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BOUNDEDNESS OF SOLUTIONS OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE IMPULSES VIA GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Resumo

Vamos considerar uma classe de equações diferenciais funcionais com retardamento e impulsos em tempo variável e investigar a limitação uniforme das soluções dessas equações através da teoria das equações diferenciais ordinárias generalizadas usando funcionais de Lyapunov.

BOUNDEDNESS OF SOLUTIONS OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE IMPULSES VIA GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we give sufficient conditions for the uniform boundedness and uniform ultimate boundedness of solutions of a class of retarded functional differential equations with impulse effects acting on variable times. We employ the theory of generalized ordinary differential equations to obtain our results. As an example, we investigate the boundedness of the solution of a circulating fuel nuclear reactor model.

1. INTRODUCTION

Impulsive differential equations are an important tool to describe the evolution of systems where the continuous development of a process is interrupted by abrupt changes of state. These equations are modelled by both differential equations, which describe the period of continuous variation of state, and additional conditions, which describe the discontinuities of first kind of a solution or of its derivatives at the moments of impulse. For example, the problem of stabilizing the solutions by imposing proper impulse controls has been used in many fields such as physics, pharmacokinetics, biotechnology, economics, chemical technology, population dynamics among others. On the other hand, there are a few results on the boundedness of solutions of impulsive retarded functional differential equations (we write impulsive RFDEs, for short). See [4, 10, 16] for instance. In particular, RFDEs with variable impulse effects have been much less studied.

In [16], I. Stamova proved several criteria for the boundedness of retarded solutions of a class of RFDEs with variable impulsive perturbations by Lyapunov's direct method. However, these criteria are valid under the assumption that the integral curves of the corresponding systems meet successively each one of the hypersurfaces exactly once.

In the present paper, we prove the same fact assuming weaker conditions. We consider that the function on the righthand side of the RFDE is Lebesgue integrable and hence not necessarily piecewise continuous. Furthermore we assume that the integral curves of the differential system meet successively each one of the hypersurfaces a finite number of times. Therefore our results encompass those from [16].

In order to get the main results, we embed our impulsive RFDE in a class of generalized ordinary differential equations (we write generalized ODEs, for short) and we develop the theory of boundedness of solutions in this setting. Then, by means of Lyapunov functionals satisfying weak Krasovskiĭ-type conditions, we get the desired results.

We consider generalized ODEs with solutions taking values in the space of regulated functions.

Let X be a Banach space and $I \subset \mathbb{R}$ be any interval of the real line. We denote by $G^-(I,X)$ the space of left continuous regulated functions $f: I \to X$, that is, $G^-(I,X)$ is the set of all functions $f: I \to X$ such that, for every compact interval $[a,b] \subset I$, $f(t-) = f(t)$ for each $t \in (a,b]$ and the right limit $f(t+)$ exists for each $t \in [a,b)$, where

$$f(t-) = \lim_{\rho \to 0^-} f(t+\rho) \quad \text{and} \quad f(t+) = \lim_{\rho \to 0^+} f(t+\rho).$$

The space $G^-([a,b],X)$ is a Banach space when endowed with the usual supremum norm. We write $C(I,X)$ to denote the space of continuous functions $f: I \to X$ and we consider the Banach space $C([a,b],X)$ equipped with the norm induced by $G^-([a,b],X)$.

2. Impulsive RFDEs

Let $t_0 \geq 0$ and $r > 0$. Given a function $y \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$, we consider $y_t \in G^-([-r,0],\mathbb{R}^n)$ defined, as usual, by

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0], \quad t \in [t_0, +\infty).$$

Consider the RFDE with impulse action

$$\begin{cases} \dot{y}(t) = f(y_t, t), & t \neq \tau_k(y(t)), \quad t \geq t_0, \\ \Delta y(t) = I_k(y(t)), & t = \tau_k(y(t)), \quad k = 1,2,\ldots, \end{cases} \tag{1}$$

subject to the initial condition

$$y_{t_0} = \phi, \tag{2}$$

where $\phi \in G^-([-r,0],\mathbb{R}^n)$. We assume that f maps each pair $(\varphi, t) \in G^-([-r,0],\mathbb{R}^n) \times [t_0,+\infty)$ to $\mathbb{R}^n$, for $k = 1,2,\ldots$, I_k maps $\mathbb{R}^n$ to itself and τ_k maps $\mathbb{R}^n$ to $(t_0,+\infty)$. Moreover

$$\Delta y(t) := y(t+) - y(t-) = y(t+) - y(t),$$

for any $t \geq t_0$.

Assume that $\tau_0(x) \equiv t_0$, for all $x \in \mathbb{R}^n$ and for each $k = 1,2,\ldots$, define the set

$$S_k = \{(t,x) \in [t_0,+\infty) \times \mathbb{R}^n : t = \tau_k(x)\}.$$

By $m(\tau_k)$ we denote the number of times at which the integral curves of system (1)-(2) meet the hypersurface S_k, $k = 1,2,\ldots$. By t_k^i we denote the i^{th} moment of time at which the integral curves of system (1)-(2) meet the hypersurface S_k, with $i = 1,\ldots,m(\tau_k)$ and $k = 1,2,\ldots$.

Throughout this paper, we shall consider the following conditions:

(C1) $\tau_k \in C(\mathbb{R}^n, (t_0,+\infty))$, $k = 1,2,\ldots$;
(C2) $t_0 < \tau_1(x) < \tau_2(x) < \ldots$, for each $x \in \mathbb{R}^n$;
(C3) $\tau_k(x) \to +\infty$ as $k \to +\infty$ uniformly on $x \in \mathbb{R}^n$;

(C4) The integral curves of system (1)-(2) meet successively each hypersurface $S_1, S_2, \ldots$ a finite number of times;

(C5) $t_k^i < t_k^{i+1}$, $i = 1, \ldots, m(\tau_k) - 1$, for all $k = 1, 2, \ldots$.

Let $PC_1 \subset G^-([t_0 - r, +\infty), \mathbb{R}^n)$ be an open set (in the topology of locally uniform convergence in $G^-([t_0 - r, +\infty), \mathbb{R}^n)$) with the following property: if y is an element of PC_1 and $\bar{t} \in [t_0, +\infty)$, then $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & t_0 - r \le t \le \bar{t}, \\ y(\bar{t}), & \bar{t} < t < +\infty, \end{cases}$$

is also an element of PC_1. In particular, any open ball in $G^-([t_0 - r, +\infty), \mathbb{R}^n)$ has this property.

We assume that $f : G^-([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$ is such that for every $y \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$, $t \mapsto f(y_t, t)$ is locally Lebesgue integrable on $t \in [t_0, +\infty)$ and moreover:

(A) There is a locally Lebesgue integrable function $M : [t_0, +\infty) \to \mathbb{R}$ such that for all $x \in PC_1$ and all $u_1, u_2 \in [t_0, +\infty)$,

$$\left| \int_{u_1}^{u_2} f(x_s, s)\, ds \right| \le \int_{u_1}^{u_2} M(s)\, ds;$$

(B) There is a locally Lebesgue integrable function $L : [t_0, +\infty) \to \mathbb{R}$ such that for all $x, y \in PC_1$ and all $u_1, u_2 \in [t_0, +\infty)$,

$$\left| \int_{u_1}^{u_2} [f(x_s, s) - f(y_s, s)]\, ds \right| \le \int_{u_1}^{u_2} L(s)\, \|x_s - y_s\|\, ds.$$

For the impulse operators $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots$, we assume the following conditions:

(A') There is a constant $K_1 > 0$ such that for all $k = 1, 2, \ldots$ and all $x \in \mathbb{R}^n$,

$$|I_k(x)| \le K_1;$$

(B') There is a constant $K_2 > 0$ such that for all $k = 1, 2, \ldots$ and all $x, y \in \mathbb{R}^n$,

$$|I_k(x) - I_k(y)| \le K_2 |x - y|.$$

We recall the definition of a solution of the initial value problem (1)-(2).

Definition 2.1. *Consider system (1)-(2), where $f : G^-([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$ is such that, for every $y \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$, $t \mapsto f(y_t, t)$ is locally Lebesgue integrable on $t \in [t_0, +\infty)$. If there is a function $y \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$ satisfying*

(i) $\dot{y}(t) = f(y_t, t)$, *for almost every* $t \in [t_0, +\infty) \setminus \{t : t = \tau_k(y(t)),\ k = 1, 2, \ldots\}$;

(ii) $y(t+) = y(t) + I_k(y(t))$, $t = \tau_k(y(t))$, $k = 1, 2, \ldots$;

(iii) $y_{t_0} = \phi$,

then y is called a solution *of (1)-(2) on $[t_0 - r, +\infty)$ with initial condition (ϕ, t_0).*

3. Generalized ordinary differential equations

Let X be a Banach space and consider the set $\Omega = O \times [t_0, +\infty)$, where $O \subset X$ is an open set. Assume that $G : \Omega \to X$ is a given X-valued function defined for all $(x, t) \in \Omega$.

Having the concept of Kurzweil integrability in mind (see, e.g., [8], [12] or the Appendix), we present the concept of generalized ordinary differential equation.

Definition 3.1. *A function* $x : [\alpha, \beta] \to X$ *is called* a solution of the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG(x, t) \tag{3}$$

in the interval $[\alpha, \beta] \subset [t_0, +\infty)$, *if* $(x(t), t) \in \Omega$ *for all* $t \in [\alpha, \beta]$ *and if the equality*

$$x(v) - x(\gamma) = \int_{\gamma}^{v} DG(x(\tau), t) \tag{4}$$

holds for every γ, $v \in [\alpha, \beta]$, *where the integral is in Kurzweil's sense.*

In particular, a function $x : [\alpha, \beta] \to X$ is a *solution of the generalized ordinary differential equation* (3) *with the initial condition* $x(t_0) = \widetilde{x}$, *on the interval* $[\alpha, \beta] \subset [t_0, +\infty)$, if $t_0 \in [\alpha, \beta]$, $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality

$$x(v) - \widetilde{x} = \int_{t_0}^{v} DG(x(\tau), t) \tag{5}$$

holds for every $v \in [\alpha, \beta]$.

Now we define a special class of functions $G : \Omega \to X$ for which we can derive interesting properties of the solutions of (3).

Definition 3.2. *A function* $G : \Omega \to X$ *belongs to the class* $\mathcal{F}(\Omega, h)$, *if there exists a nondecreasing function* $h : [t_0, +\infty) \to \mathbb{R}$ *such that*

$$\|G(x, s_2) - G(x, s_1)\| \leq |h(s_2) - h(s_1)| \tag{6}$$

for all (x, s_2), $(x, s_1) \in \Omega$ *and*

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) + G(y, s_1)\| \leq \|x - y\| |h(s_2) - h(s_1)| \tag{7}$$

for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in \Omega$.

By Definition 3.1 and by the definition of the Kurzweil integral (see [8], [12] or the Appendix, for instance), if $G : \Omega \to X$ satisfies (6) and $x : [\alpha, \beta] \to X$ is a solution of (3) with $[\alpha, \beta] \subset [t_0, +\infty)$, then the inequality

$$\|x(s_1) - x(s_2)\| \leq |h(s_2) - h(s_1)| \tag{8}$$

holds for every $s_1, s_2 \in [\alpha, \beta]$. See [12], Lemma 3.10 for a proof of this fact. Besides, if $\mathrm{var}_{\alpha}^{\beta}(x)$ denotes the variation of a function $x : [\alpha, \beta] \to X$ on the interval $[\alpha, \beta]$, then it follows from (8) that x is of bounded variation on $[\alpha, \beta]$ and

$$\mathrm{var}_{\alpha}^{\beta} x \leq h(\beta) - h(\alpha) < +\infty.$$

Then it is clear that every point in $[\alpha, \beta]$ at which the function h is continuous is a continuity point of the solution $x : [\alpha, \beta] \to X$.

Now we present a result on the existence of the Kurzweil integral involved in the definition of a solution of the generalized ODE (3) (see Definition 3.1). This result is a particular case of Corollary 3.16 from [12].

Proposition 3.1. *Let $G \in \mathcal{F}(\Omega, h)$. Suppose $[\alpha, \beta] \subset [t_0, +\infty)$ and $x : [\alpha, \beta] \to X$ is a function of bounded variation on $[\alpha, \beta]$ and $(x(s), s) \in \Omega$ for every $s \in [\alpha, \beta]$. Then the integral $\int_\alpha^\beta DG(x(\tau), t)$ exists and the function $s \mapsto \int_\alpha^s DG(x(\tau), t) \in X$ is of bounded variation.*

The next result can be found in [12], Lemma 3.12. It describes the descontinuities of a solution of (3), provided G satisfies (6).

Proposition 3.2. *If $[\alpha, \beta] \subset [t_0, +\infty)$ and $x : [\alpha, \beta] \to X$ is a solution of (3) and $G : \Omega \to X$ satisfies condition (6), then*

$$x(\sigma+) - x(\sigma) = G(x(\sigma), \sigma+) - G(x(\sigma), \sigma), \quad \text{for } \sigma \in [\alpha, \beta)$$

and

$$x(\sigma) - x(\sigma-) = G(x(\sigma), \sigma) - G(x(\sigma), \sigma-), \quad \text{for } \sigma \in (\alpha, \beta],$$

where

$$G(x, \sigma+) = \lim_{s \to \sigma+} G(x, s), \quad \text{for } \sigma \in [\alpha, \beta)$$

and

$$G(x, \sigma-) = \lim_{s \to \sigma-} G(x, s), \quad \text{for } \sigma \in (\alpha, \beta].$$

The next result concerns existence of a solution of (3). Uniqueness is obtained provided an initial condition is given. See [3], Theorem 2.15, for a proof.

Theorem 3.1 (Local existence and uniqueness). *Let $G \in \mathcal{F}(\Omega, h)$, where the function h is non-decreasing and left continuous. If for every $(\tilde{x}, t_0) \in \Omega$ such that for $\tilde{x}_+ = \tilde{x} + G(\tilde{x}, t_0+) - G(\tilde{x}, t_0)$ we have $(\tilde{x}_+, t_0) \in \Omega$, then there exists $\Delta > 0$ such that there exists a unique solution $x : [t_0, t_0 + \Delta] \to X$ of the generalized ODE (3) for which $x(t_0) = \tilde{x}$.*

The assumption that the function h is left continuous in Theorem 3.1 implies, by (8), that the solutions of (3) are also left continuous. Given a solution x of (3), the limit $x(\sigma-)$ exists for every σ in the domain of x. This follows again by (8). Moreover, by Proposition 3.2, we have the relation

$$x(\sigma) = x(\sigma-) + G(x(\sigma), \sigma) - G(x(\sigma), \sigma-)$$

which describes the discontinuity of a given solution x of (3).

4. Boundedness of solutions of generalized ODEs

The main results of this section concern the boundedness of the unique solution of an initial value problem for a generalized ODE whose righthand side belongs to $\mathcal{F}(\Omega, h)$.

Let X be a Banach space and set $\Omega = O \times [t_0, +\infty)$, where $O \subset X$ is an open set. We assume that $G \in \mathcal{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a non-decreasing and left continuous function.

Consider the generalized ODE

$$\frac{dx}{d\tau} = DG(x, t) \tag{9}$$

subject to the initial condition

$$x(t_0) = z_0, \tag{10}$$

where $t_0 \geq 0$ and $z_0 \in O$. Let $x(t) = x(t, t_0, z_0)$ be the solution of (9)-(10) defined on the interval $[t_0, +\infty)$.

Definition 4.1. *The solution $x(t) = x(t, t_0, z_0)$ of system (9)-(10) is said to be*

(*i*) Uniformly bounded, *if for every $\alpha > 0$, there exists $M = M(\alpha) > 0$ such that if*

$$\|z_0\| < \alpha,$$

then

$$\|x(t)\| < M, \quad \textit{for all} \quad t \geq t_0.$$

(*ii*) Quasi-uniformly ultimately bounded, *if there exists a constant $B > 0$ such that for every $\alpha > 0$, there exists a constant $T = T(\alpha) > 0$ such that if*

$$\|z_0\| < \alpha,$$

then

$$\|x(t)\| < B, \quad \textit{for all} \quad t \geq t_0 + T.$$

(*iii*) Uniformly ultimately bounded, *if it is uniformly bounded and quasi-uniformly ultimately bounded.*

In the sequel, we use Lyapunov functionals to obtain boundedness results. But before that, we mention an auxiliary result whose proof follows by straightforward adaptation of the proof of Lemma 10.12 from [12] with obvious adaptations to Banach-space valued functions.

Lemma 4.1. *Let $G \in \mathcal{F}(\Omega, h)$. Suppose $V : [t_0, +\infty) \times X \to \mathbb{R}$ is such that $V(\cdot, x) : [t_0, +\infty) \to \mathbb{R}$ is left continuous on $(t_0, +\infty)$ for $x \in X$ and satisfies*

$$|V(t, z) - V(t, y)| \leq K\|z - y\|, \quad z, y \in X, \ t \in [t_0, +\infty),$$

where $K > 0$ is a constant. Suppose, in addition, that there is a function $\Phi : X \to \mathbb{R}$ such that for every solution $x : [a, b] \to X$ of (9) with $[a, b] \subset [t_0, +\infty)$, we have

$$\dot{V}(t, x(t)) = \limsup_{\eta \to 0^+} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leq \Phi(x(t)), \quad t \in [a, b].$$

If $\overline{x} : [\gamma, v] \to X$, $t_0 \leq \gamma < v < +\infty$ is left continuous on $(\gamma, v]$ and of bounded variation on $[\gamma, v]$, then

$$V(v, \overline{x}(v)) - V(\gamma, \overline{x}(\gamma)) \leq K\, var_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), t)\right) + M(v - \gamma),$$

where $M = \sup_{t \in [\gamma, v]} \Phi(\overline{x}(t))$.

Theorem 4.1. *Let $V : [t_0, +\infty) \times X \to \mathbb{R}$ be such that $V(\cdot, x) : [t_0, +\infty) \to \mathbb{R}$ is left continuous on $(t_0, +\infty)$ for $x \in X$ and the following conditions hold:*

(i) *$V(t, 0) = 0$, for each $t \in [t_0, +\infty)$;*

(ii) *For each $a > 0$, there is a constant $K_a > 0$ such that*

$$|V(t, z) - V(t, y)| \leq K_a \|z - y\|, \quad t \in [t_0, +\infty), \quad y, z \in B_a,$$

where $B_a = \{x \in X : \|x\| < a\}$;

(iii) *There is a monotone increasing function $b : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $b(0) = 0$ and $b(s) \to +\infty$ as $s \to +\infty$, such that*

$$V(t, z) \geq b(\|z\|), \quad t \in [t_0, +\infty) \text{ and } z \in X;$$

(iv) *The right derivative of V along every solution $x : [\gamma, v] \to X$ of (9), where $[\gamma, v] \subset [t_0, +\infty)$, is non-positive, that is,*

$$\dot{V}(t, x(t)) = \limsup_{\eta \to 0+} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leq 0,$$

for each $t \in [\gamma, v]$.

Then the solution $x(t) = x(t, t_0, z_0)$ of initial value problem (9)-(10) is uniformly bounded.

Proof. Let $x : [t_0, +\infty) \to X$ be the solution of (9)-(10). We assert that $V(t, x(t)) \leq V(t_0, x(t_0))$ for all $t \geq t_0$. In fact, take $t > t_0$ and $a = 2 \sup_{s \in [t_0, t]} \|x(s)\|$. Note that x is left continuous on $(t_0, +\infty)$ and of bounded variation on $[t_0, t]$ by (8).

By item (ii), there exists $K_a > 0$ such that

$$|V(\xi, z) - V(\xi, y)| \leq K_a \|z - y\|$$

for all $\xi \in [t_0, t]$ and all $z, y \in B_a$. Thus, by Lemma 4.1, we have

$$V(t, x(t)) \leq V(t_0, x(t_0)) + K_a \operatorname{var}_{t_0}^t \left(x(s) - \int_{t_0}^s DG(x(\tau), t)\right).$$

On the other hand, since x is a solution of (9), it follows from Definition 3 that

$$\operatorname{var}_{t_0}^t \left(x(s) - \int_{t_0}^s DG(x(\tau), t)\right) = 0. \tag{11}$$

Thus $V(t, x(t)) \leq V(t_0, x(t_0))$ and, since t is arbitrary,

$$V(t, x(t)) \leq V(t_0, x(t_0)), \quad \text{for all} \quad t \geq t_0. \tag{12}$$

Now, let $\alpha > 0$ be such that

$$\|x(t_0)\| < \alpha. \tag{13}$$

Since $z_0 = x(t_0) \in B_a$, by (i) and (ii), we have

$$V(t_0, x(t_0)) \leq |V(t_0, x(t_0))| \leq K_a \|x(t_0)\|. \tag{14}$$

Also, since $b(s) \to +\infty$ as $s \to +\infty$, we can choose a positive number $M = M(\alpha)$ such that

$$K_a \alpha < b(M). \tag{15}$$

Then, by (12), (14), (13) and (15), we obtain

$$V(t, x(t)) < b(M), \quad \text{for all } \ t \geq t_0. \tag{16}$$

Now, we need to prove that

$$\|x(t)\| < M, \quad \text{for} \quad t \geq t_0.$$

Suppose the contrary, that is, assume that there exists $t^* \geq t_0$ such that $\|x(t^*)\| \geq M$. Then item (iii) implies

$$V(t^*, x(t^*)) \geq b(\|x(t^*)\|) \geq b(M),$$

which contradicts (16). Hence $\|x(t)\| < M$ for all $t \geq t_0$ and the result follows. □

The next result provides sufficient conditions for the unique solution of system (9) - (10) to be uniformly ultimately bounded.

Theorem 4.2. *Assume that* $V : [t_0, +\infty) \times X \to \mathbb{R}$ *satisfies conditions* (i), (ii) *and* (iii) *from Theorem 4.1. Suppose there is a continuous function* $\Phi : X \to \mathbb{R}$, *with* $\Phi(0) = 0$ *and* $\Phi(x) > 0$ *for* $x \neq 0$, *such that for every solution* $x : [\gamma, v] \to X$ *of* (9), *where* $[\gamma, v] \subset [t_0, +\infty)$, *we have*

$$\dot{V}(t, x(t)) \leq -\Phi(x(t)), \quad t \in [\gamma, v]. \tag{17}$$

Then the solution $x(t) = x(t, t_0, z_0)$ *of initial value problem* (9)-(10) *is uniformly ultimately bounded.*

Proof. By (17), $\dot{V}(t, x(t)) \leq 0$. Thus Theorem 4.1 implies that the solution $x(t) = x(t, t_0, z_0)$ of (9)-(10) defined on the interval $[t_0, +\infty)$ is uniformly bounded. Therefore it remains to prove that $x(t, t_0, z_0)$ is quasi-uniformly ultimately bounded.

Since the solution $x(t)$ of (9)-(10) is uniformly bounded, given $\alpha > 0$, there exists a positive number $M = M(\alpha)$ such that if

$$\|z_0\| < \alpha, \tag{18}$$

then

$$\|x(t)\| < M, \quad \text{for all } \ t \geq t_0. \tag{19}$$

Let $[\bar{t}, +\infty) \subset [t_0, +\infty)$ and define $y : [\bar{t}, +\infty) \to X$ by $y(t) = x(t)$ for all $t \in [\bar{t}, +\infty)$. Note that if $\|y(\bar{t})\| < \rho$, where $\rho > 0$, then there exists $B > \rho$ such that

$$\|y(t)\| < B, \quad \text{for all } \ t \geq \bar{t}, \tag{20}$$

since (19) holds.

Consider ρ, α and B as above and take $\lambda = \min\{\alpha, \rho\}$. Let $a = 2\|x(t_0)\|$ and define

$$N := \sup\{-\Phi(w) : \lambda \leq \|w\| \leq B\} < 0$$

and

$$T(\alpha) := \frac{-2K_a\alpha}{N} > 0. \tag{21}$$

We want to prove that $\|x(t)\| < B$, for all $t \geq t_0 + T(\alpha)$. In order to do that, we first assert that there exists $t^* \in \left[t_0 + \frac{T(\alpha)}{2}, t_0 + T(\alpha)\right]$ such that $\|x(t^*)\| < \lambda$. Indeed. Suppose the contrary, that is, $\|x(s)\| \geq \lambda$, for every $s \in \left[t_0 + \frac{T(\alpha)}{2}, t_0 + T(\alpha)\right]$. Taking $\overline{a} = \sup_{s\in\left[t_0+\frac{T(\alpha)}{2}, t_0+T(\alpha)\right]} \|x(s)\|$, we know that there exists $K_{\overline{a}} > 0$ such that

$$|V(t, y) - V(t, z)| \leq K_{\overline{a}} \|z - y\|,$$

for all $t \in \left[t_0 + \frac{T(\alpha)}{2}, t_0 + T(\alpha)\right]$ and all $z, y \in B_{\overline{a}}$. Then, by Lemma 4.1, equation (11), conditions (i), (ii) and (iii) from Theorem 4.1 and also by (18) and (21), we have

$$\begin{aligned}
&V(t_0 + T(\alpha), x(t_0 + T(\alpha))) \leq \\
\leq\ & V\left(t_0 + \frac{T(\alpha)}{2}, x\left(t_0 + \frac{T(\alpha)}{2}\right)\right) + K_{\overline{a}} \operatorname{var}_{t_0+\frac{T(\alpha)}{2}}^{t_0+T(\alpha)} \left(x(s) - \int_{t_0+\frac{T(\alpha)}{2}}^{s} DG(x(\tau), t)\right) + \\
+\ & \frac{T(\alpha)}{2} \sup\left\{-\Phi(x(t)) : t_0 + \frac{T(\alpha)}{2} \leq t \leq t_0 + T(\alpha)\right\} \\
\leq\ & V(t_0, x(t_0)) + \frac{T(\alpha)}{2} \sup\{-\Phi(w) : \lambda \leq \|w\| \leq B\} \\
\leq\ & K_a\|x(t_0)\| - K_a\alpha < K_a\alpha - K_a\alpha = 0.
\end{aligned}$$

On the other hand, by condition (iii) from Theorem 4.1,

$$V(t_0 + T(\alpha), x(t_0 + T(\alpha))) \geq b\left(\|x(t_0 + T(\alpha))\|\right) \geq b(\lambda) > 0,$$

which is a contradiction and hence the assertion holds. Thus $\|x(t)\| < B$, for $t \geq t^*$, since (20) holds for $\overline{t} = t^*$. Also, $\|x(t)\| < B$, for $t > t_0 + T(\alpha)$, since $t^* \in \left[t_0 + \frac{T(\alpha)}{2}, t_0 + T(\alpha)\right]$. Therefore the solution $x(t) = x(t, t_0, z_0)$ of (9)- (10) is quasi-uniformly ultimately bounded and the proof is complete. □

5. Boundedness of solutions of impulsive retarded systems

Now we turn our attention to impulsive retarded functional differential equations. We will establish results on the boundedness of solutions of these equations by the theory of generalized ODEs.

Let $t_0 \geq 0$ and $r > 0$. Given $y \in PC_1$ and $t \in [t_0, +\infty)$, we define

$$F(y,t)(\vartheta) = \begin{cases} 0, & t_0 - r \leq \vartheta \leq t_0, \\ \int_{t_0}^{\vartheta} f(y_s, s)\, ds, & t_0 \leq \vartheta \leq t < +\infty, \\ \int_{t_0}^{t} f(y_s, s)\, ds, & t_0 \leq t \leq \vartheta < +\infty, \end{cases} \tag{22}$$

and

$$J(y,t)(\vartheta) = \sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i(t) H_k^i(\vartheta) I_k(y(t_k^i)), \tag{23}$$

where $\vartheta \in [t_0 - r, +\infty)$ and H_k^i denotes the left continuous Heavyside function concentrated at t_k^i, that is,

$$H_k^i(t) = \begin{cases} 0, & \text{for } t_0 \leq t \leq t_k^i, \\ 1, & \text{for } t > t_k^i. \end{cases}$$

Taking $F(y,t)$ and $J(y,t)$ given by (22) and (23), we define

$$G(y,t)(\vartheta) = F(y,t)(\vartheta) + J(y,t)(\vartheta), \tag{24}$$

for $y \in PC_1$, $t \in [t_0, +\infty)$ and $\vartheta \in [t_0 - r, -\infty)$. Clearly the values of $G(y,t)$ belong to $G^-([t_0 - r, +\infty), \mathbb{R}^n)$, that is,

$$G : PC_1 \times [t_0, +\infty) \to G^-([t_0 - r, +\infty), \mathbb{R}^n).$$

Moreover, for $s_1, s_2 \in [t_0, +\infty)$ and $x, y \in PC_1$ we have

$$\|G(x, s_2) - G(x, s_1)\| \leq |h(s_2) - h(s_1)| \tag{25}$$

and

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) - G(y, s_1)\| \leq \|x - y\| |h(s_2) - h(s_1)|, \tag{26}$$

where

$$h(t) = \int_{t_0}^{t} [M(s) + L(s)] ds + \max\{K_1, K_2\} \sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i(t), \quad t \in [t_0, +\infty)$$

is a nondecreasing real function which is continuous from the left at every point, continuous at $t \neq t_k^i$ and $h(t_k^i+)$ exists for $k = 1, 2, \ldots$ and $i = 1, 2, \ldots$. For details, see [3].

According to (25) and (26), the function G defined by (24) belongs to the class $\mathcal{F}(\Omega, h)$, where $\Omega = PC_1 \times [t_0, +\infty)$.

Consider the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG(x, t), \tag{27}$$

where G is given by (24). The next result gives a one-to-one relation between the solution of the impulsive RFDE (1) and the solution of the generalized ODE (27), with initial condition depending on the initial condition of (1). A proof of this fact can be carried out by following the ideas of Theorems 3.4 and 3.5 from [3].

Theorem 5.1 (Correspondence of equations).

(i) *Consider system* (1)-(2), *where* $f : G^-([-r,0],\mathbb{R}^n) \times [t_0,+\infty) \to \mathbb{R}^n$, *for each* $y \in G^-([t_0-r,+\infty),\mathbb{R}^n)$, $t \mapsto f(y_t,t)$ *is locally Lebesgue integrable over* $[t_0,+\infty)$ *and conditions* (A), (B), (A'), (B') *are fulfilled. Let* $y : [t_0-r,+\infty) \to \mathbb{R}^n$ *be the solution of the impulsive RFDE* (1) *on the interval* $[t_0,+\infty)$. *Given* $t \in [t_0,+\infty)$, *let*

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & t_0 - r \le \vartheta \le t, \\ y(t), & t \le \vartheta < +\infty. \end{cases}$$

Then $x(t) \in G^-([t_0-r,+\infty),\mathbb{R}^n)$ *and* x *is the solution of* (27) *on* $[t_0,+\infty)$, *with* G *given by* (24).

(ii) *Reciprocally, let* x *be the solution of* (27), *with* G *given by* (24), *on the interval* $[t_0,+\infty)$ *satisfying the initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & t_0 - r \le \vartheta \le t_0, \\ x(t_0)(t_0), & t_0 \le \vartheta < +\infty. \end{cases}$$

For every $\vartheta \in [t_0-r,+\infty)$, *define*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \le \vartheta \le t_0, \\ x(\vartheta)(\vartheta), & t_0 \le \vartheta < +\infty. \end{cases}$$

Then $y : [t_0-r,+\infty) \to \mathbb{R}^n$ *is a solution of* (1)-(2) *on* $[t_0-r,+\infty)$.

By Theorem 3.1, for $\widetilde{x} \in PC_1$ the condition

$$\widetilde{x}+ = \widetilde{x} + G(\widetilde{x},t_0+) - G(\widetilde{x},t_0) \in PC_1,$$

is needed, since it assures that the solution of the initial value problem for the generalized ODE (27) does not jump out of the set PC_1 immediately after the moment t_0. In our setting, where G is given by (24), we have $G(\widetilde{x},t_0^+) - G(\widetilde{x},t_0) = 0$, since $t_0 < t_k^i$, $i = 1,\dots,m(\tau_k)$, $k = 1,2,\dots$, that is, t_0 is not a moment of impulse.

Let $y : [t_0-r,+\infty) \to \mathbb{R}^n$ be the solution of the initial value problem (1)-(2). We write $y(t) = y(t,t_0,\phi)$.

Definition 5.1. *The solution* $y(t) = y(t,t_0,\phi)$ *of system* (1)-(2) *is said to be*

(*i*) Uniformly bounded, *if for every* $\alpha > 0$, *there exists* $M = M(\alpha) > 0$ *such that if*

$$\|\phi\| < \alpha,$$

then

$$|y(t)| < M, \quad t \ge t_0.$$

(*ii*) Quasi-uniformly ultimately bounded, *if there exists a constant* $B > 0$ *such that for every* $\alpha > 0$, *there exists a constant* $T = T(\alpha) > 0$ *such that if*

$$\|\phi\| < \alpha,$$

then

$$|y(t)| < B, \quad t \geq t_0 + T(\alpha).$$

(*iii*) Uniformly ultimately bounded, *if it is uniformly bounded and quasi-uniformly ultimately bounded.*

We will apply Theorem 5.1 together with Theorems 4.1 and 4.2 to obtain results on the boundedness of the solution of problem (1)-(2).

Given $t \geq t_0$ and a function $\psi \in G^-([-r,0],\mathbb{R}^n)$, consider equation (1) with initial condition $y_t = \psi$. This initial value problem admits a unique local solution $y : [t-r,v] \to \mathbb{R}^n$ with $[t-r,v] \subset [t-r,+\infty)$ (see [2], Theorem 2.1). Then, by Theorem 5.1(i), we can find a solution $x : [t,v] \to G^-([t,v],\mathbb{R}^n)$ of the generalized ODE (27), with initial condition $x(t) = \widetilde{x}$, where $\widetilde{x}(\tau) = \psi(\tau - t)$, $t - r \leq \tau \leq t$, and $\widetilde{x}(\tau) = \psi(0)$, $\tau \geq 0$. Then $x(t)(t+\theta) = y(t+\theta)$ for all $\theta \in [-r,0]$ and, hence, $(x(t))_t = y_t$. In this case, we write $y_{t+\eta} = y_{t+\eta}(t,\psi)$ for every $\eta \geq 0$. Then for $U : [t_0, +\infty) \times G^-([-r,0],\mathbb{R}^n) \to \mathbb{R}$, we define

$$D^+U(t,\psi) = \limsup_{\eta \to 0+} \frac{U(t+\eta, y_{t+\eta}(t,\psi)) - U(t, y_t(t,\psi))}{\eta}, \quad t \geq t_0.$$

On the other hand, given $t \geq t_0$, if $\widetilde{x} \in G^-([t-r,+\infty),\mathbb{R}^n)$ is such that $\widetilde{x}(\tau) = \psi(\tau - t)$, $t - r \leq \tau \leq t$, and $\widetilde{x}(\tau) = \psi(0)$, $\tau \geq t$, then there exists a unique solution $x : [t,\overline{v}] \to G^-([t,\overline{v}],\mathbb{R}^n)$ of the generalized ODE (27) such that $x(t) = \widetilde{x}$, with $[t,\overline{v}] \subset [t_0,+\infty)$. By Theorem 5.1($ii$), we can find a solution $y : [t-r,\overline{v}] \to \mathbb{R}^n$ of (1) which satisfies $y_t = \psi$, and is described in terms of x. Then, we write $x_\psi(t)$ instead of $x(t)$ and we have $y_t(t,\psi) = (x_\psi(t))_t = \psi$. Consequently, $(t, x_\psi(t)) \mapsto (t, y_t(t,\psi))$ is a one-to-one mapping and we can define a function $V : [t_0,+\infty) \times G^-([t_0 - r,+\infty),\mathbb{R}^n) \to \mathbb{R}$ by

$$V(t, x_\psi(t)) = U(t, y_t(t,\psi)). \tag{28}$$

Hence

$$D^+U(t,\psi) = \limsup_{\eta \to 0+} \frac{V(t+\eta, x_\psi(t+\eta)) - V(t, x_\psi(t))}{\eta}, \quad t \geq t_0. \tag{29}$$

Lemma 5.1. *Let $U : [t_0, +\infty) \times G^-([-r,0],\mathbb{R}^n) \to \mathbb{R}$ and assume that the following conditions hold:*

(i) $U(t,0) = 0$, *for all $t \in [t_0,+\infty)$;*

(ii) *For each $a > 0$, there is a constant $K_a > 0$ such that*

$$|U(t,\psi) - U(t,\overline{\psi})| \leq K_a\|\psi - \overline{\psi}\|, \quad t \in [t_0,+\infty), \ \ \psi,\overline{\psi} \in \mathcal{B}_a.$$

where $\mathcal{B}_a = \{\phi \in G^-([-r,0],\mathbb{R}^n) : \|\phi\| < a\}$. Then the function V defined by (28) *satisfies $V(t,0) = 0$ for all $t \in [t_0,+\infty)$, and*

$$|V(t,x) - V(t,\overline{x})| \leq K_a\|x - \overline{x}\|,$$

for all $t \geq t_0$ and all $x, \overline{x} \in B_a$, where $B_a = \{\psi \in G^-([t_0 - r,+\infty),\mathbb{R}^n) : \|\psi\| < a\}$.

Proof. Given $t \geq t_0$, let $y, \overline{y}, \widehat{y} : [t-r, +\infty) \to \mathbb{R}^n$ be solutions of equation (1) with initial conditions $y_t = \psi$, $\overline{y}_t = \overline{\psi}$ and $\widehat{y}_t = 0$. Suppose $x, \overline{x}, \widehat{x}$ are solutions on $[t, +\infty)$ of the generalized ODE (27) given by Theorem 5.1(i) and corresponding to y,$\overline{y}$ and $\widehat{y}$ respectively. Then $(x(t))_t = y_t = \psi$, $(\overline{x}(t))_t = \overline{y}_t = \overline{\psi}$ and $(\widehat{x}(t))_t = \widehat{y}_t = 0$.

Notice that since f satisfies (A) and (B) and I_k satisfies (A') and (B') for $k = 1, 2, \ldots$, the function G in equation (27) belongs to $\mathcal{F}(\Omega, h)$.

Let $V : [t_0, +\infty) \times G^-([t_0 - r, +\infty), \mathbb{R}^n) \to \mathbb{R}$ be given by (28). By condition (i), we have

$$0 = U(t, 0) = U(t, \widehat{y}_t(t, 0)) = V(t, \widehat{x}(t)) = V(t, 0),$$

since $\widehat{x}(t)$ is such that $\widehat{x}(t)(\tau) = 0$ for all τ (see Theorem 5.1(i)), that is, $\widehat{x}(t) \equiv 0$. By condition (ii), for $a = 2\max\{\|\psi\|, \|\overline{\psi}\|\}$, we have

$$\begin{aligned} |V(t, x_\psi(t)) - V(t, \overline{x}_{\overline{\psi}}(t))| &= |U(t, y_t(t, \psi)) - U(t, \overline{y}_t(t, \overline{\psi}))| \\ &= |U(t, \psi) - U(t, \overline{\psi})| \\ &\leq K_a\|\psi - \overline{\psi}\| = K_a\|x_\psi(t) - \overline{x}_{\overline{\psi}}(t)\|, \end{aligned}$$

where we applied Theorem 5.1(i) to obtain the last equality. Then it is clear that given $t \geq t_0$ and $z, \overline{z} \in B_a$, there exist solutions x and $\overline{x}$ of the generalized ODE (27) and functions $\psi, \overline{\psi} \in G^-([-r, 0], \mathbb{R}^n)$ such that $z = x_\psi(t)$, $(x_\psi(t))_t = y_t(t, \psi)$, $\overline{z} = \overline{x}_{\overline{\psi}}(t)$ and $(\overline{x}_{\overline{\psi}}(t))_t = \overline{y}_t(t, \overline{\psi})$. Since

$$\|\psi\| = \|y_t(t, \psi)\| = \|x_\psi(t)\| = \|z\| < a$$

and

$$\|\overline{\psi}\| = \|\overline{y}_t(t, \overline{\psi})\| = \|\overline{x}_{\overline{\psi}}(t)\| = \|\overline{z}\| < a,$$

then

$$|V(t, z) - V(t, \overline{z})| \leq K_a\|z - \overline{z}\|, \qquad z, \overline{z} \in B_a.$$

Finally, since t is arbitrary, the result follows. □

With the previous notation, we now are able to prove the next two results concerning the boundedness of the solution of (1)-(2), provided (A), (B), (A'), (B') are fulfilled.

Theorem 5.2. *Consider system* (1)-(2) *and suppose conditions* (A), (B), (A'), (B') *are fulfilled. Let* $U : [t_0, +\infty) \times G^-([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ *be left continuous on* $(t_0, +\infty)$ *and assume that* U *satisfies the following conditions:*

(i) $U(t, 0) = 0$, $t \in [t_0, +\infty)$;

(ii) *For each* $a > 0$, *there is a constant* $K_a > 0$ *such that*

$$|U(t, \psi) - U(t, \overline{\psi})| \leq K_a\|\psi - \overline{\psi}\|, \quad t \in [t_0, +\infty), \quad \psi, \overline{\psi} \in B_a;$$

(iii) *There is a monotone increasing function* $b : \mathbb{R}_+ \to \mathbb{R}_+$, *such that* $b(0) = 0$, $b(s) \to +\infty$ *as* $s \to +\infty$ *and*

$$U(t, \psi) \geq b(\|\psi\|),$$

for all $t \geq t_0$ *and for all* $\psi \in G^-([-r, 0], \mathbb{R}^n)$;

(iv) *The inequality*

$$D^+U(t,\psi)\leq 0$$

holds for $t\geq t_0$ *and* $\psi\in G^-([-r,0],\mathbb{R}^n)$.

Then the solution $y(t)=y(t,t_0,\phi)$ *of system* (1)-(2) *is uniformly bounded.*

Proof. Notice that, since f satisfies (A) and (B) and I_k satisfies (A') and (B') for $k=1,2,\ldots$, the function G from equation (27) belongs to $\mathcal{F}(\Omega,h)$.

Considering the function $V:[t_0,+\infty)\times G^-([t_0-r,+\infty),\mathbb{R}^n)\to\mathbb{R}$ given by (28), we have

$$V(t,0)=0 \;\text{ for }\; t\in[t_0,+\infty)$$

and

$$|V(t,x)-V(t,\overline{x})|\leq K_a\|x-\overline{x}\|$$

for $t\in[t_0,+\infty)$ and $x,\overline{x}\in B_a$ (see Lemma 5.1).

By condition (iii), we have

$$b(\|y_t\|)=b(\|\psi\|)\leq U(t,\psi)=U(t,y_t(t,\psi))=V(t,x_\psi(t))$$

for a solution y of (1) which satisfies $y_t=\psi$, where $t\geq t_0$.

On the other hand,

$$\|x_\psi(t)\|=\sup_{t-r\leq\tau<+\infty}|x_\psi(t)(\tau)|=\sup_{t-r\leq\tau\leq t}|y(\tau)|=\sup_{\theta\in[-r,0]}|y_t(\theta)|=\|y_t\|,$$

where we applied Theorem 5.1(i) to obtain the second equality. Hence

$$V(t,x_\psi(t))\geq b\left(\|x_\psi(t)\|\right)$$

and by previous arguments (see Lemma 5.1), we have

$$V(t,z)\geq b\left(\|z\|\right),$$

for every $z\in G^-([t_0-r,+\infty),\mathbb{R}^n)$.

Thus the function V satisfies conditions (i), (ii) and (iii) from Theorem 4.1.

Assume that $x:[t,+\infty)\to G^-([t,+\infty),\mathbb{R}^n)$ is a solution of (27) such that $(x(t))_t=\psi$, where $t\in[t_0,+\infty)$. By (29), we have

$$\dot{V}(t,x_\psi(t))=D^+U(t,\psi)\leq 0.$$

This implies that condition (iv) from Theorem 4.1 is satisfied.

Now, let $y:[t_0-r,+\infty)\to\mathbb{R}^n$ be the solution of system (1)-(2) and let x be the solution on $[t_0,+\infty)$ of the generalized ODE (27) given by Theorem 5.1(i), satisfying the initial condition $x(t_0)=z_0$, where

$$z_0(\tau)=\begin{cases}\phi(\tau-t_0),\; t_0-r\leq\tau\leq t_0,\\ \phi(0),\;\;\tau\geq t_0.\end{cases}\tag{30}$$

Thus x can be written as

$$x(s)(\tau) = \begin{cases} y(\tau), & t_0 - r \le \tau \le s, \\ y(s), & \tau \ge s, \end{cases} \tag{31}$$

for $s \ge t_0$.

Let $\alpha > 0$ be such that

$$\|\phi\| < \alpha. \tag{32}$$

Note that

$$\|z_0\| = \sup_{t_0 - r \le \tau < +\infty} \|z_0(\tau)\| = \|\phi\|, \tag{33}$$

by (30). Therefore (32) and (33) imply

$$\|z_0\| < \alpha. \tag{34}$$

By Theorem 4.1, x is uniformly bounded. Hence there exists $M > 0$ such that

$$\|x(t)\| < M, \quad \text{for any } t \ge t_0. \tag{35}$$

Thus $\|x(t)\| < M$, for all $t \in [t_0, b]$, where b is any element of $[t_0, +\infty)$. In particular, $\|x(b)\| < M$. But (31) implies that for any $t \in [t_0, b]$,

$$\begin{aligned} |y(t)| \le \|y_t\| &= \sup_{-r \le \theta \le 0} |y(t+\theta)| \le \sup_{t_0 - r \le \tau \le b} |y(\tau)| \\ &= \sup_{t_0 - r \le \tau \le b} |x(b)(\tau)| = \|x(b)\| < M. \end{aligned} \tag{36}$$

Thus the solution $y(t) = y(t, t_0, \phi)$ of (1)-(2) is uniformly bounded and we finished the proof. □

The next theorem concerns the uniform ultimate boundedness of the unique solution of (1)-(2).

Theorem 5.3. *Consider system* (1)-(2), *where* (A), (B), (A'), (B') *are fulfilled. Assume that* $U : [t_0, +\infty) \times G^-([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ *satisfies conditions* (*i*) *to* (*iii*) *from Theorem 5.2. Suppose there is a continuous function* $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying* $\Lambda(0) = 0$ *and* $\Lambda(x) > 0$ *if* $x \neq 0$, *such that for every* $\psi \in G^-([-r, 0], \mathbb{R}^n)$, *we have*

$$D^+U(t, \psi) \le -\Lambda(\|\psi\|), \qquad t \ge t_0. \tag{37}$$

Then the solution $y(t) = y(t, t_0, \phi)$ *of* (1)-(2) *is uniformly ultimately bounded.*

Proof. We assume the notation of the previous theorem.

Suppose $V : [t_0, +\infty) \times G^-([t_0 - r, +\infty), \mathbb{R}^n) \to \mathbb{R}$ is given by (28). Then the hypotheses of Theorem 4.1 are fulfilled.

Let $\Phi : G^-([t_0 - r, +\infty), \mathbb{R}^n) \to \mathbb{R}$ be defined by

$$\Phi(z) = \Lambda(\|z\|), \quad z \in G^-([t_0 - r, +\infty), \mathbb{R}^n).$$

Then Φ is continuous, $\Phi(0) = 0$ and $\Phi(z) > 0$, for $z \neq 0$.

Assume that $x : [t, +\infty) \to G^-([t-r, +\infty), \mathbb{R}^n)$ is a solution of (27) such that $(x(t))_t = \psi$, where $t \in [t_0, +\infty)$, and suppose $y : [t - r, +\infty) \to \mathbb{R}^n$ is the solution of (1), with $y_t = \psi$, given by Theorem 22(ii). By (28) and (37), we have

$$\dot{V}(t, x_\psi(t)) = D^+U(t, y_t) = D^+U(t, \psi) \leq -\Lambda(\|\psi\|) = -\Lambda(\|y_t\|).$$

However

$$\begin{aligned} \|y_t\| &= \sup_{-r\leq\theta\leq 0} |y(t+\theta)| = \sup_{t-r\leq\tau\leq t} |y(\tau)| \\ &= \sup_{t-r\leq\tau\leq t} |x_\psi(t)(\tau)| = \sup_{t-r\leq\tau<+\infty} |x_\psi(t)(\tau)| \\ &= \|x_\psi(t)\|. \end{aligned}$$

Therefore

$$\dot{V}(t, x_\psi(t)) \leq -\Lambda(\|y_t\|) = -\Lambda(\|x_\psi(t)\|) = -\Phi(x_\psi(t)),$$

and the hypotheses of Theorem 4.2 are satisfied.

Now, consider $y : [t_0 - r, +\infty) \to \mathbb{R}^n$ as the solution of system (1)-(2) and let x be the solution on $[t_0, +\infty)$ of the generalized ODE (27), given by Theorem 5.1(i), satisfying the initial condition $x(t_0) = z_0$, where

$$z_0(\tau) = \begin{cases} \phi(\tau - t_0), \; t_0 - r \leq \tau \leq t_0, \\ \phi(0), \; \tau \geq t_0. \end{cases} \tag{38}$$

Thus, for $s \geq t_0$, x can be written as

$$x(s)(\tau) = \begin{cases} y(\tau), \; t_0 - r \leq \tau < s, \\ y(s), \; \tau \geq s. \end{cases} \tag{39}$$

Hence x is uniformly ultimately bounded and this means that there exists $B > 0$ such that for every $\alpha > 0$, there exists $T = T(\alpha) > 0$ such that, if

$$\|z_0\| < \alpha \tag{40}$$

then

$$\|x(t)\| < B, \qquad \text{for any } t \geq t_0 + T(\alpha). \tag{41}$$

Assume that

$$\|\phi\| < \alpha. \tag{42}$$

We have to prove that

$$|y(t)| < B, \text{ for any } t \geq t_0 + T(\alpha). \tag{43}$$

But this is immediate by the proof of Theorem 5.2. By (42), we obtain (40) as in (33). Finally (43) holds, since we have (36) as in Theorem 5.2 and because of (41). The proof is complete. □

6. Example

In [1], the author studies the equation

$$y'(t) = -\int_{t-r}^{t} p(t-s)g(y(s))ds \tag{44}$$

in the theory of a circulating fuel nuclear reactor. In such model, y is the neutron density. It is well-known that this is a good model in one-dimensional viscoelasticity in which y is the strain and p is the relaxation function.

We consider equation (44) with an impulse condition and, consequently, we obtain boundedness of the solution. Thus, consider the impulsive RFDE

$$\begin{cases} y' = -\int_{t-r}^{t} p(t-s)g(y(s))ds, & t \neq \tau_k(y(t)), \quad t \geq 0, \\ \Delta y\,(t) = d_k, & t = \tau_k(y(t)), \quad k = 1, 2, \ldots, \end{cases} \tag{45}$$

subject to the initial condition

$$y_0 = \phi, \tag{46}$$

where $r > 0$, $\phi \in G^{-}([-r,0],\mathbb{R}^n)$, $p : \mathbb{R} \to \mathbb{R}_+$ is a Lebesgue integrable function such that $p(u) \leq B$ for all $u \in \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}_+$ is such that $|g(x) - g(y)| \leq \mathcal{K}|x - y|$ for all $x, y \in \mathbb{R}$ and there exits a function $m : \mathbb{R} \to \mathbb{R}$ Lebesgue integrable such that

$$\left|\int_{s_1}^{s_2} g(y(s))ds\right| \leq \int_{s_1}^{s_2} m(s)ds,$$

for all $s_1, s_2 \in \mathbb{R}$, for $k = 1, 2, \ldots$, $\{d_k\}$ is a sequence of non-positive constants which is bounded from below, τ_k maps $\mathbb{R}$ to $(0, +\infty)$ and τ_k satisfies $(C_1) - (C_5)$.

Consider $\Delta y\,(t) := y\,(t+) - y\,(t-) = y\,(t+) - y\,(t)$, for any $t \geq 0$. It is easy to check that the function $I_k : \mathbb{R}^n \to \mathbb{R}^n$ given by $I_k(y) = d_k$, for $y \in \mathbb{R}^n$, satisfies (A') and (B'), for $k = 1, 2, \ldots$.

For each $t \geq 0$, let $f(y_t, t) = -\int_{t-r}^{t} p(t-s)g(y(s))ds$. We will show that f satisfies conditions (A) and (B).

(A) Given $y \in PC_1$ and $u_1, u_2 \in [0, +\infty)$, we have

$$\left|\int_{u_1}^{u_2} f(y_s, s)ds\right| = \left|\int_{u_1}^{u_2}\left(-\int_{s-r}^{s} p(s-u)g(y(u))du\right)ds\right| \leq$$

$$\leq \int_{u_1}^{u_2}\left(\int_{s-r}^{s} |p(s-u)g(y(u))|du\right)ds \leq B\int_{u_1}^{u_2}\left|\int_{s-r}^{s} g(y(u))du\right|ds \leq$$

$$\leq B\int_{u_1}^{u_2}\left(\int_{s-r}^{s} m(u)du\right)ds = \int_{u_1}^{u_2} M(s)ds,$$

where $M(s) = B\int_{s-r}^{s} m(u)du$. Thus condition (A) holds.

(B) Given $x, y \in PC_1$ and $u_1, u_2 \in [0, +\infty)$, we have

$$\left|\int_{u_1}^{u_2} f(x_s, s) - f(y_s, s)ds\right| = \left|\int_{u_1}^{u_2}\left(-\int_{s-r}^{s} p(s-u)(g(x(u)) - g(y(u)))du\right) ds\right| \leq$$

$$\leq \left|\int_{u_1}^{u_2}\left(\int_{s-r}^{s} |p(s-u)|\mathcal{K}\|x(u) - y(u)\|du\right) ds\right| =$$

$$= \left|\int_{u_1}^{u_2}\left(\int_{r}^{0} |p(\tau)|\mathcal{K}\|x(s-\tau) - y(s-\tau)\|d\tau\right) ds\right| \leq$$

$$\leq \int_{u_1}^{u_2} \mathcal{K}\|x_s - y_s\|\left(\int_{0}^{r} |p(\tau)|d\tau\right) ds \leq$$

$$\leq \int_{u_1}^{u_2} B\mathcal{K}r\|x_s - y_s\|ds = \int_{u_1}^{u_2} L(s)\|x_s - y_s\|ds,$$

where $L(s) = B\mathcal{K}r$. Thus f satisfies (B).

Define a function $W : \mathbb{R} \to \mathbb{R}$ by $W(y) = \frac{y^3}{3}$ and let $y(t)$ be a solution of (45). For $t \neq t_k^i$, we have

$$D^+W(y(t)) = W'(y(t))y'(t) = y^2(t)y'(t) = -y^2(t)\int_{t-r}^{t} p(t-s)g(y(s))ds \leq 0,$$

since p and g are non-negative functions.

Note that $W(y(t_k^i+)) = W(y(t_k^i) + d_k) \leq W(y(t_k^i))$, since W is an increasing function and $d_k < 0$. Then, for $\eta > 0$ sufficiently small, we have $W(y(t_k^i + \eta)) \leq W(y(t_k^i))$, by the continuity of W. Thus, for $t = t_k^i$, we have

$$D^+W(y(t)) = \limsup_{\eta \to 0+} \frac{W(y(t+\eta)) - W(y(t))}{\eta} \leq 0.$$

Now, define a function $U : [0, +\infty) \times G^-([-r, 0], \mathbb{R}^n) \to \mathbb{R}_+$ by

$$U(t, \psi) = \sup_{-r \leq \theta \leq 0} W(\psi(\theta)) = \frac{1}{3}\sup_{-r \leq \theta \leq 0} \psi^3(\theta) = \frac{1}{3}\left(\sup_{-r \leq \theta \leq 0} \psi(\theta)\right)^3 = \frac{1}{3}\|\psi\|^3.$$

We will show that the function U satisfies the conditions from Theorem 5.2.

(i) By the definition of U, it is clear that $U(t, 0) = 0$ for all $t \geq 0$.

(ii) Let $\psi, \varphi \in \mathcal{B}_\rho = \{\Psi \in G^-([-r, 0], \mathbb{R}^n) : \|\Psi\| < \rho\}$. Then

$$|U(t, \psi) - U(t, \varphi)| \leq \frac{1}{3}\left|\|\psi\| - \|\phi\|\right|\left[\|\psi\|^2 + \|\varphi\|\|\psi\| + \|\varphi\|^2\right] < \rho^2\|\psi - \varphi\|,$$

for all $t \geq 0$.

(iii) Given $t \geq 0$ and a function $\psi \in G^-([-r, 0], \mathbb{R}^n)$, we have

$$U(t, \psi) = \frac{1}{3}\|\psi\|^3 = b(\|\psi\|),$$

where $b(s) = \frac{s^3}{3}$.

(*iv*) Given $t \geq 0$ and a function $\psi \in G^-([-r,0],\mathbb{R}^n)$, by considering the solution y of (45) defined on $[t-r,+\infty]$ such that $y_t = \psi$, we have

$$U(t,\psi) = U(t,y_t) = \sup_{-r\leq\theta\leq 0} W(y_t(\theta)) = W(y_t(\theta_0)).$$

We want to show that $D^+U(t,y_t) \leq 0$. We consider two cases: when $\theta_0 = 0$ and otherwise. At first, we assume that $\theta_0 = 0$. In this case,

$$D^+U(t,y_t) = D^+W(y(t)) \leq 0.$$

Now, we consider $-r \leq \theta_0 < 0$. Since $\sup_{-r\leq\theta\leq 0} W(y_t(\theta)) = W(y_t(\theta_0))$, for $\eta > 0$ sufficiently small, we have

$$\sup_{-r\leq\theta\leq 0} W(y_{t+\eta}(\theta)) = \sup_{-r\leq\theta\leq 0} W(y_t(\theta)).$$

Consequently,

$$D^+U(t,y_t) = \limsup_{\eta\to 0^+} \frac{\sup_{-r\leq\theta\leq 0} W(y_{t+\eta}(\theta)) - \sup_{-r\leq\theta\leq 0} W(y_t(\theta))}{\eta} = 0.$$

Then Theorem 5.2 implies the solution $y(t) = y(t,0,\phi)$ of (45)-(46) is uniformly bounded.

7. Appendix

In this part of our paper, we present the concept of integrability Kurzweil.

A *tagged division* of a compact interval $[a,b] \subset \mathbb{R}$ is a finite collection

$$\{(\tau_i,[s_{i-1},s_i]) : i = 1,2,\ldots,k\},$$

where $a = s_0 \leq s_1 \leq \ldots \leq s_k = b$ is a division of $[a,b]$ and $\tau_i \in [s_{i-1},s_i]$, $i = 1,2,\ldots,k$.

A *gauge* on $[a,b]$ is any function $\delta : [a,b] \to (0,+\infty)$. Given a gauge δ on $[a,b]$, a tagged division $d = (\tau_i,[s_{i-1},s_i])$ of $[a,b]$ is *δ-fine* if, for every i,

$$[s_{i-1},s_i] \subset \{t \in [a,b] : |t-\tau_i| < \delta(\tau_i)\}.$$

Let X be a Banach space. Now, we define the type of integration which belongs to Jaroslav Kurzweil.

Definition 7.1. *A function $U(\tau,t) : [a,b]\times[a,b] \to X$ is* Kurzweil integrable *over $[a,b]$, if there is a unique element $I \in X$ such that given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine tagged division $d = (\tau_i,[s_{i-1},s_i])$ of $[a,b]$, we have*

$$\|S(U,d) - I\| < \varepsilon,$$

where $S(U,d) = \sum_i [U(\tau_i,s_i) - U(\tau_i,s_{i-1})]$. In this case, we write $I = \int_a^b DU(\tau,t)$ and use the convention $\int_a^b DU(\tau,t) = -\int_b^a DU(\tau,t)$, whenever $b < a$.

The Kurzweil integral was described extensively in Chapter I of [12] for the case $X = \mathbb{R}^n$ (see Definition 1.2 in [12]).

For some basic facts of the Kurzweil integration theory and of the theory of generalized ODEs, see [12].

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