2.

i	x_i	y_i	z_i	x'_i	y'_i	z'_i	map
1	−5	0	0	−1	0	0	b_1
2	5	0	0	1	0	0	b_2
3	−3	0	0	3	0	0	b_3
4	−7	0	0	7	0	0	c_4
5	−3	2	0	3	2	0	a_1
6	−3	4	0	3	4	0	a_2
7	−3	6	0	3	6	0	a_3
8	−3	8	0	3	8	0	a_4
9	−3	10	0	3	10	0	a_5
10	−5	10	0	5	10	0	a_6

i	x_i	y_i	z_i	x'_i	y'_i	z'_i	map
11	−7	10	0	7	10	0	c_1
12	15	0	0	21	0	0	c_3
13	15	2	0	21	2	0	a_{10}
14	15	4	0	21	4	0	a_9
15	15	6	0	21	6	0	a_8
16	15	8	0	21	8	0	a_7
17	15	10	0	21	10	0	c_2
18	−7	−4	0	15	0	−4	c_7
19	−5	−4	0	13	0	−4	M_2
20	−3	−4	0	11	2	−4	c_{10}

i	x_i	y_i	z_i	x'_i	y'_i	z'_i	map
21	−3	2	−4	11	2	−4	M_1
22	−3	4	−4	11	4	−4	c_8
23	−1	4	−4	9	4	−4	D_1
24	1	4	−4	5	4	−4	D_2
25	3	4	−4	7	4	−4	D_3
26	−7	10	−4	15	10	−4	c_5
27	−5	10	−4	13	10	−4	m_2
28	−3	10	−4	11	10	−4	c_9
29	−3	12	−4	11	12	−4	m_1
30	−3	14	−4	11	14	−4	c_6

i	x_i	y_i	z_i	x'_i	y'_i	z'_i	map
31	−1	14	−4	−1	14	−4	d_1
32	1	14	−4	−1	14	−4	d_2
33	3	14	−4	−1	14	−4	d_3
34	7	0	−4	−1	0	−4	A_{13}
35	7	10	−4	−1	10	−4	a_{13}
36	−7	0	−2	−1	0	−2	A_{11}
37	15	0	−2	−1	0	−2	A_{12}
38	7	0	−6	−1	0	−6	A_{14}
39	7	0	−8	−1	0	−8	c_{12}
40	9	0	−8	−1	0	−8	D_4

i	x_i	y_i	z_i	x'_i	y'_i	z'_i	map
41	11	0	−8	−1	0	−8	D_5
42	13	0	−8	−1	0	−8	D_6
43	−7	10	−2	−1	10	−2	a_{11}
44	15	10	−2	−1	10	−2	a_{12}
45	7	10	−6	−1	10	−6	a_{14}
46	7	10	−8	−1	10	−8	c_{11}
47	9	10	−8	−1	10	−8	d_4
48	11	10	−8	−1	10	−8	d_5
49	13	10	−8	−1	10	−8	d_6

Table 2

The centers of the isometric spheres S_i and S'_i, and the corresponding side pairing transformations.

It is easy to see that K is a linearization of the graph in Bestvina-Cooper's example. For instance, the bridge of K is the segment $[p_3, p'_3]$.

3.2 Consider a family of 2-spheres $T = \{S_i, S'_i : 1 \leq i \leq 49\}$, all of radius one, centered at the points p_i, p'_i respectively. One sees that adjacent spheres touch.

Let P be the complement in $\mathbf{E}^3$ of the union of all the closed balls bounded by the spheres S_i, S'_i.

Next we will define the side pairing transformations for P.

For each pair (S, S') from T define the Möbius transformation $h_S: \mathbf{E}^3 \longrightarrow \mathbf{E}^3$ as follows

$$h_S = J_s \circ I_S,$$

where I_S is the inversion in S, and J_S is the reflection in the bisector of the centers of S and S'. Then h_S is hyperbolic, and S and S' are the isometric spheres of h_S and h_S^{-1}, respectively. One can easily verify that for each S from T h_S maps the points of contact of S to those of S'. Table 2 provides also the notations of all the side pairing transformations. For instance, b_1 corresponds to the pair (S_1, S'_1). We denote the set of these side pairing transformations for P as W.

3.3 Let G be the group generated by these side pairing transformations.

We next prove that the group G is Kleinian, and that P is a fundamental domain for G.

To this end, we will list all the cycles of points of contact and verify that each cyclic transformation is parabolic.

First of all, we observe that each cycle of points of contact $c = \{c_0, \ldots, c_{k-1}\}$ lies in the same plane L_c as the centers of the isometric spheres of the transformations $g_1, \ldots, g_k$, where $h = g_k \circ \ldots \circ g_1$ is the cyclic transformation related to the cycle c. Since $g_1, \ldots, g_k$ are hyperbolic, the plane L_c is invariant under $g_1, \ldots, g_k$.

Note that any Möbius transformation from M(3) is parabolic if and only if its restriction to any invariant plane is parabolic as an element of M(2). Thus, we can use convenient coordinates in each such plane to verify whether the cyclic transformations are parabolic or not.

3.3.1 The cycles with cyclic transformations $h_1 = b_3^{-1} b_2^{-1} b_3 b_1$ and $h_2 = b_1^{-1} b_2 b_3$ have the same structure as the analogues cycles in the first example, where it was verified that h_1 and h_2 are parabolic; see (1) and (2) in section 2.1.1.

3.3.2 Let us now consider the two point cycles. The cyclic transformations in this case are of the form $h = g_1 g_2^{-1}$, where g_1 and g_2 are transformations from W.

Let L be a plane invariant under g_1 and g_2 passing through the centers of the isometric spheres of g_1 and g_2. We identify L with the complex plane $\mathbf{C}$ and call the intersection $L \cap P$ a *slice* of P

corresponding to L. All the slices we need are shown in Figures 11 – 15. The arrows show the side pairing transformations.

Figures 11–15.

Now we are going to write down the matrices of the restrictions of the elements g_1, g_2, and h to the corresponding invariant plane L.

We have the following cases to consider.

For the first case, we have:

i	S_i	S'_i	map
1	$-a$	a	g_1
2	$-a-2$	$a+2$	g_2

The first case: Here $a > 0$.

Hence

$$g_1 = \begin{pmatrix} a & a^2-1 \\ 1 & a \end{pmatrix}, \quad g_2 = \begin{pmatrix} a+2 & (a+2)^2-1 \\ 1 & a+2 \end{pmatrix}.$$

and

$$h = g_1 g_2^{-1} = \begin{pmatrix} 1+2a & 2(a+1)^2 \\ 2 & -3-2a \end{pmatrix},$$

which is parabolic.

Below we list the pairs of the transformations (g_1, g_2) and the planes L invariant under g_1 and g_2 corresponding to this case (see slices).

- $L = \{z = 0\}$; (a_6, c_1), (a_5, a_6).
- $L = \{z = -4\}$; (m_2, c_5), (d_1, c_6), (c_9, m_2), (M_2, c_7), (c_{10}, M_2), (D_1, c_8).
- $L = \{z = -8\}$; (d_4, c_{11}), (D_4, c_{12}).

For the second case, we have:

i	S_i	S'_i	map
1	$-a+i$	$a+i$	g_1
2	$-a-i$	$a-i$	g_2

The second case: Here $a > 0$

Hence

$$g_1 = \begin{pmatrix} a+i & a^2 \\ 1 & a-i \end{pmatrix}, \quad g_2 = \begin{pmatrix} a-i & a^2 \\ 1 & a+i \end{pmatrix}.$$

and

$$h = g_1 g_2^{-1} = \begin{pmatrix} -1+2ai & -2a^2 i \\ 2i & -1-2ai \end{pmatrix},$$

which is parabolic.

This corresponds to the following pairs (g_1, g_2) and the invariant planes L:

- $L = \{z = 0\}$; (a_1, b_3), (a_2, a_1), (a_3, a_2), (a_4, a_3), (a_5, a_4), (c_2, a_7), (a_7, a_8), (a_8, a_9), (a_9, a_{10}), (a_{10}, c_3).
- $L = \{z = -4\}$; (m_1, c_9), (c_6, m_1), (M_1, c_{10}), (c_8, M_1).
- $L = \{y = 10\}$; (c_{11}, a_{14}), (a_{14}, a_{13}), (a_{12}, c_2), (a_{11}, c_1).
- $L = \{y = 0\}$; (c_{12}, A_{14}), (A_{14}, A_{13}), (A_{12}, c_3), (A_{11}, c_4).

3.3.3 Consider now the four point cycles. We apply the same procedure as in section 3.3.2. The cyclic transformations in this case have the form $h = g_3^{-1} g_2^{-1} g_4 g_1$, where:

i	S_i	S_i'	map
1	$a+i$	$b+i$	g_1
2	$c+i$	$d+i$	g_2
3	$a-i$	$c-i$	g_3
4	$b-i$	$d-i$	g_4

Here $a, b, c, d \in \mathbb{R}$.

The matrices of g_i are:

$$g_1 = \begin{pmatrix} b+i & -ab-(a+b)i \\ 1 & -a-i \end{pmatrix}, \quad g_2 = \begin{pmatrix} d+i & -cd-(c+d)i \\ 1 & -c-i \end{pmatrix},$$

$$g_3 = \begin{pmatrix} c-i & -ac+(a+c)i \\ 1 & -a+i \end{pmatrix}, \quad g_4 = \begin{pmatrix} d-i & -bd+(b+d)i \\ 1 & -b+i \end{pmatrix}.$$

Hence

$h = g_3^{-1} g_2^{-1} g_4 g_1 = \begin{pmatrix} 1+4ai & -4a^2 i \\ 4i & 1-4ai \end{pmatrix}$, which is parabolic.

This corresponds to the following 4-tuples (g_1, g_2, g_3, g_4) and the invariant planes L:

- $L = \{y = 0\}$; $(c_7, A_{13}, A_{11}, A_{12})$.
- $L = \{y = 10\}$; $(c_5, a_{13}, a_{11}, a_{12})$.

3.3.4 Finally, we have the five point cycles. In this case the cyclic transformations are of the form $h = g_2^{-1} g_3 g_2 g_3^{-1} g_1 4$, where we have the following:

i	S_i	S'_i	map
1	-5	5	g_1
2	-3	1	g_2
3	-1	3	g_3

with

$$g_1 = \begin{pmatrix} 5 & 24 \\ 1 & 5 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

Thus, we obtain that

$$h = \begin{pmatrix} -19 & -80 \\ 5 & 21 \end{pmatrix},$$

which is parabolic.

This corresponds to the following 3-tuples (g_1, g_2, g_3) and the invariant planes L:

- $L = \{z = -4\}$; (d_1, d_2, d_3), (D_1, D_2, D_3).
- $L = \{z = -8\}$; (d_4, d_5, d_6), (D_4, D_5, D_6).

3.4 It is seen that we have listed all the cycles of points of contact. We have also verified that all the cyclic transformations are parabolic. Therefore, it follows from Poincaré's Polyhedron Theorem that the group G is Kleinian, and P is its fundamental domain. In the next section we will prove the the limit set of the group G is a wild Cantor set, that is, G is an FST-group.

Remark. It follows from Theorem 1.2 that the group G is of type (9,40).

4 The groups H and G are Fake Schottky Type groups

In this section we show that both Kleinian groups H and G constructed in sections 2 and 3 have wild Cantor set limit sets.

4.1 We recall that a Kleinian group $\Gamma \subset M(n)$ is said to be *geometrically finite* if it has a hyperbolic fundamental domain in $\mathbb{H}^{n+1}$ with a finite number of sides.

4.2 Let Γ and Γ' be Kleinian groups. We say that an isomorphism $\phi\colon \Gamma \longrightarrow \Gamma'$ is *type preserving* if it carries parabolic elements of Γ bijectively onto parabolic elements of Γ'.

We will need the following theorem.

Theorem 4.1 (Tukia [11]) *Let Γ and Γ' be geometrically finite Kleinian groups. Let $\phi\colon \Gamma \longrightarrow \Gamma'$ be a type preserving isomorphism. Then there is an homeomorphism $f_\phi\colon L(\Gamma) \longrightarrow L(\Gamma')$ of the limit sets inducing ϕ.*

Corollary 4.1 *Let Γ be a geometrically finite Kleinian group. Assume that there is a type preserving isomorphism $\phi\colon \Gamma \longrightarrow \Gamma'$, where Γ' is a non-elementary ST-group. Then the limit set $L(\Gamma)$ of the group Γ is a Cantor set.*

4.3

Proposition 4.1 *The limit set of the group H constructed in section 2 is a Cantor set.*

Proof: Let Γ be the Kleinian group constructed in section 2.1.1. Consider the groups $\tilde\Gamma_1 = f_1\Gamma f_1^{-1}$, $\tilde\Gamma_2 = f_2\Gamma f_2^{-1}$, $\tilde\Gamma_3 = f_3\Gamma f_3^{-1}$, $\tilde\Gamma_4 = f_4\Gamma f_4^{-1}$, where $f_i \in \mathbf{PSL}(2, \mathbf{C})$. One easily sees that we can choose the elements f_i in such a way that the fundamental domains $\tilde F_i = f_i(D)$ of the groups $\tilde\Gamma_i$ are located as in Figure 16.

Figure 16.

Then applying Klein's Combination Theorem, we obtain that $\tilde\Gamma = \langle \tilde\Gamma_1, \tilde\Gamma_2, \tilde\Gamma_3, \tilde\Gamma_4\rangle$ generated by $\tilde\Gamma_i$ is a Kleinian group. Its fundamental domain $\tilde F = \bigcap_{i=1} \tilde F_i$ is the complement of all the closed discs bounded by the circles shown in Figure 16.

The same argument as in section 2.2 show that $\tilde\Gamma$ is an ST-group. Therefore, in particular, the limit set $L(\tilde\Gamma)$ of $\tilde\Gamma$ is a Cantor set.

We know that $\Gamma = \langle g_1, \ldots, g_{13}\rangle$ (see section 2.1). Take the following generators of the group $\tilde\Gamma$:

- $f_1(g_1)f_1^{-1}, \ldots, f_1(g_{13})f_1^{-1}$;
- $f_2(g_1)f_2^{-1}, \ldots, f_2(g_{13})f_2^{-1}$;
- $f_3(g_1)f_3^{-1}, \ldots, f_3(g_{13})f_3^{-1}$;
- $f_4(g_1)f_4^{-1}, \ldots, f_4(g_{13})f_4^{-1}$.

Let us denote them as $a_{ij} = f_i(g_j)f_i^{-1}$.

Now let us consider the natural extension of $\tilde\Gamma$ to the action on $\bar{\mathbf{E}}^3$ and keep old notations for the group and its generators.

Recall that the group H constructed in section 2.3 looks like $H = \langle \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\rangle$, where $\Gamma_i = h_i\Gamma^* h_i^{-1}$, $i = 1, 2, 3, 4$..

Take the following generators of the group H:

- $h_1(g_1^*)h_1^{-1}, \ldots, h_1(g_{13}^*)h_1^{-1}$;
- $h_2(g_1^*)h_2^{-1}, \ldots, h_2(g_{13}^*)h_2^{-1}$;
- $h_3(g_1^*)h_3^{-1}, \ldots, h_3(g_{13}^*)h_3^{-1}$;
- $h_4(g_1^*)h_4^{-1}, \ldots, h_4(g_{13}^*)h_4^{-1}$;

where g_i^*, $i = 1, 2, \ldots, 13$ are the generators of Γ^* (see section 2.3).

We let denote these generators as $b_{ij} = h_i(g_j^*)h_i^{-1}$.

It is easy to see that the cycles of points of contact and the cyclic transformations of the groups H and $\bar{\Gamma}$ have the same structure. By applying Theorem 1.2, we obtain that the assignment

$$\phi : a_{ij} \mapsto b_{ij},$$

$i = 1, 2, 3, 4$, $j = 1, 2, \ldots, 13$, defines a type preserving isomorphism $\phi: \bar{\Gamma} \longrightarrow H$. Then Corollary 4.1 implies that the limit set $L(H)$ of the group H is a Cantor set.

4.3.1

Proposition 4.2 *The limit set of the group G constructed in section 3 is a Cantor set.*

Proof: Let us consider the group G' acting on the plane generated by the hyperbolic transformations shown in Figure 17. This figure also shows the isometric circles of all these generators. As usual, each generator h' of G' is the composition $J \circ I$, where I is the inversion in the isometric circle of h' and J is the reflection in the bisector of the centers of the isometric circles of h' and $(h')^{-1}$.

By the same arguments as in section 2, we conclude that the complement of all the closed discs bounded by the circles shown in this figure is a fundamental domain for G', and that G' is an ST-group. Also, one can verify that G' is a free group of the same rank as G, and that the cycles of points of contact and the cyclic transformations of the groups G and G' have the same structure. Now we can finish the proof following the same lines as in the proof of proposition 4.1.

Figure 17.

4.4 In this section we show that the limit sets of the groups H and G are wild Cantor sets.

First of all, we recall the following well-known fact.

Proposition 4.3 *If $L \subset \bar{\mathbb{E}}^3$ is a tame Cantor set, then $\bar{\mathbb{E}}^3 \setminus L$ is simply connected.*

Thus, if we establish that the regular sets $R(H)$ and $R(G)$ are not simply connected, we will obtain the result we need.

4.4.1 We start with the group H.

Proposition 4.4 *Let S be a side of F, where F is the fundamental domain for H constructed in section 2.3.2. Then the inclusion $S \subset \bar{F}$ ($\bar{F}$ is the closure of F in $R(H)$) induces a monomorphism $\pi_1(S) \to \pi_1(\bar{F})$.*

Proof: Consider the graph K_H formed by the spines K_i of the groups Γ_i (see section 2.3.2 and Figure 8), and let us compute the fundamental group of $\bar{E}^3 \setminus K_H$. To this end, consider a projection K'_H of K_H into the plane L which is in general position with respect to K_H. To K'_H we associate arrows whose directions are shown in Figure 18.

Figure 18.

We designate them with letters a_i, b_i, c_i, d_i, β_i, $i = 1,2,3,4$.

Then, by applying a standard procedure for writing down a presentation of the fundamental group of a graph (see, for instance, Bing [2]), we obtain that the group $\pi_1(\bar{E}^3 \setminus K_H)$ has the following presentation.

Generators: a_i, b_i, c_i, d_i, β_i, $i = 1,2,3,4$.

Relations:

(a) at branching points:

$\beta_1 a_3 a_1^{-1}$, $\beta_4 a_4 a_2^{-1}$,
$\beta_1 b_3 b_1^{-1}$, $\beta_2 b_4 b_2^{-1}$,
$\beta_3 c_3 c_1^{-1}$, $\beta_2 c_4 c_2^{-1}$,
$\beta_3 d_3 d_1^{-1}$, $\beta_4 d_4 d_2^{-1}$.

(b) at crossing points:

$a_1 a_4 a_1^{-1} a_2^{-1}$, $a_3 a_4 a_1^{-1} a_4^{-1}$,
$b_1 b_4 b_1^{-1} b_2^{-1}$, $b_3 b_4 b_1^{-1} b_4^{-1}$,
$c_1 c_4 c_1^{-1} c_2^{-1}$, $c_3 c_4 c_1^{-1} c_4^{-1}$,
$d_1 d_4 d_1^{-1} d_2^{-1}$, $d_3 d_4 d_1^{-1} d_4^{-1}$,

Now we are going to show that all the elements a_i, b_i, c_i, d_i, β_i are non-trivial.

To prove this, let us consider the group A having the following presentation:

$$A = \{x_1, x_2, x_3, x_4, y : y x_3 x_1^{-1} = y x_4 x_2^{-1} = x_1 x_4 x_1^{-1} x_2^{-1} = x_3 x_4 x_1^{-1} x_4^{-1} = 1\}.$$

We may simplify this presentation.

From the relations $x_1 x_4 x_1^{-1} x_2^{-1} = x_3 x_4 x_1^{-1} x_4^{-1} = 1$, we get $x_2 = x_1 x_4 x_1^{-1}$ and $x_3 = x_4 x_1 x_4^{-1}$. Then the relations $y x_3 x_1^{-1} = y x_4 x_2^{-1} = 1$ are equivalent to $y = [x_1, x_4]$. That is, the group A is free on the generators x_1 and x_4. Observe that x_2, x_3 and y are nontrivial elements of A.

Consider the map of $\pi_1(\bar{E}^3 \setminus K)$ onto the group A given for all i by :

$$a_i,\ b_i,\ c_i,\ d_i \mapsto x_i, \quad \beta_i \mapsto y.$$

One sees that this map defines a homomorphism. We have that all the elements a_i, b_i, c_i, d_i, β_i are mapped to nontrivial elements of A and, therefore, are non-trivial.

Remark: It is clear that $\beta_1 = \beta_2 = \beta_3 = \beta_4$, see Figure 18. Therefore, the group $\pi_1(\bar{E}^3 \setminus K_H)$ has generators $a_1, a_4, b_1, b_4, c_1, c_4, d_1, d_4$, and relations $[a_1, a_4] = [b_1, b_4] = [c_1, c_4] = [d_1, d_4]$.

Consider now the boundary of $\bar{F}$. We see that each component of $\partial \bar{F}$ is either a 2-punctured or a 3-punctured sphere. Also, note that each non-trivial simple loop on $\partial \bar{F}$ is homotopic to a small linking circle around the edge of K_H, and, hence, its homotopy class is an element from the set $\{a_i^{\pm}, b_i^{\pm}, c_i^{\pm}, d_i^{\pm}\}$. It proves that for each component $S \subset \partial \bar{F}$ the inclusion homomorphism $\pi_1(S) \to \pi_1(\bar{F})$ is injective.

Proposition 4.5 *The regular set $R(H)$ of the group H is not simply connected.*

Proof: We know that $R(H) = \bigcup_{\gamma \in H} \gamma(\bar{F})$. The pair $(\gamma(\bar{F}), \gamma(\partial \bar{F}))$ is homeomorphic to $(\bar{F}, \partial \bar{F})$. Therefore, for each component $S_\gamma \subset \gamma(\partial \bar{F})$, $\pi_1(S_\gamma) \to \pi_1(\gamma(\bar{F}))$ is a monomorphism. Thus, $R(H)$ is the union of manifolds with incompressible boundary glued along their boundaries. Using Van Kampen's Theorem and an easy induction, we obtain that $\pi_1(R(H))$ is a non-trivial group; moreover $\pi_1(R(H))$ is infinitely generated.

Summarizing, we have the following theorem.

Theorem 4.2 *The limit set of the group H is a wild Cantor set.*

4.4.2 In this section we consider the group G.

Theorem 4.3 *The limit set of the group G is a wild Cantor set.*

Proof: First of all, we note that the fundamental group of $\bar{E}^3 \setminus K$ is isomorphic to the group A in Proposition 4.4, where K is the spine of the group G, see section 3.1. In particular, a small linking circle around the bridge of K represents the commutator $\beta = [b, c]$. Therefore, following the lines in section 4.4.1, we obtain the proof of the theorem.

4.5 In this section we present another proof of the fact that the limit sets of the groups H and G are wild Cantor sets. Besides, this result will be used in section 5.

4.5.1 We start with recalling the following.

Theorem 4.4 *Let L be a tame Cantor set in $\bar{E}^3$. Then $\bar{E}^3 \setminus L$ is not aspheric in dimension 2, that is, $\pi_2(\bar{E}^3 \setminus L) \neq 0$.*

Proof: This property actually is always true of a Cantor set L in $\bar{E}^3$ with simply connected complement. One then can apply the Hurewicz Isomorphism Theorem and the Sphere Theorem.

Theorem 4.5 (J.H.C. Whitehead [12]) *Let $P = P_1 \cup P_2$, $P_{12} = P_1 \cap P_2$, where P, P_1 and P_2 are connected polyhedra, and suppose that*

1. $\pi_2(P_i) = 0$, $i = 1,2$;

2. *any loop in P_{12} which is homotopic to a point in P_1 or in P_2 is homotopic to a point in P_{12}.*

Then $\pi_2(P) = 0$.

4.5.2

Theorem 4.6 *The regular sets $R(H)$ and $R(G)$ of the groups H and G are aspheric in dimension 2.*

Proof: The proof follows immediately by induction from the results in sections 4.4.1, 4.4.2 and Theorem 4.5. For instance, for the group H, we have that $R(H) = \bigcup_{\gamma\in H}\gamma(\bar{F})$, and we have already proved that ∂F is incompressible in $\bar{F}$; besides, $\bar{F}$ is aspheric because of the Sphere Theorem.

4.6 In this section we compare the fundamental and the 1-homology groups of the manifold $M(H) = R(H)/H$ and the manifold $M(\tilde{\Gamma}) = R(\tilde{\Gamma})/\tilde{\Gamma}$, where $\tilde{\Gamma}$ is the ST–group constructed in Proposition 4.1. Also, we compare the manifolds $M(G)$ and $M(G')$, where G and G' are the groups from Proposition 4.2.

We start with the groups H and $\tilde{\Gamma}$. Recall that $\tilde{\Gamma}$ is an ST-group, while H is an FST-group. It has been already verified that H and $\tilde{\Gamma}$ have the same type. It follows from results in sections 2.2 and 4.2.3 that the group $\tilde{\Gamma}$ is of type (4,48).

4.6.1 Since $R(\tilde{\Gamma})$ is simply connected, the fundamental group $\pi_1(M(\tilde{\Gamma}))$ is isomorphic to $\tilde{\Gamma}$. It implies that $\pi_1(M(\tilde{\Gamma}))$ is a free group of rank 52, and $H_1(M(\tilde{\Gamma}), \mathbb{Z})$ is a free abelian group of rank 52.

4.6.2 Now let us consider the manifold $M(H)$.

$M(H)$ is a 3-manifold that can be obtained from the closure of the fundamental domain F by glueing the equivalent points on the boundary ∂F of $\bar{F}$. We have already proved that for each component $S \subset \partial F$ the homomorphism $\pi_1(S) \to \pi_1(\bar{F})$ is a monomorphism, therefore, $\pi_1(M(H))$ is an HNN–extension of the fundamental group of F.

Recall that $\pi_1(F)$ has the following presentation.

Generators: a_i, b_i, c_i, d_i, β_i, $i = 1,2,3,4$.

Relations:

(a) $\beta_1 a_3 a_1^{-1}$, $\beta_4 a_4 a_2^{-1}$, $\beta_1 b_3 b_1^{-1}$, $\beta_2 b_4 b_2^{-1}$, $\beta_3 c_3 c_1^{-1}$, $\beta_2 c_4 c_2^{-1}$, $\beta_3 d_3 d_1^{-1}$, $\beta_4 d_4 d_2^{-1}$.

(b) $a_1a_4a_1^{-1}a_2^{-1}$, $a_3a_4a_1^{-1}a_4^{-1}$, $b_1b_4b_1^{-1}b_2^{-1}$, $b_3b_4b_1^{-1}b_4^{-1}$, $c_1c_4c_1^{-1}c_2^{-1}$, $c_3c_4c_1^{-1}c_4^{-1}$, $d_1d_4d_1^{-1}d_2^{-1}$, $d_3d_4d_1^{-1}d_4^{-1}$.

Let $a_{ij} = h_i(g_j^*)h_i^{-1}$ $(i = 1,2,3,4,\ 1 \le j \le 13)$ be the generators of H as in section 4.2.3. Let γ_{ij} be a path in $\bar{F}$ connecting equivalent points on the sides S_{ij} and S'_{ij} which are equivalent under a_{ij}. Consider the natural projection $\bar{F} \xrightarrow{p} M(H)$. Then the image $p(\gamma_{ij})$ is a loop in $M(H)$.

We denote $p(\gamma_{1j})$ as A_j, $p(\gamma_{2j})$ as B_j, $p(\gamma_{3j})$ as C_j, $p(\gamma_{4j})$ as D_j. We also denote $p(a_i)$, $p(b_i)$, $p(c_i)$, $p(d_i)$ and $p(\beta_i)$ as $\bar{a}_i$, $\bar{b}_i$, $\bar{c}_i$, $\bar{d}_i$, $\bar{\beta}_i$, respectively.

Then we have the following presentation of $\pi_1(M(H))$.

Generators: $\bar{a}_i$, $\bar{b}_i$, $\bar{c}_i$, $\bar{d}_i$, $\bar{\beta}_i$, $i = 1,2,3,4$; A_j, B_j, C_j, D_j, $1 \le j \le 13$.

The relations are devided into two groups:

(1) Old relations which come from $\pi_1(F)$:

(a) $\bar{\beta}_1\bar{a}_3\bar{a}_1^{-1}$, $\bar{\beta}_4\bar{a}_4\bar{a}_2^{-1}$, $\bar{\beta}_1\bar{b}_3\bar{b}_1^{-1}$, $\bar{\beta}_2\bar{b}_4\bar{b}_2^{-1}$, $\bar{\beta}_3\bar{c}_3\bar{c}_1^{-1}$, $\bar{\beta}_2\bar{c}_4\bar{c}_2^{-1}$, $\bar{\beta}_3\bar{d}_3\bar{d}_1^{-1}$, $\bar{\beta}_4\bar{d}_4\bar{d}_2^{-1}$.

(b) $\bar{a}_1\bar{a}_4\bar{a}_1^{-1}\bar{a}_2^{-1}$, $\bar{a}_3\bar{a}_4\bar{a}_1^{-1}\bar{a}_4^{-1}$, $\bar{b}_1\bar{b}_4\bar{b}_1^{-1}\bar{b}_2^{-1}$, $\bar{b}_3\bar{b}_4\bar{b}_1^{-1}\bar{b}_4^{-1}$, $\bar{c}_1\bar{c}_4\bar{c}_1^{-1}\bar{c}_2^{-1}$, $\bar{c}_3\bar{c}_4\bar{c}_1^{-1}\bar{c}_4^{-1}$, $\bar{d}_1\bar{d}_4\bar{d}_1^{-1}\bar{d}_2^{-1}$, $\bar{d}_3\bar{d}_4\bar{d}_1^{-1}\bar{d}_4^{-1}$.

(2) New relations which come from the identifications of the sides of $\bar{F}$:

- $\bar{a}_2 = A_2\bar{\beta}A_2^{-1}$, $\bar{d}_2 = A_3\bar{\beta}A_3^{-1}$, $\bar{\beta} = A_1\bar{\beta}A_1^{-1}$, $\bar{a}_2 = A_1\bar{d}_2A_1^{-1}$, $\bar{a}_4 = A_1\bar{d}_4A_1^{-1}$, $\bar{a}_2 = A_k\bar{d}_2A_k^{-1}$ $(k = 4,5,6)$, $\bar{a}_4 = A_k\bar{d}_4A_k^{-1}$ $(7 \le k \le 13)$;
- $\bar{b}_1 = B_2\bar{\beta}B_2^{-1}$, $\bar{a}_1 = B_3\bar{\beta}B_3^{-1}$, $\bar{\beta} = B_1\bar{\beta}B_1^{-1}$, $\bar{b}_1 = B_1\bar{a}_1B_1^{-1}$, $\bar{b}_3 = B_1\bar{a}_3B_1^{-1}$, $\bar{b}_1 = B_k\bar{a}_1B_k^{-1}$ $(4 \le k \le 9)$, $\bar{b}_3 = B_k\bar{a}_3B_k^{-1}$ $(10 \le k \le 13)$;
- $\bar{c}_2 = C_2\bar{\beta}C_2^{-1}$, $\bar{b}_2 = C_3\bar{\beta}C_3^{-1}$, $\bar{\beta} = C_1\bar{\beta}C_1^{-1}$, $\bar{c}_2 = C_1\bar{b}_2C_1^{-1}$, $\bar{c}_4 = C_1\bar{b}_4C_1^{-1}$, $\bar{c}_2 = C_k\bar{b}_2C_k^{-1}$ $(k = 4,5,6)$, $\bar{c}_4 = C_k\bar{b}_4C_k^{-1}$ $(7 \le k \le 13)$;
- $\bar{d}_1 = D_2\bar{\beta}D_2^{-1}$, $\bar{c}_1 = D_3\bar{\beta}D_3^{-1}$, $\bar{\beta} = D_1\bar{\beta}D_1^{-1}$, $\bar{d}_1 = D_1\bar{c}_1D_1^{-1}$, $\bar{d}_3 = D_1\bar{c}_3D_1^{-1}$, $\bar{d}_1 = D_k\bar{c}_1D_k^{-1}$ $(4 \le k \le 9)$, $\bar{d}_3 = D_k\bar{c}_3D_k^{-1}$ $(10 \le k \le 13)$;

Here $\beta = \beta_1 = \beta_2 = \beta_3 = \beta_4$.

To find a presentation of the group $H_1(M(H), \mathbf{Z})$, we recall that this group is the abelianization of $\pi_1(M(H))$. Therefore, from the relations above together with all the commutator relations needed we can deduce the following relations:

$$1 = \bar{\beta} = \bar{a}_i = \bar{b}_i = \bar{c}_i = \bar{d}_i, \quad i = 1,2,3,4;$$

that implies that $H_1(M(H), \mathbf{Z})$ is free abelian of rank 52 generated by the letters A_i, B_i, C_i and D_i.

4.6.3 For the manifold $M(G)$, we have the following presentation of the fundamental group.

As in the previous example, we first compute the fundamental group $\pi_1(P)$ of the fundamental domain P of the group G. It has the following presentation.

Generators: a, b, c, d, e, f, g, h, i, j.

Relations: $acb^{-1} = aed^{-1} = 1$; $fcb^{-1}c^{-1} = 1$; $geh^{-1}e^{-1} = ged^{-1}e^{-1} = 1$; $ecj^{-1}c^{-1} = icj^{-1}c^{-1} = ich^{-1}c^{-1} = 1$.

It is an easy exercise to verify that this presentation can be reduced to one with the generators c and e and with no relations.

To find a presentation of $\pi_1(M(G))$, we follow the same procedure as before. We will denote the image of the loops generating $\pi_1(F)$ under the natural projection map $F \to M(G)$ by the same letters. We denote by γ_i a path in $\bar{F}$ connecting equivalent points of the isometric spheres S_i and S_i', $1 \leq i \leq 49$. Table 3 gives the loops on S_k, S_k' and the corresponding side pairing transformations.

k	S_k	map	S_k'
1	b	b_1	a
2	d	b_2	a
3	a	b_3	a
3	b	b_3	d
3	c	b_3	e
4	b	c_4	d
5	c	a_1	e
6	c	a_2	e
7	c	a_3	e
8	c	a_4	e
9	c	a_5	e
10	c	a_6	e

k	S_k	map	S_k'
11	c	c_1	e
12	c	c_2	j
13	c	a_7	j
14	c	a_8	j
15	c	a_9	j
16	c	a_{10}	j
17	c	c_3	j
18	e	a_{13}	j
19	c	c_5	c
20	c	m_2	c

k	S_k	map	S_k'
21	c	c_9	c
22	c	m_1	c
23	c	c_6	c
24	c	d_1	c
25	c	d_2	c
26	c	d_3	c
27	d	A_{13}	j
28	b	c_7	c
29	b	M_2	c
30	b	c_{10}	c

k	S_k	map	S_k'
31	b	M_1	c
32	b	c_8	c
33	b	D_1	c
34	f	D_3	c
35	f	D_2	f
36	e	c_{11}	j
37	e	d_4	j
38	e	d_5	j
39	e	d_6	j
40	g	c_{12}	i

k	S_k	map	S_k'
41	g	D_4	i
42	h	D_5	h
43	h	D_6	i
44	b	A_{11}	d
45	c	A_{12}	j
46	g	A_{14}	j
47	c	a_{11}	e
48	c	a_{12}	j
49	e	a_{14}	j

Table 3: Loops on S_k and S_k' and the corresponding side pairing transformations.

We denote also by γ_i the loop in $M(G)$ which is the image under the natural projection $F \to M(G)$ of the path γ_i in $\bar{F}$. So $\pi_1(M(G))$ has the following presentation.

Generators: a, b, c, d, e, f, g, h, i, j, and γ_k ($1 \leq k \leq 49$). (See Table 3 relating the loops γ_k to the elements of G and the loops $a, \ldots, j$.)

The relations are:

(1) Old relations coming from the fundamental group of the fundamental domain:
$acb^{-1} = aed^{-1} = 1$; $fcb^{-1}c^{-1} = 1$; $geh^{-1}e^{-1} = ged^{-1}e^{-1} = 1$; $ecj^{-1}c^{-1} = icj^{-1}c^{-1} = ich^{-1}e^{-1} = 1$.

(2) New relations coming from group identifications of the sides of the fundamental domain:
$b = \gamma_1 a \gamma_1^{-1}$, $d = \gamma_2 a \gamma_2^{-1}$, $a = \gamma_3 a \gamma_3^{-1}$,
$b = \gamma_3 d \gamma_3^{-1}$, $c = \gamma_3 e \gamma_3^{-1}$,
$c = \gamma_k e \gamma_k^{-1}$, $k = 5, 6, 7, 8, 9, 10, 11, 47$;
$c = \gamma_k j \gamma_k^{-1}$, $k = 12, 13, 14, 15, 16, 17, 45, 48$;
$c = \gamma_k c \gamma_k^{-1}$, $k = 19, 20, 21, 22, 23, 24, 25, 26$;
$e = \gamma_k j \gamma_k^{-1}$, $k = 18, 36, 37, 38, 39, 49$;
$b = \gamma_k c \gamma_k^{-1}$, $k = 28, 29, 30, 31, 32, 33$;
$g = \gamma_k i \gamma_k^{-1}$, $k = 40, 41$;
$b = \gamma_k d \gamma_k^{-1}$, $k = 4, 44$;
$d = \gamma_{27} j \gamma_{27}^{-1}$, $f = \gamma_{34} c \gamma_{34}^{-1}$, $f = \gamma_{35} f \gamma_{35}^{-1}$, $g = \gamma_{46} j \gamma_{46}^{-1}$,
$h = \gamma_{42} h \gamma_{42}^{-1}$, $h = \gamma_{43} i \gamma_{43}^{-1}$.

Observe that the letters a, b, c, d, e, f, g, h, i and j and the relations (1) above can be reduced to the letters c and e and no relations, because we can deduce the relations $a = c^{-1}ece^{-1}$, $b = c^{-1}ece^{-1}c$, $d = h = j = c^{-1}ec$, $f = ece^{-1}$, $g = ec^{-1}ece^{-1}$, and $i = e$.

To compute $H_1(M(G), \mathbf{Z})$, we have to add the commutator relations to the ones above. From these and from $c = \gamma_5 e \gamma_5^{-1}$, we get $c = e$. From $b = c^{-1}ece^{-1}c$, we get $b = a$. From $b = \gamma_1 a \gamma_1^{-1}$, we get $b = a$. But from $a = c^{-1}ecc^{-1}$, we get $a = 1$. This implies that $H_1(M(G), \mathbf{Z})$ is free abelian of rank 49. It can be presented as a free abelian group generated by the letters γ_k ($1 \leq k \leq 49$).

Remark. Let A be an ST-group of the type (r,s) acting on $\bar{E}^3$. Then the manifold $M(A) = R(A)/A$ is homeomorphic to the connected sum of r Hopf manifolds $S^2 \times S^1$ and s solid open tori $E^2 \times S^1$.

We call a manifold M an *ST-manifold* of type (r,s) if M is homeomorphic to the manifold M(A) above.

Summarizing, we have the following theorem.

Theorem 4.7 *The manifolds $M(H)$ and $M(G)$ have the same 1-homology groups as the corresponding ST-manifolds.*

5 Constructing inequivalent FST-groups of the same rank

5.1 We say that the actions of two FST-groups Γ_1 and Γ_2 in $M(n)$ are *equivalent* (or more shortly, Γ_1 and Γ_2 are equivalent) if there is a homeomorphism $h\colon \bar{E}^n \longrightarrow \bar{E}^n$ such that $\Gamma_2 = h \circ \Gamma_1 \circ h^{-1}$. Otherwise, the actions of Γ_1 and Γ_2 are *inequivalent*.

It is clear that if the actions of Γ_1 and Γ_2 are equivalent, then the manifolds $M_1 = R(\Gamma_1)/\Gamma_1$ and $M_2 = R(\Gamma_2)/\Gamma_2$ are homeomorphic.

5.2 The objective of this section is to show that there are a lot of inequivalent FST-groups of the same rank and type acting on $\bar{E}^3$; more precisely, we will prove the following.

Theorem 5.1 *For any integer $N \geq 2$ there exist at least N inequivalent FST-groups acting on $\bar{E}^3$ having the same rank $k = k(N)$ and the same type.*

5.3 Construction

Let H be the group constructed in section 2.3.2 , and F be its fundamental spherical polyhedron (see Figure 9.) Take a Möbius transformation $g \in M(3)$ and consider the group $\Gamma_g = g \circ H \circ g^{-1}$. Then $F_g = g(F)$ is a fundamental domain for Γ_g. Let T_g be a 2-torus in $\bar{E}^3$ such that the boundary ∂F_g lies in the interior of T_g. (See Figure 19.)

Figure 19.

Let $N \geq 2$ be given. Take Möbius transformations $g_{ij} \in M(3)$, $i, j = 1, \ldots, N$, and consider the following groups $\Gamma_{ij} = g_{ij} \circ H \circ g_{ij}^{-1}$ with the fundamental domains $F_{ij} = g_{ij}(F)$. Let T_{ij} be the tori associated to the group Γ_{ij} as above. Let

$$\begin{aligned} \Gamma_1 &= \langle \Gamma_{11}, \Gamma_{12}, \ldots, \Gamma_{1N} \rangle \\ \Gamma_2 &= \langle \Gamma_{21}, \Gamma_{22}, \ldots, \Gamma_{2N} \rangle \\ \vdots \;\; & \;\;\vdots \qquad\qquad \vdots \\ \Gamma_N &= \langle \Gamma_{N1}, \Gamma_{N2}, \ldots, \Gamma_{NN} \rangle \end{aligned}$$

be the groups generated by the groups listed in the parentheses.

It is easy to see that we can choose the transformations g_{ij} in such a way that the tori T_{ij} are situated as shown in Figure 20.

Figure 20.

Having chosen such transformations g_{ij}, we can apply Klein's Combination Theorem to conclude that all the groups Γ_i are Kleinian, $F_i = \bigcap_{j=1}^{N} F_{ij}$ is a fundamental domain for Γ_i, and that Γ_i is a free group of rank $k = N \cdot \mathrm{rank}(\Gamma)$.

By applying the same arguments as in section 4, we can conclude that all the groups Γ_i are FST-groups. Moreover, one sees that they are all of the same type.

Let $M_i = R(\Gamma_i)/\Gamma_i$. We know that the manifold $M = R(H)/H$ is aspheric (see section 4.5). Then it follows easily from Milnor's Decomposition Theorem [7] that all the manifolds M_i are mutually non-homeomorphic. This proves the theorem.

Remark 1. Following the same lines as in section 4.6, we obtain that all the manifolds M_i above have isomorphic first homology groups; $H_1(M_i, \mathbf{Z})$ is a free abelian group of rank k.

Remark 2. It is easy to see that the regular sets $R(\Gamma_i)$ of the groups Γ_i are mutually non-homeomorphic. For instance, this follows from the fact that they have non-isomorphic second homotopy groups considered as π_1-modules. This implies that their limit sets are non-equivalent Cantor sets in $\bar{\mathbf{E}}^3$.

Remark 3. We also point out topological distinctions between the limit sets of the FST-groups constructed in sections 2 and 3 and the limit sets of the FST-groups constructed in this section.

The FST-groups in sections 2 and 3 have the property that every proper sub-Cantor set of their limit set has simply connected complement. The limit sets of the FST-groups in this section do not share this property, because these groups have proper FST-subgroups.

6 Extension

In this section we will prove that the natural extensions of the FST-groups G and H we constructed in sections 2 and 3 to the action in $\bar{\mathbf{E}}^4$ are ST-groups.

6.1

Lemma 6.1 *Let $A \subset \mathrm{M}(n)$ be an ST-group acting on $\bar{\mathbf{E}}^n$. Then the natural extension $A^* \subset \mathrm{M}(n+1)$ of the group A is also an ST-group.*

Lemma 6.2 *Let A_1 and A_2 be ST-groups acting on $\bar{\mathbf{E}}^n$. Assume that there exist fundamental domains F_1 and F_2 for A_1 and A_2 such that $F_1 \cup F_2 = \bar{\mathbf{E}}^n$ and $F = F_1 \cap F_2 \neq \emptyset$. Besides, suppose that there exists an $(n-1)$-dimensional locally flat topological sphere $S \subset F$ which separates the boundaries ∂F_1 and ∂F_2 of F_2 and F_2, that is, ∂F_1 and ∂F_2 lie in distinct components of $\bar{\mathbf{E}}^n \setminus S$. Then the group $A = \langle A_1, A_2 \rangle$ is an ST–group.*

The proof of these lemmas is left to the reader.

6.2 In this section we use the notations of section 2.3.2.

Let Γ_i^* and H^* denote the extensions of the groups Γ_i and H to the action in $\bar{\mathbf{E}}^4$. Let P_i^* and F^* be the spherical polyhedra in $\bar{\mathbf{E}}^4$ formed by the 3-spheres spanning the 2-spheres on the boundaries of the spherical polyhedra P_i and F respectively. Then it is clear that $F^* = P_1^* \cap P_2^* \cap P_3^* \cap P_4^*$. Let us note that F^* is a fundamental domain for H^*, and P_i^* is a fundamental domain for Γ_i^*.

Let us remark that the 1-link formed by the spines K_i of the groups Γ_i^*, $i = 1,2,3,4$, is splittable in $\bar{E}^4$. In particular, there are disjoint locally flat compact 4-balls B_1, B_2, B_3, B_4 containing ∂P_1^*, ∂P_2^*, ∂P_3^* and ∂P_4^* respectively. Let $S_i = \partial B_i$.

It follows from lemma 6.1 that the groups Γ_i^* are ST-groups. Now applying lemma 6.2 inductively and the remark, above we conclude that the group H^* is an ST-group.

6.3 In this section we use the notations of sections 3.3 and 4.3.1.

Let G^* and G'^* denote the extensions of the groups G and G' to the action on $\bar{E}^4$. Let P^* and P'^* be the spherical polyhedra in $\bar{E}^4$ spanning the polyhedra of G and G' respectively. Since the spines of the groups G^* and G'^* are unknotted in $\bar{E}^4$, it follows that there exists an orientation preserving homeomorphism $h: \bar{P}^* \longrightarrow \bar{P}'^*$, where $\bar{P}^*$ and $\bar{P}'^*$ are the closures of P^* and P'^* in $R(G^*)$ and $R(G'^*)$ respectively, satisfying the following conditions:

1. h maps bijectively the sides of $\bar{P}^*$ onto the sides of $\bar{P}'^*$;

2. If the sides S and S' of $\bar{P}^*$ are paired by the side pairing transformation $T \in G^*$, $T(S) = S'$, then the sides $h(S)$ and $h(S')$ are paired by the side pairing transformation $\phi(T) \in G'^*$, where ϕ is the isomorphism constructed in section 4.3;

3. The following diagram is commutative

$$\begin{array}{ccc} S & \xrightarrow{T} & S' \\ \downarrow h & & \downarrow h \\ h(S) & \xrightarrow{\phi(T)} & h(S') \end{array}$$

Now h extends equivariantly to all of $\bar{E}^4$, that is, the groups G^* and G'^* are conjugated.

It follows from lemma 6.1 that the group G'^* is an ST-group. Then the above implies that G^* is also an ST-group.

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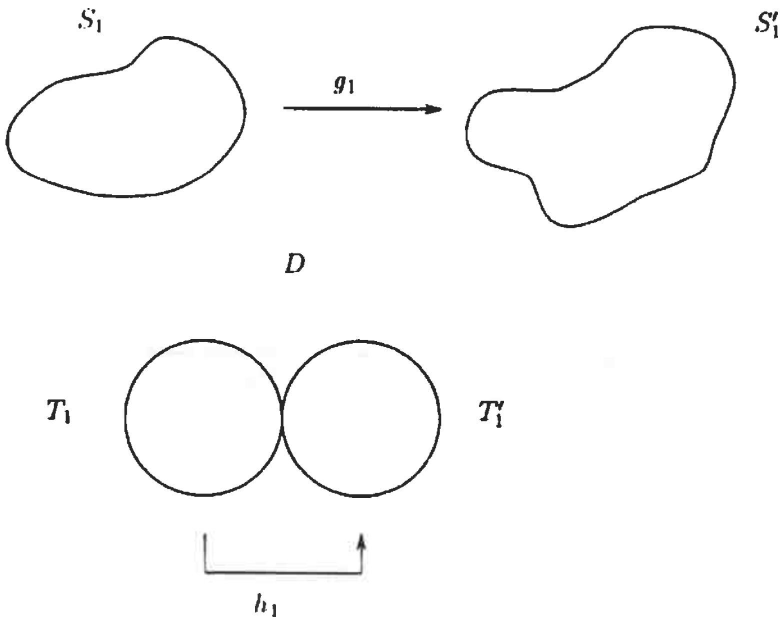
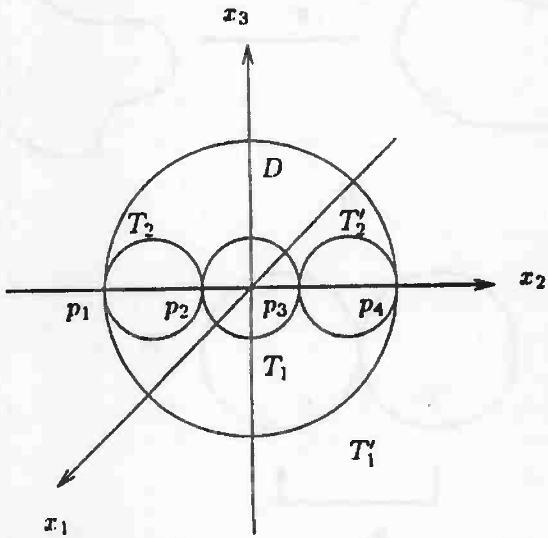


Figure 1: Fundamental domain for an ST-group of type (1, 1).



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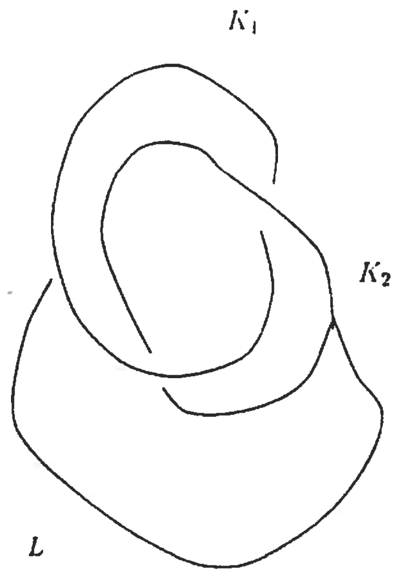

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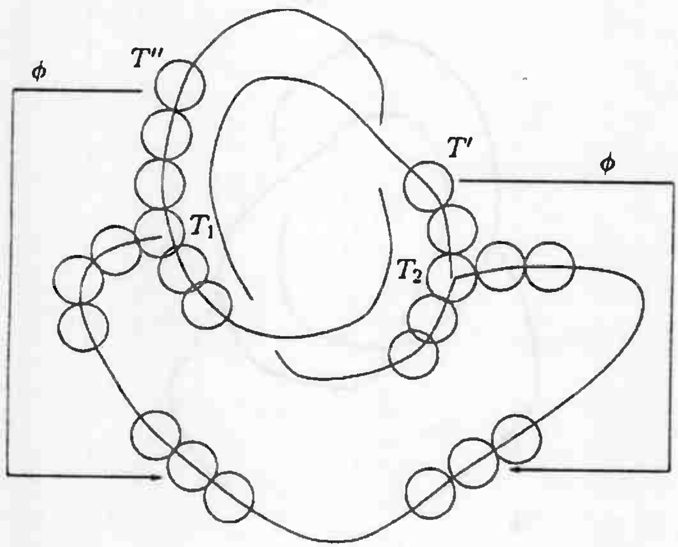

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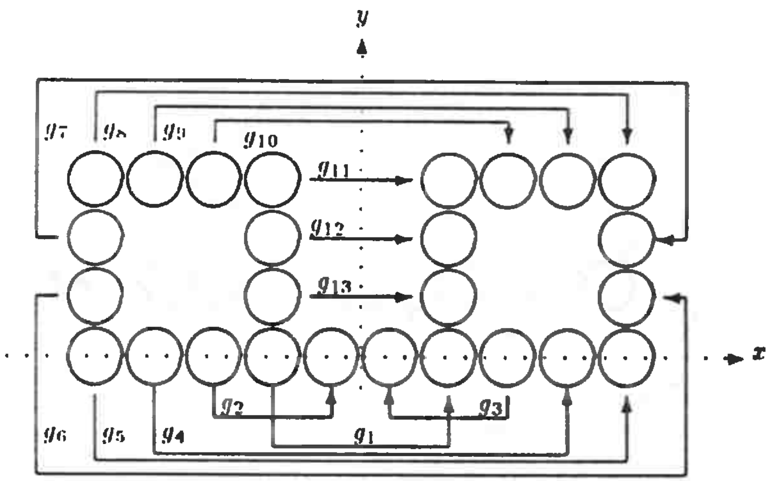
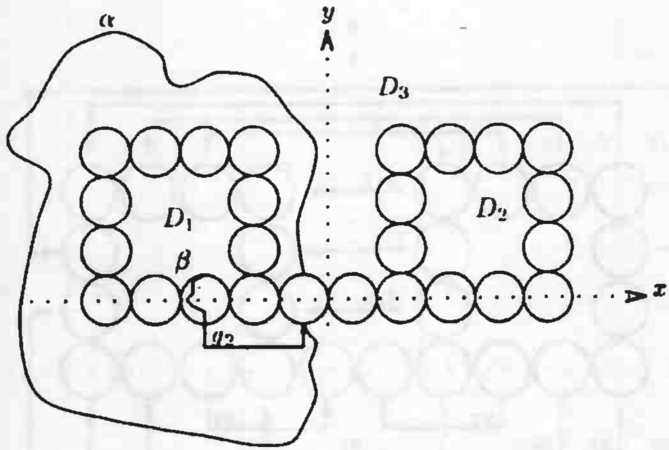


Figure 5: Fundamental domain for Γ.



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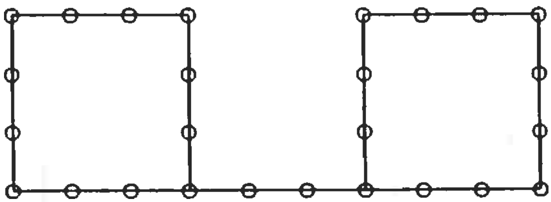


Figure 7: Spine of Γ.

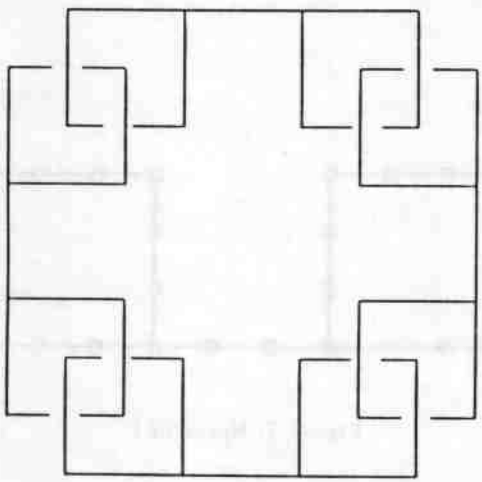


Figure 8: Link of spines.

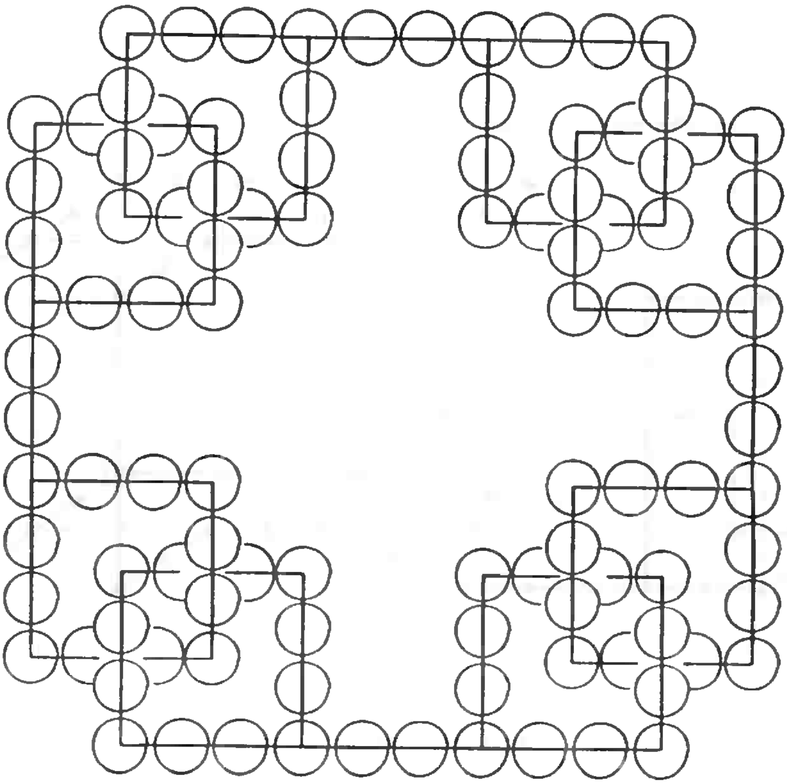


Figure 9: Fundamental domain for H.

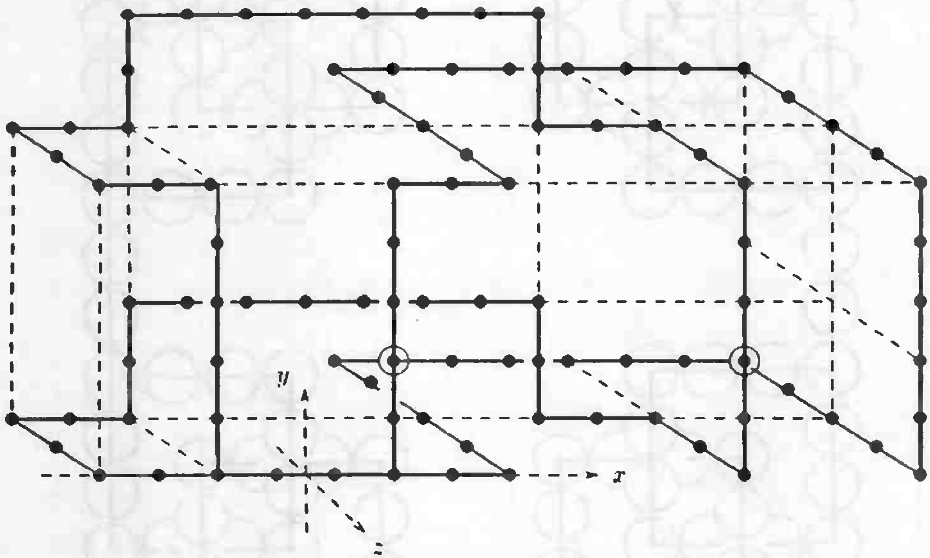


Figure 10: Spine of G.

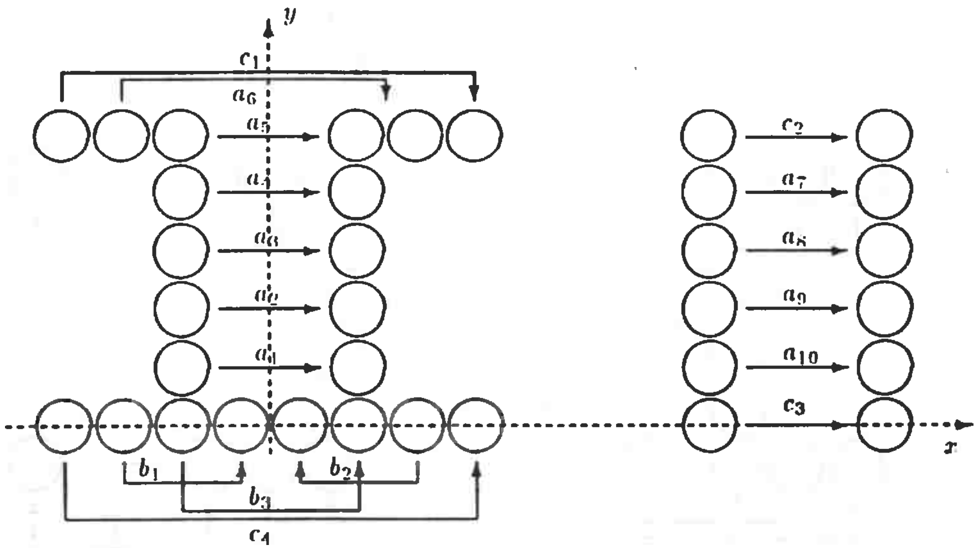


Figure 11: Slice by the plane $z = 0$.

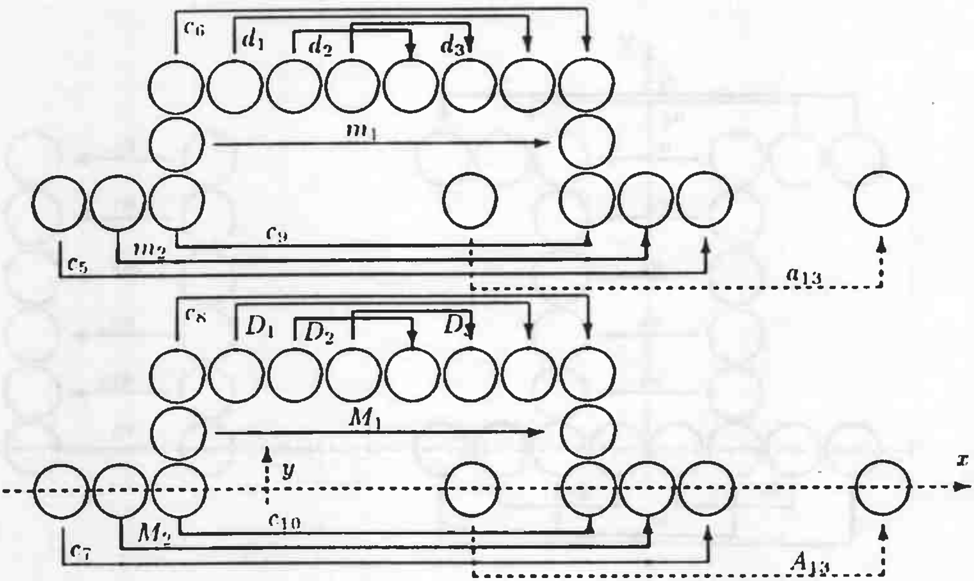


Figure 12: Slice by the plane $z = -4$.

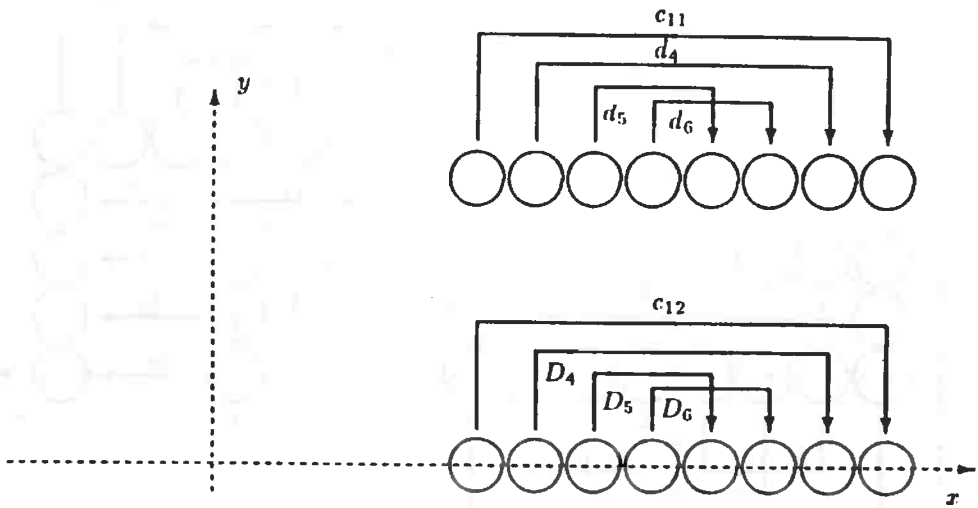


Figure 13: Slice by the plane $z = -8$.

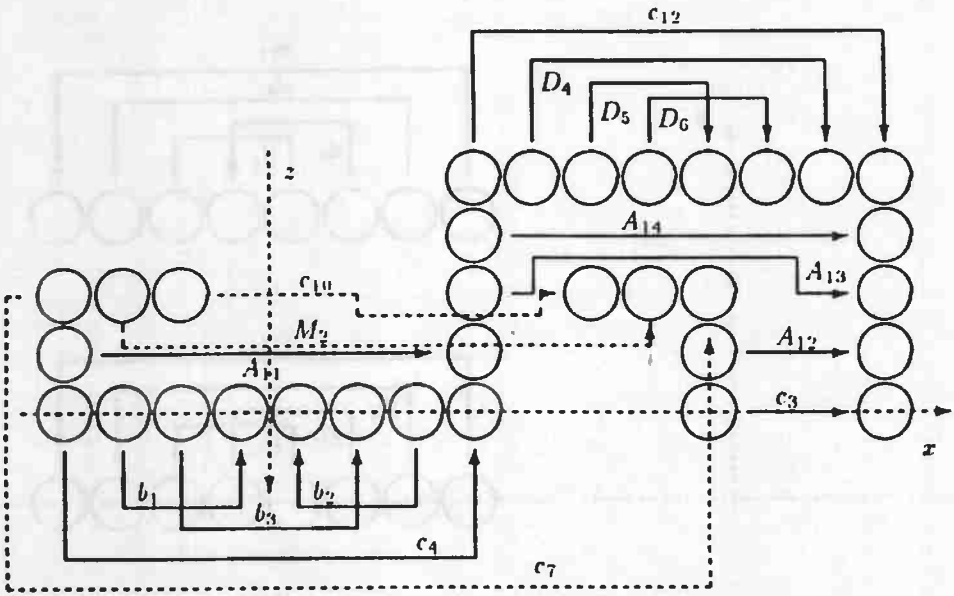


Figure 14: Slice by the plane $y = 0$.

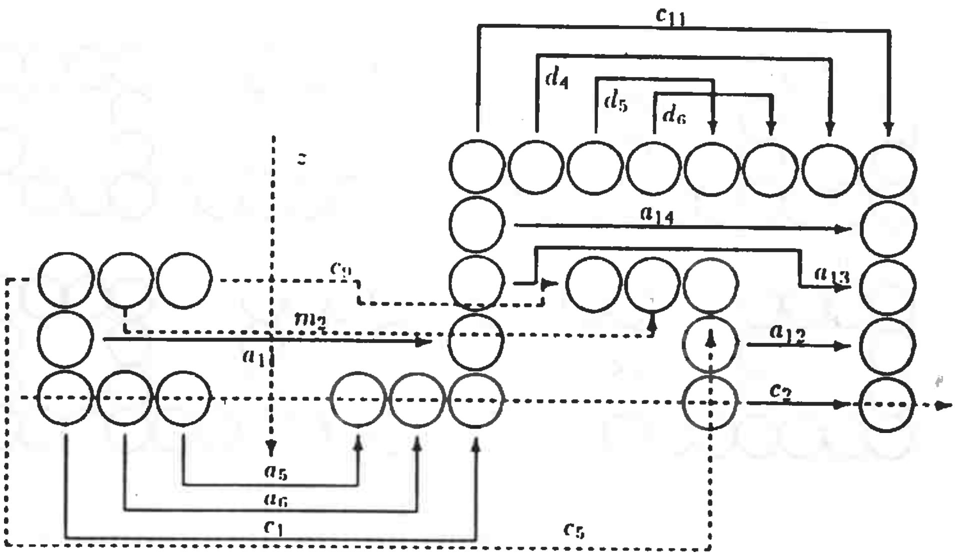
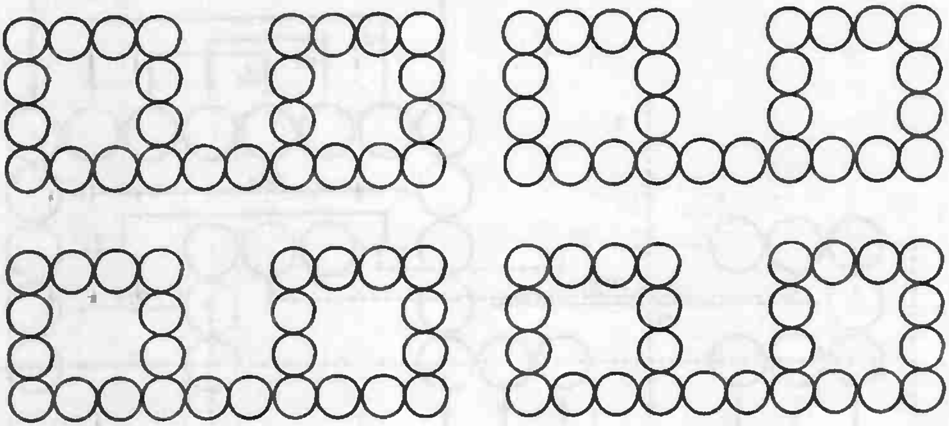


Figure 15: Slice by the plane $y = 10$.



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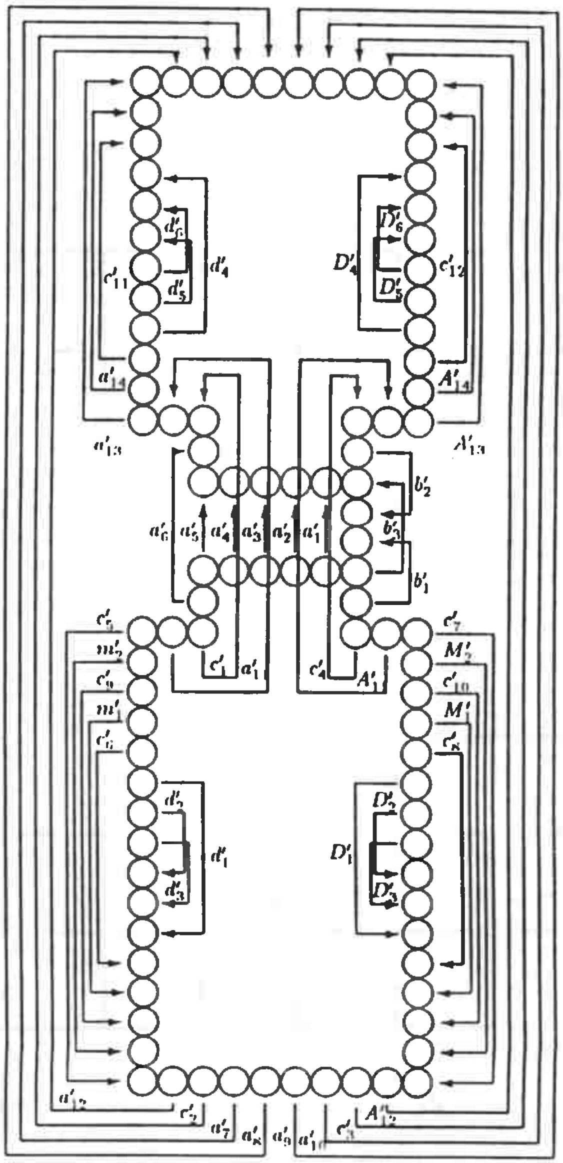
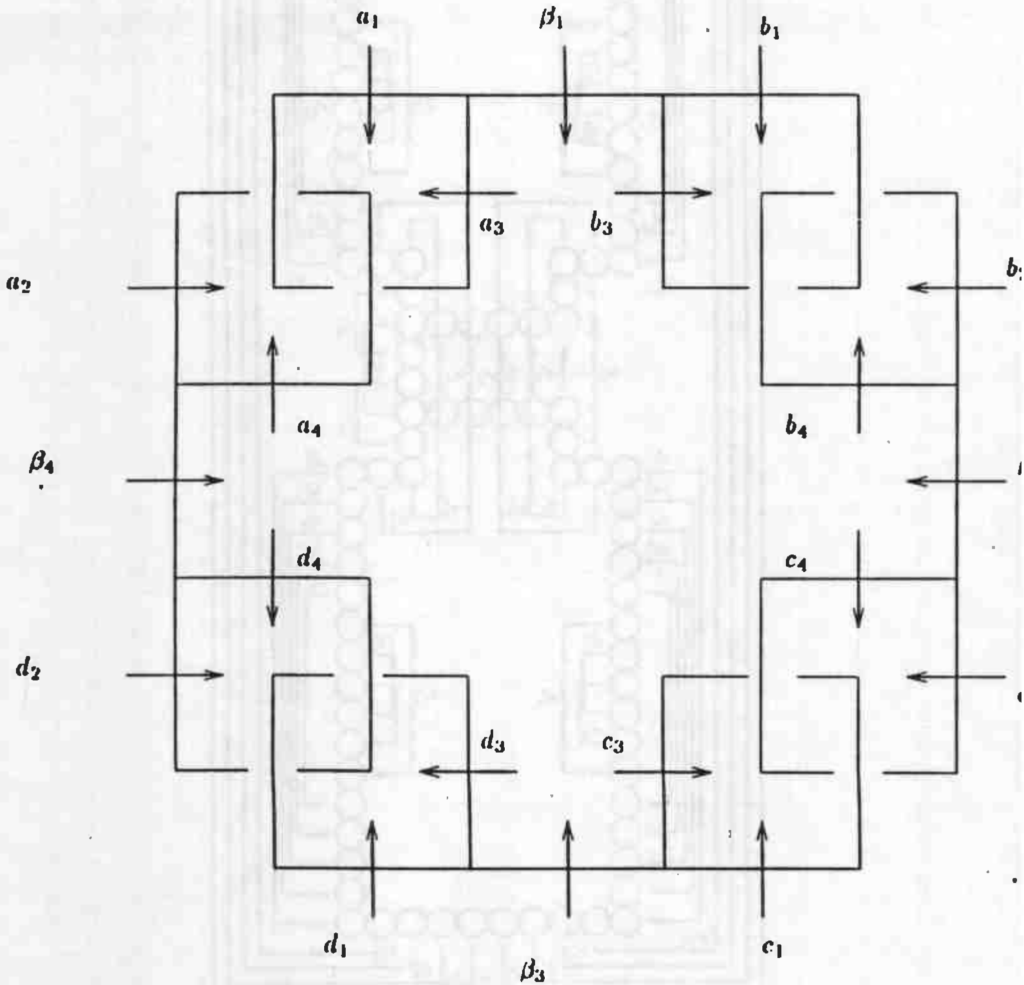


Figure 17: Fundamental domain and generators for G''



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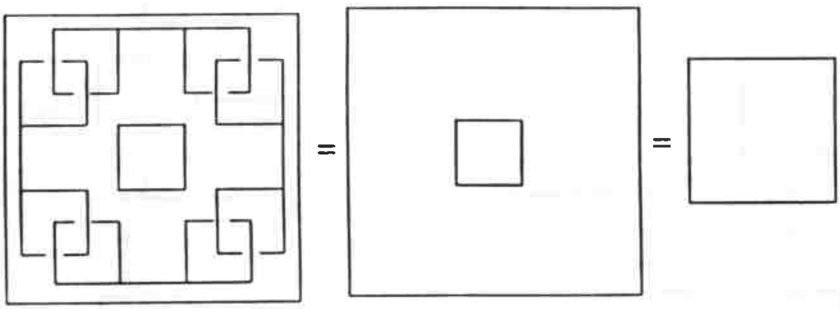

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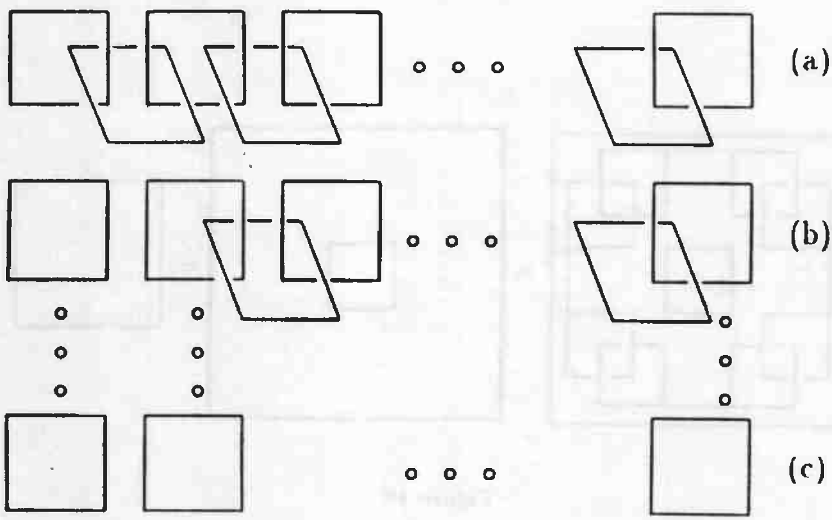

Figure 20:

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