OSCILLATING AND PERIODIC SOLUTIONS OF EQUATIONS OF THE TYPE $\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0$

HAMILTON LUIZ GUIDORIZZI

Instituto de Matemática e Estatística
Universidade de São Paulo, Caixa Postal 20570
(Ag. Iguatemi), CEP 01498, São Paulo, Brasil

Abstract

In this paper we consider equations of the type $\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0$ *and study their qualitative behavior from the point of view of oscillation and periodicity of the solutions.*

1. Introduction

In [1] and [2], Utz establishes sufficient conditions for existence of periodic solutions for equations of the types

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$$

and

$$\ddot{x} + Q(\dot{x}) + g(x) = 0.$$

In this paper, working with the non-usual positive definite function

$$V_{\alpha,\beta}(x,y) = \int_0^y \frac{s}{\alpha s^2 + \beta s + 1} ds + \int_0^x g(u)du$$

we establish sufficient conditions for existence of periodic solutions for the equation

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0$$

and, also, sufficient conditions for solutions to be oscillating.

2. The Positive Definite Function $V_{\alpha,\beta}$.

Auxilliary Lemmas.

Throughout this work we assume f_1, f_2 and g as functions of $\mathbf{R}$ in $\mathbf{R}$ satisfying the following conditions:

a) f_1, f_2 and g are of class C^1;

b) $xg(x) > 0$ for $x \neq 0$;

c) $\int_0^{+\infty} g(x) = +\infty = \int_0^{-\infty} g(x)$.

Let $\alpha, \beta \in \mathbf{R}$, with $\alpha > 0$. We indicate by $\Omega_{\alpha,\beta}$ the following open set:

$$\Omega_{\alpha,\beta} = \mathbf{R}^2 \quad \text{if} \quad \beta^2 - 4\alpha < 0,$$

$$\Omega_{\alpha,\beta} = \{(x,y) \in \mathbf{R}^2 \mid y > \rho_1\} \quad \text{if} \quad \beta > 0 \quad \text{and} \quad \beta^2 - 4\alpha \geq 0$$

and

$$\Omega_{\alpha,\beta} = \{(x,y) \in \mathbf{R}^2 \mid y < \rho_2\} \quad \text{if} \quad \beta < 0 \quad \text{and} \quad \beta^2 - 4\alpha \geq 0,$$

where

$$\rho_1 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha}}{2\alpha} \quad \text{and} \quad \rho_2 = \frac{-\beta - \sqrt{\beta^2 - 4\alpha}}{2\alpha} .$$

Consider the positive definite function

$$V_{\alpha,\beta} : \Omega_{\alpha,\beta} \to \mathbf{R}$$

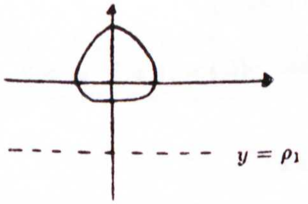
given by

$$V_{\alpha,\beta}(x,y) = \int_0^y \frac{s}{\alpha s^2 + \beta s + 1} ds + \int_0^x g(u)du .$$

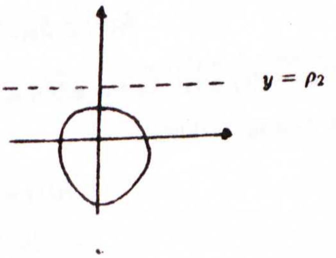
It can be easily verified that the level curves of $V_{\alpha,\beta}$ are all closed curves and that $V_{\alpha,\beta}(x,0)$ is strictly increased in $[0,+\infty[$.

For $\beta^2 - 4\alpha < 0$, $\Omega_{\alpha,\beta} = \mathbf{R}^2$ and, for each $(x_0, y_0) \in \mathbf{R}^2$, $V_{\alpha,\beta}(x,y) = V_{\alpha,\beta}(x_0,y_0)$ is a closed curve. For $\beta^2 - 4\alpha \geq 0$, such curves show the following aspect

$\beta < 0$



$\beta < 0$



The equation

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0 \tag{2.1}$$

is equivalent to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f_1(x)y - f_2(x)y^2 - g(x). \end{cases} \tag{2.1}$$

The condition (a) ensures existence and uniqueness of solution for (2.2). The condition b) ensures that $(0,0)$ is the only point of equilibrium for (2.2). It can be immediately verified that the derivative of $V_{\alpha,\beta}$ relative to the system (2.2) is:

$$\dot{V}_{\alpha,\beta}(x,y) = \frac{y^3[\beta g(x) - f_1(x)] + y^2[\alpha g(x) - f_2(x)]}{\alpha y^2 + \beta y + 1}. \tag{2.3}$$

We observe $\dot{V}_{\alpha,\beta}(x,y)$ has the same sign of $y^2[\beta g(x) - f_1(x)] + y^3[\alpha g(x) - f_2(x)]$, because $\alpha y^2 + \beta y + 1 > 0$ on $\Omega_{\alpha,\beta}$.

We also shall consider the positive definite function

$$W(x,y) = \frac{y^2}{2} + \int_0^x g(u)du, \quad (x,y) \in \mathbf{R}^2. \tag{2.4}$$

whose derivative relative to the system (2.2) is

$$\dot{W}(x,y) = -[y^3 f_2(x) + y^2 f_1(x)].$$

It can be easily verified that the level curves of W are all closed curves and $W(x,0)$ is strictly increasing in $[0,+\infty[$.

Lemma 1. Assume there are $b > 0$, $\alpha > 0$ and $\beta < 0$, with $\beta^2 - 4\alpha < 0$, such that, for all $x \geq b$,

$$f_1(x) \geq \beta g(x) \quad \text{and} \quad f_2(x) \geq \alpha g(x).$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (2.2), so that $\gamma(t_0) = (b, y_1)$, $y_1 > 0$. Then, there is $t_1 > t_0$ such that

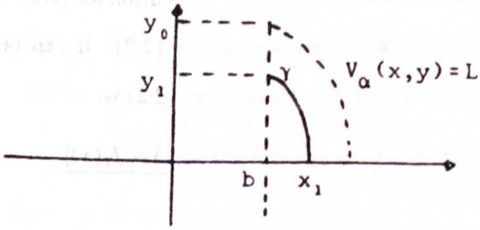
$$\gamma(t_1) = (x_1, 0), \quad x_1 > b.$$

Proof.

Let

$$K = \{(x, y) \in \mathbf{R}^2 \mid x \geq b,\ y \geq 0 \text{ and } V_{\alpha,\beta}(x, y) \leq L\},$$

where $L = V_{\alpha,\beta}(b, y_0)$, with $y_1 < y_0$.



From $\dot{x}(t_0) = y_1 > 0$, it follows there is $t_2 > t_0$ so that

$$\gamma(t) \in K, \quad t_0 \leq t \leq t_2.$$

On the other hand, being $\dot{x}(t) > 0$ on the half plane $y > 0$ and $(0,0)$ the only point of equilibrium, there must exist $t_3 > t_2$ such that

$$\gamma(t_3) \notin K.$$

Let

$$t_1 = \max\{u > t_0 \mid \gamma(t) \in K,\ t_0 \leq t \leq u\}.$$

From hypothesis and from (2.3) it follows

$$\dot{V}_{\alpha,\beta}(\gamma(t)) \leq 0, \quad t_0 \leq t \leq t_1.$$

Because $V_{\alpha,\beta}(\gamma(t_0)) = V_{\alpha,\beta}(b, y_1) < L$, it follows that $V_{\alpha,\beta}(\gamma(t_1)) < L$. So, $\gamma(t_1)$ does not belong to the arc given by

$$x \geq b, \; y \geq 0 \quad \text{and} \quad V_{\alpha,\beta}(x, y) = L.$$

Because $\dot{x}(t) > 0$ on the $y > 0$ half plane, it follows

$$\gamma(t_1) = (x_1, 0), \quad x_1 > b.$$

Remark 1. The lemma 1 remains valid if the hypothesis: "there are $b > 0$, $\alpha > 0$ and $\beta < 0$, with $\beta^2 - 4\alpha < 0$, such that, for all $x \geq b$,

$$f_1(x) \geq \beta g(x) \quad \text{and} \quad f_2(x) \geq \alpha g(x)\text{''}$$

is replaced by: "there is $b > 0$ such that, for all $x \geq b$, $f_1(x) \geq 0$ and $f_2(x) \geq 0$". In this case, we work with the positive definite function W given in (2.4).

In a similar way, we can demonstrate the following lemmas.

Lemma 2. Assume there are $b > 0$, $\alpha > 0$ and $\beta > 0$, with $\beta^2 - 4\alpha \geq 0$, such that, for all $x \geq b$,

$$f_1(x) \geq \beta g(x) \quad \text{and} \quad f_2(x) \leq \alpha g(x).$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (2.2) such that $\gamma(t_0) = (x_1, 0)$, with $x_1 > b$. Let $x_2 > x_1$, $L = V_{\alpha,\beta}(x_2, 0)$ and

$$K = \{(x, y) \in \mathbf{R}^2 \mid x \geq b, \; y \leq 0 \text{ and } V_{\alpha,\beta}(x, y) \leq L\}.$$

Then, there is $t_1 > t_0$ such that

$$\gamma(t) \in K, \quad t_0 \leq t \leq t_1,$$

$$\text{and} \quad \gamma(t_1) = (b, y_2), \quad \text{with } \rho_1 < y_2 < 0,$$

$$\text{where} \quad \rho_1 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha}}{2\alpha}.$$

Lemma 3. Assume there are $a < 0$, $\alpha < 0$ and $\beta > 0$, with $\beta^2 - 4\alpha < 0$, such that, for all $x \le a$,

$$f_1(x) \ge \beta g(x) \quad \text{and} \quad f_2(x) \le \alpha g(x).$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (2.2), so that $\gamma(t_0) = (a, y_1)$, with $y_1 < 0$. Then, there is $t_1 > t_0$ such that $\gamma(t_1) = (x_2, 0)$, with $x_2 < a$.

Remark 2. The lemma 3 remains valid if the hypothesis: "there are $a < 0$, $\alpha > 0$ and $\beta > 0$, with $\beta^2 - 4\alpha < 0$, such that, for all $x \le a$, $f_1(x) \ge \beta g(x)$ and $f_2(x) \le \alpha g(x)$" is replaced by: "there is $a < 0$ such that, for all $x \le a$, $f_1(x) \ge 0$ and $f_2(x) \le 0$".

Before presenting the next lemmas, we observe if $\alpha < 0$ there is $r > 0$ such that, for all (x_0, y_0), with $|x_0| \le r$ and $|y_0| \le r$,

$$V_{\alpha,\beta}(x,y) = V_{\alpha,\beta}(x,y)$$

is a closed curve, and $V_{\alpha,\beta}(x,0)$ is strictly increasing in $[0,r]$. We, also, observe that the equation

$$\frac{dy}{dx} = -f_1(x) - f_2(x)y - \frac{g(x)}{y}$$

does not admit vertical asymptote; consequently, the system (2.2) does not admit it too.

Lemma 4. Assume there are $r > 0$, $\alpha < 0$ and $\beta > 0$ such that, for $0 < x < r$,

$$f_1(x) \le \beta g(x) \quad \text{and} \quad f_2(x) \ge \alpha g(x).$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (2.2) so that $\gamma(t_0) = (x_1, 0)$, $x_1 > 0$. Then, there is $t_1 > t_0$ such that $\gamma(t_1) = (0, y_1)$, with $y_1 < 0$.

Lemma 5. Assume there are $r < 0$, $\alpha < 0$ and $\beta < 0$ such that, for $r < x < 0$,

$$f_1(x) \le \beta g(x) \quad \text{and} \quad f_2(x) \le \alpha g(x).$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (2.2), so that $\gamma(t_0) = (x_1, 0)$, $x_1 < 0$. Then, there is $t_1 > t_0$ such that

$$\gamma(t_1) = (0, y_1), \quad \text{with } y_1 > 0.$$

3. Sufficient Conditions for Oscillating Solutions

A solution $x(t)$ of equation (2.1) is oscillating if there is a sequence $(t_n)_{n\geq 1}$ tending monotonically to $+\infty$ such that $x(t_n) = 0$ for $n \geq 1$.

Theorem 1. Suppose f_1, f_2 and g satisfy the conditions a), b) and c) of the previous section. Assume, also, that the following conditions are satisfied:

1) There are $b > 0$, $\alpha > 0$ and $\beta < 0$, with $\beta^2 - 4\alpha < 0$, such that, for all $x \geq b$,

$$f_1(x) \geq \beta g(x) \quad \text{and} \quad f_2(x) \geq \alpha g(x)$$

or

there is $b > 0$ such that, for all $x \geq b$,

$$f_1(x) \geq 0 \quad \text{and} \quad f_2(x) \geq 0;$$

2) There are $a < 0$, $\alpha_3 > 0$ and $\beta_3 > 0$, with $\beta_3^2 - 4\alpha_3 < 0$ such that, for all $x \leq a$,

$$f_1(x) \geq \beta_3 g(x) \quad \text{and} \quad f_2(x) \leq \alpha_3 g(x)$$

or

there is $a < 0$ such that, for all $x \leq a$,

$$f_1(x) \geq 0 \quad \text{and} \quad f_2(x) \leq 0;$$

3) There are $r_1 > 0$, $\alpha_1 < 0$ and $\beta_1 > 0$ such that, for $0 < x < r_1$,

$$f_1(x) \leq \beta_1 g(x) \quad \text{and} \quad f_2(x) \geq \alpha_1 g(x);$$

4) There are $r_2 < 0$, $\alpha_2 < 0$ and $\beta_2 < 0$ such that, for $r_2 < x < 0$,

$$f_1(x) \leq \beta_2 g(x) \quad \text{and} \quad f_2(x) \leq \alpha_2 g(x).$$

Then, any non-trivial solution of the equation (2.1) will be oscillating.

Proof.

Let $\gamma(t) = (x(t), y(t))$ be a solution of (2.2) such that

$$\gamma(t_0) = (x_0, 0), \quad x_0 > 0.$$

There is, from hypotheses 2-3 and lemmas 3-4, a smallest $t_1 > t_0$ such that

$$\gamma(t_1) = (x_1, 0), \quad x_1 < 0.$$

There is, from hypotheses 1-4 and lemmas 1-5, a smallest $t_2 > t_1$ such that

$$\gamma(t_2) = (x_2, 0), \quad x_2 > 0.$$

So, any non-trivial solution of (2.1) is oscillating. ■

A special situation occurs when $f_2(x) = \alpha g(x)$ for some $\alpha > 0$, as the next theorem shows.

Theorem 2. Suppose f_1 and g satisfy the conditions a), b) and c) of the previous section. Assume, also, that the following conditions are satisfied:

1) There is $\alpha > 0$ such that, for all x,

$$f_2(x) = \alpha g(x);$$

2) There is $\beta \in \mathbf{R}$, with $\beta^2 - 4\alpha < 0$, such that

$$f_1(x) > \beta g(x), \quad x \neq 0;$$

3) There are $r > 0$ and $\beta_1 > 0$ such that, for $0 < |x| < r$,

$$f_1(x) \leq \beta_1 g(x).$$

Then, any non-trivial solution $x(t)$ of the equation (2.1) will be oscillating and $\gamma(t) = (x(t), \dot{x}(t))$ approaches to the origin as $t \to +\infty$.

Proof.

It is enough to observe that

$$\dot{V}_{\alpha,\beta}(x,y) = \frac{[\beta g(x) - f_1(x)]}{\alpha y^2 + \beta y + 1} y^2 < 0. \quad \text{for } x \neq 0 \text{ and } y \neq 0.$$

to get to the conclusions that any solution of (2.2) approaches to the origin when $t \to +\infty$. Based in hypothesis 3, we conclude that every solution starting at $(x_0, 0)$, $x_0 > 0$, crosses the negative y half-axis and every solution starting at $(x_1, 0)$. $x_1 < 0$, crosses the positive y half-axis. Therefore, all non-trivial solutions of (2.1) will be oscillating. ■

Remark 3. If, in theorem 2, the hypothesis $\beta^2 - 4\alpha < 0$ in 2) is replaced by $\beta^2 - 4\alpha \geq 0$, we can assert only that any non-trivial solution passing by $(x_0, y_0) \in \Omega_{\alpha,\beta}$ is oscillating and approaches to the origin as $t \to +\infty$.

Example 1. Consider the equation

$$\ddot{x} + (\delta x^2 + \beta x)\dot{x} + \alpha x \dot{x}^2 + x = 0$$

where $\delta \geq 0$, $\alpha > 0$ and β are given. We have

a) If $\delta = 0$ and $\beta^2 - 4\alpha < 0$, then every non-trivial solution is periodic. If $\delta = 0$ and $\beta^2 - 4\alpha \geq 0$. then every non-trivial solution passing by $(x_0, y_0) \in \Omega_{\alpha,\beta}$ is periodic.

b) If $\delta > 0$ and $\beta^2 - 4\alpha > 0$, then evey non-trivial solution is oscillating and $(x(t). \dot{x}(t))$ approaches to the origin when $t \to +\infty$. If $\delta > 0$ and $\beta^2 - 4\alpha \geq 0$, then every non-trivial solution $x(t)$ passing by $(x_0, y_0) \in \Omega_{\alpha,\beta}$ is oscillating and $(x(t). \dot{x}(t))$ approaches to the origin when $t \to +\infty$.

Solution.

a) It is enough to observe that

$$\dot{V}_{\alpha,\beta}(x,y) = 0, \quad (x,y) \in \Omega_{\alpha,\beta}$$

and $V_{\alpha,\beta}(x,y) = V_{\alpha,\beta}(x_0,y_0)$ is a closed curve for all $(x_0,y_0) \in \mathbf{R}^2$ if $\beta^2 - 4\alpha < 0$, and is a closed curve for all $(x_0,y_0) \in \Omega_{\alpha,\beta}$ if $\beta^2 - 4\alpha \geq 0$.

b) We have

$$\delta x^2 + \beta x > \beta x, \quad x \neq 0$$

and there is $r > 0$ and $\beta_1 > 0$ such that

$$\delta x^2 + \beta x < \beta_1 |x|, \quad 0 < |x| < r.$$

The conclusion follows from theorem 2 and remark 3.

4. Sufficient Conditions for Existence of Periodic Solutions

Theorem 3. Suppose that f_1, f_2 and g satisfy the conditons a), b) and c) of the section 2. Assume, also, that the following conditions are satisfied:

1) There are $b > 0$, $\alpha > 0$ and $\beta > 0$, with $\beta^2 - 4\alpha \geq 0$, such that, for all $x \geq b$,

$$f_1(x) \geq \beta g(x) \quad \text{and} \quad 0 \leq f_2(x) \leq \alpha g(x);$$

2) There are $a < 0$, $\alpha_1 > 0$ and $\beta_1 > 0$, with $\beta_1^2 - 4\alpha_1 < 0$, such that, for all $x \leq a$,

$$f_1(x) \geq \beta_1 g(x) \quad \text{and} \quad f_2(x) \leq \alpha_1 g(x)$$

or

there is $a < 0$ such that, for all $x \leq a$,

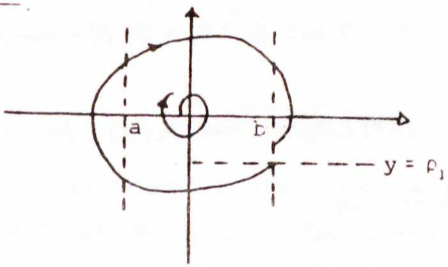
$$f_1(x) \geq 0 \quad \text{and} \quad f_2(x) \leq 0;$$

3) The origin $(0,0)$ is repulsive.

Then the equation (2.1) admits at least one non-trivial periodic solution.

Proof.

Theorem 1 ensures that any non-trivial solution of the equation (2.1) is oscillating. From lemma 2 and hypothesis 1, every solution of (2.2) starting at (b, y_0), $y_0 > 0$, crosses again the straight line $x = b$ at a point (b, y_1), with $\rho_1 < y_1 < 0$, where $\rho_1 = \dfrac{-\beta + \sqrt{\beta^2 - 4\alpha}}{2\beta}$.



From the theorem of Poincaré–Bendixson, the equation will admit at least one periodic solution. ■

Remark 4. It can be immediately verified that a sufficient condition for that the origin $(0,0)$ be repulsive is $f_1(0) < 0$ and $(0,0)$ a local center of the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f_2(x)y^2 - g(x). \end{cases}$$

The condition $xg(x) > 0$, $x \neq 0$ ensures that $(0,0)$ is a local center for the system above (see [1]). So, $f_1(0) < 0$ is a sufficient condition for the origin to be repulsive.

Remark 5. If $f_2(x) = \alpha g(x)$, $x \in \mathbf{R}$, for some $\alpha > 0$, then the following condition is sufficient for $(0,0)$ to be repulsive: there is $r > 0$ and $\beta_2 \in \mathbf{R}$ such that, for $0 < |x| < r$,

$$f_1(x) < \beta_2 g(x).$$

Remark 6. The theorem 3 remains valid if the hypothesis 1 and 2 are replaced, respectively, by:

1') There are $b > 0$, $\alpha > 0$ and $\beta < 0$, with $\beta^2 - 4\alpha < 0$, such that, for all $x \geq b$,

$$f_1(x) \geq \beta g(x) \quad \text{and} \quad f_2(x) \geq \alpha g(x)$$

or

there is $b > 0$ such that, for all $x \geq b$,

$$f_1(x) \geq 0 \quad \text{and} \quad f_2(x) \geq 0;$$

2') There are $a < 0$, $\alpha_1 > 0$ and $\beta_1 < 0$, with $\beta_1^2 - 4\alpha_1 \geq 0$, such that, for all $x \leq a$,

$$f_1(x) \geq \beta_1 g(x) \quad \text{and} \quad \alpha_1 g(x) \leq f_2(x) \leq 0.$$

Example 2. Consider the equation

$$\ddot{x} + P(x)\dot{x} + \alpha x \dot{x}^2 + x = 0$$

where $\alpha > 0$, $P(x) = \sum_{k=0}^{2n} a_k x^k$, $n \geq 2$, $a_{2n} > 0$, $a_2 < 0$ if $a_0 = 0$, and $a_0 \leq 0$. It follows immediately from theorem 3 and remark 5 that the equation admits at least one non-trivial periodic solution.

Example 3. Consider the equation

$$\ddot{x} + (x^6 - 1)\dot{x} + (x^3 + x - 1)\dot{x}^2 + x^5 = 0.$$

Let $f_1(x) = x^6 - 1$, $f_2(x) = x^3 + x - 1$ and $g(x) = x^5$. We have

1) $f_1(x) \geq 3g(x)$ and $0 \leq f_2(x) \leq g(x)$ for $x \geq 4$, with $\beta^2 - 4\alpha > 0$ ($\beta = 3$ and $\alpha = 1$);

2) $f_1(x) \geq 0$ and $f_2(x) \leq 0$ for $x \leq -1$;

3) $f_1(0) < 0$.

From theorem 3 and remark 4, the equation admits at least one non-trivial periodic solution.

Example 4. From theorem 3 and remark 4, it follows immediately that the equation

$$\ddot{x} + (e^x - 2)\dot{x} + (x^3 - x^2 - 1)\dot{x}^2 + x^3 = 0$$

admits at least one non-trivial periodic solution.

Remark 7. The hypothesis 2, in theorem 3, can be weakened. Let $y = y(x)$ be the solution of

$$\frac{dy}{dx} = -f_1(x) - f_2(x)y - \frac{g(x)}{y}$$

passing by (b, ρ_1), where

$$\rho_1 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha}}{2\alpha} ,$$

being α and β as in hypothesis 1. Assume that such solution crosses the straight line $x = a$ at the point (a, y_2), with $y_2 < \rho_1$. Then, there will be $a < x_0 \leq b$ such that $y(x_0) = \rho_1$ and, for $a \leq x < x_0$, $y(x) < \rho_1$. So, for $a \leq x \leq x_0$,

$$\frac{dy}{dx} \leq |f_1(x)| - |f_2(x)|y + \frac{|g(x)|}{|\rho_1|} .$$

Let $A > 0$ be such that, for $a \leq x \leq b$,

$$A \geq |f_1(x)| + \frac{|g(x)|}{|\rho_1|} \quad \text{and} \quad A \geq |f_2(x)| .$$

Then, for $a \leq x \leq x_0$,

$$\frac{dy}{dx} + Ay \leq A.$$

It follows that

$$\frac{d}{dx}[y e^{Ax}] \leq A e^{Ax}$$

and therefore

$$y(x_0)e^{Ax_0} - y(a)e^{Aa} \leq \int_a^{x_0} Ae^{Ax}dx.$$

Hence

$$-y(a) \leq (-\rho_1 + 1)e^{A(x_0-a)} - 1$$

and, therefore,

$$-y(a) \leq (-\rho_1 + 1)e^{A(b-a)} - 1.$$

Then, the hypothesis 2 can be replaced by:

2_1) There is $a < 0$ such that, for every $x \in [c, a]$,

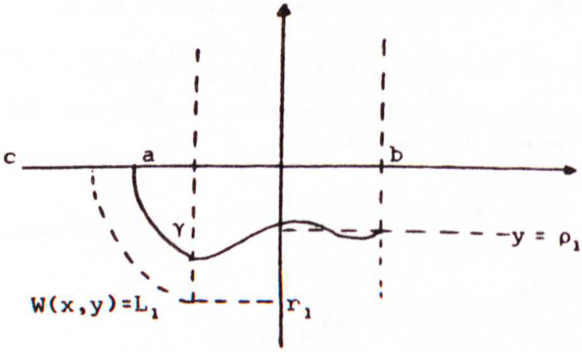
$$f_1(x) \geq 0 \quad \text{and} \quad f_2(x) \leq 0$$

where

$$W(a, r_1) = W(c, 0) = L_1$$

with $r_1 < 0$ and $|r_1| \geq (-\rho_1 + 1)e^{A(b-a)} - 1$, being, for every $x \in [a.b]$,

$$A \geq |f_1(x)| + \frac{|g(x)|}{|\rho_1|} \quad \text{and} \quad A \geq |f_2(x)|\ .$$



Or by:

2_2) There are $a < 0$, $\alpha_1 > 0$ and $\beta_1 > 0$, with $\beta_1^2 - 4\alpha_1 < 0$, such that, for every $x \in [c, a]$,

$$f_1(x) \geq \beta_1 g(x) \quad \text{and} \quad f_2(x) \leq \alpha_1 g(x)$$

where

$$V_{\alpha_1, \beta_1}(a, r_1) = V_{\alpha_1, \beta_1}(c, 0)$$

with $r_1 < 0$ and $|r_1| \geq (-\rho_1 + 1)e^{A(b-a)} - 1$, being, for every $x \in [c,a]$,

$$A \geq |f_1(x)| + \frac{|g(x)|}{|\rho_1|} \quad \text{and} \quad A \geq |f_2(x)|\,.$$

Example 5. The equation

$$\ddot{x} + (x^3 + 13x^2 + 2x - \frac{1}{4})\dot{x} + (x - \frac{1}{4})\dot{x}^2 + x = 0$$

admits at least one non-trivial periodic solution.

Solution.

Let $f_1(x) = x^3 + 13x^2 + 2x - \frac{1}{4}$, $f_2(x) = x - \frac{1}{4}$ and $g(x) = x$.

We have

1) $f_1(x) \geq \beta g(x)$ and $f_2(x) \leq \alpha g(x)$, for $x \geq b$, where $\beta = 2$, $\alpha = 1$ and $b = \frac{1}{4}$. So, $\beta^2 - 4\alpha = 0$ and $\rho_1 = -1$.

2_1) $f_1(x) \geq 0$ and $f_2(x) \leq 0$, for every $x \in [c,a]$, where $c = -12$ and $a = -\frac{1}{4}$. Let $A = 2$. So, $A \geq |f_1(x)| + \dfrac{|g(x)|}{|\rho_1|}$ and $A \geq |f_2(x)|$, for $a \leq x \leq b$. Let $r_1 > 0$ such that

$$\frac{r_1^2}{2} + \frac{a^2}{2} = \frac{c^2}{2} \quad (W(c,0) = W(a, r_1)).$$

Thus, $|r_1| \geq (-\rho_1 + 1)e^{A(b-a)} - 1$.

3) $f_1(0) < 0$.

From theorem 3 and remark 7 the equation admits at least one non-trivial periodic solution.

References

[1] UTZ, W.R. *Periodic solutions of a nonlinear second order differential equation.* SIAM J. APPL. MATH. 19 (1970), 56-59.

[2] UTZ, W.R. *Periodic solutions of second order differential equations with nonlinear, nondifferentiable damping.* SIAM J. APPL. MATH. 31 (1976), 504-510.