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RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH WHITE NOISE PERTURBATIONS

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ABSTRACT

Given a retarded functional differential equation, a stable equilibrium point and a bounded neighbourhood contained in its basin of attraction, one perturbs the differential equation with small white noise and proves a large deviation result, namely, solutions starting in the neighbourhood very likely escape from it. The end of the escaping path is described.

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I. Preliminaries and statement of the main result

Let C be the Banach space of all continuous paths $\varphi:[-1,0]\to R^n$ with respect to the norm $\|\varphi\| = \max\{|\varphi(\theta)|, -1 \le \theta \le 0\}$, where $|\cdot|$ is the Euclidean norm in R^n. For a continuous function $x:[t_o-1, t_o+A)\to R^n$, $A>0$, and a real number t in $[t_o, t_o+A)$, we write x_t to denote the element in C given by $x_t(\theta) = x(t+\theta)$, $-1 \le \theta \le 0$; then, the function $t \in [t_o, A) \mapsto x_t \in C$ is continuous.

We call a retarded functional differential equation a relation of the form

$$(1,0) \qquad \dot{x}(t) = f(x_t)$$

where $f: C \to \mathbb{R}^n$ is a continuous function (see [Ha]).

An important example is given by the integrodifferential equation

$$\dot{x}(t) = F(x(t), x(t-1)) - \int_{-1}^{0} a(-\theta) g(x(t+\theta)) d\theta$$

where $F: R^n \times R^n \to R^n$, $g: \mathbb{R}^n \to R^n$ and $a:[0,1]\to R$ are continuous functions.

For technical reasons, we will suppose that f is continuously differentiable and that $\|f\|_1 = \max\{\sup_{\varphi\in C}|f(\varphi)|, \sup_{\varphi\in C}\|Df(\varphi)\|\}$ is finite, where $\|Df(\varphi)\| = \sup\{|Df(\varphi)\psi| : \|\psi\| = 1\}$.

Equation (1,0) defines the semigroup solution $(S(t))_{t\ge 0}$ — the flow — on C by

$$S(t)\varphi = x_t \ , \ t \ge 0 \ , \ \varphi \in C$$

where x is the solution of (1,0) in $[0,+\infty)$ which starts in φ, i.e., $x_0 = \varphi$ or $x(\theta) = \varphi(\theta)$, for all θ in $[-1,0]$. We know that $S(t): C \to C$ is continuously differentiable, $S(t+s)=S(t)S(s)$ for all $t,s \geq 0$, $S(0)$ = identity of C and also that for any fixed φ in C, and any ψ in C, $\lim_{t\to 0} \left[\frac{\partial}{\partial\varphi}(S(t)\varphi)\right]\psi = \psi$ in C.

Let $W^{1,2}$ be the subset of all functions $\varphi: [-1,0] \to R^n$ absolutely continuous on $[-1,0]$ such that $\dot\varphi \in L_2$ i.e. $\|\dot\varphi\|_{L_2} = \left(\int_{-1}^{0} |\dot\varphi(\theta)|^2 d\theta\right)^{1/2} < \infty$. With respect to the norm $\|\varphi\|_{1,2} = \left(|\varphi(0)|^2 + \int_{-1}^{0} |\dot\varphi(\theta)|^2 d\theta\right)^{1/2}$, $W^{1,2}$ is a Hilbert space and the inclusion $W^{1,2} \to C$ is continuous. We know that $W^{1,2}$ is invariant under the flow $(S(t))_{t\geq 0}$ and that $S(t): W^{1,2} \to W^{1,2}$ remains continuously differentiable.

Later, we will suppose that the function $\varphi = 0$ in C is an equilibrium point of Equation (1,0), that is, $f(0) = 0$, which attracts its small neighborhoods. We know that there exist positive constant K and a such that $\|x_t(\varphi)\| \leq K\, e^{-at}\|\varphi\|$ for all $t \geq 0$ and all φ in a sufficiently small neighborhood of 0. The above estimate is obtained (see [Ha]) by considering the linearized variational equation about $\psi = 0$: $\dot y(t) = f'(0)y_t$.

Let (Ω,F,P) be a probability space and let $w(t):\Omega\to R^n$, $t \geq 0$, be the Brownian motion in R^n. We remember [F-W] that: 1º) $w(0)=0$; 2º) w is continuous in $t \in [0,\infty)$ with probability 1; 3º) w has stationary independent increments; 4º) for each $t > 0$ and each borelian $A \subset R^n$ we have $P\{w(t) \in A\} = \int_A \frac{1}{\sqrt{2\pi t}}\, e^{-|x|^2/2t}\, dx$.

Let us consider the perturbed stochastic differential equation

$$(1,\varepsilon) \quad X^{\varepsilon}(t) = f(X_t^{\varepsilon}) + \varepsilon\, \dot{w}(t), \ t > 0$$

for which a solution through $\varphi \in C$ at time $t = 0$ is meant a continuous random variable $X^{\varepsilon}(t,\omega)$, $t \geq -1$, $\omega \in \Omega$ such that $X^{\varepsilon}(\theta,\omega) = \varphi(\theta)$ (with probability one) for all $\theta \in [-1,0]$ and such that for all $t \geq 0$, we have

$$X^{\varepsilon}(t) = \varphi(0) + \int_0^t f(X_s^{\varepsilon})ds + \varepsilon\, w(t)$$

also with probability one.

We know that, for each $\varphi \in C$, there exists one and only one solution of Equation $(1,\varepsilon)$ through φ defined for all $t \geq 0$. We will prove that:

I.1. Proposition - *Given an interval* $[0,T]$ *and a function* $\varphi \in C$, *then, during the time interval* $[0,T]$, *the solution* $X^{\varepsilon}(t)$ *of equation* $(1,\varepsilon)$ *through* φ *at time* $t = 0$ *very likely follows the solution of Equation* $(1,0)$ *through* φ *at time* $t = 0$; *more precisely, for any* $\delta > 0$,

$$\lim_{\varepsilon \to 0} P \left\{ \sup_{t \in [-1,T]} |X^{\varepsilon}(t) - x(t)| > \delta \right\} = 0$$

or, equivalently,

$$\lim_{\varepsilon \to 0} P \left\{ \sup_{t \in [0,T]} \|X_t^{\varepsilon} - x_t\| > \delta \right\} = 0.$$

Proof: It is easy to see that

$$\|X_t^\varepsilon - x_t\| \le \|f\|_1 \int_0^1 \|X_s^\varepsilon - x_s\| ds + \varepsilon |w(t)|,$$

so, Gronwall's inequality implies that

$$\|X_t^\varepsilon - x_t\| \le \varepsilon\, e^{T\|f\|_1} \sup_{t\in[0,T]} |w(t)|.$$

Using the classical inequality (see [F.W])

$$P\left\{ \sup_{0\le t\le T} \varepsilon|w(t)| > \eta \right\} \le C_1\, e^{-C_2/\varepsilon^2}$$

we find that

$$P\left\{ \sup_{0\le t\le T} \|X_t^\varepsilon - x_t\| > \delta \right\} \le C_1\, e^{-C_2'/\varepsilon^2}$$

for appropriate constants C_1 e C_2 ; this last inequality clearly implies the conclusion of our proposition.

In order to estimate the probability that a solution X_t^ε of the Equation $(1,\varepsilon)$ belongs to a neighborhood of a fixed path $\{x(t),\ T_1 \le t \le T_2\}$, we introduce the action functional associated to the random process X_t^ε and a quasipotential extending Freidlin and Wentzell's construction for perturbed vector fields [see F-W].

Associated to the Brownian motion there is an action functional

$$S:\ C[T_1,T_2] \to [0,\infty]$$

defined by

$$S(\gamma) = \frac{1}{2}\int_{T_1}^{T_2} |\dot{\gamma}(t)|^2 dt$$

if $\gamma \in W^{1,2}[T_1,T_2]$ and $S(\gamma) = +\infty$ otherwise.

In a similar way as Freidlin and Wentzell do for vector fields, let us define the normalized action functional associated to the process X_t^ε by

$$S_{T_1T_2}(\gamma) = \frac{1}{2}\int_{T_1}^{T_2} |\dot{\gamma}(t) - f(\gamma_t)|^2 dt$$

Without loss of generality, we can assume that either $T_1 = 0$ and $T_2 > 0$ or $T_2 = 1$ and $T_1 < 0$.

Let us denote by ρ_T the distance between two continuous functions x and y in $C[-1,T]$:

$$\rho_T(x,y) = \max_{t\in[-1,T]} |x(t)-y(t)|$$

and, for $\Phi \subset C[-1,T]$, let us define:

$$\rho_T(x,\Phi) = \inf\{\rho_T(x,y) : y \in \Phi\}.$$

The following is an extension of a theorem of Freidlin and Wentzell, which we will use many times:

I.2. Theorem - *Let* $X^\varepsilon(\varphi)$ *be a solution of the perturbed equation* $(1,\varepsilon)$. *Then, given* $T > 1$, $\delta > 0$, $\beta > 0$ *and* $s_o > 0$ *we have:*

(a) *There exists* $\varepsilon_o > 0$ *such that for all* $\varepsilon \in (0,\varepsilon_o)$ *the inequality*

$$P\{\rho_T(x^\varepsilon(\varphi)-\gamma) < \delta\} \geq \exp\{-\varepsilon^{-2}(S(\gamma)+\beta)\}$$

holds for all $\varphi \in W^{1,2}$ *and* $\gamma \in W^{1,2}[-1,T]$ *such that* $\gamma_o = \varphi$ *and* $S(\gamma) \leq s_o$.

(b) *There exists* $\varepsilon_o > 0$ *such that for all* $\varepsilon \in (0,\varepsilon_o)$, *all* $s \in [0,s_o]$ *and all* $\varphi \in W^{1,2}$, *we have*

$$P\{\rho_T(X^\varepsilon(\varphi),\Phi(s,\varphi) \geq \delta\} \leq \exp\{-\varepsilon^{-2}(s-\beta)\}$$

where $\Phi(s,\varphi) = \{\gamma \in W^{1,2}[-1,T]: \gamma_o = \varphi$ *and* $S(\gamma) \leq s\}$.

The *quasipotential* of Equation (1,0) with respect to the origin 0 in $W^{1,2}$ is, by definition, the functional

$$V(0,\varphi) = \inf\{S_{T1,T2}(\gamma): T_1 < T_2,\ \gamma \in W^{1,2}[T_1-1,T_2], \gamma_{T_1}=0, \gamma_{T_2}=\varphi\}.$$

It is clear that $V(0,\varphi) \geq 0$ for all $\varphi \in W^{1,2}$ and that $V(0,\varphi)$ is continuous in φ; moreover, if 0 is an equilibrium of equation (1,0), then $V(0,0) = 0$.

The name quasipotential comes from the fact that for gradient systems in R^n, $x(t) = -\text{grad } U(x(t))$, with 0 as an attractor, the quasipotential is twice the potential U, if we stay in the basin of the attractor.

The following theorem, as in the non retarded case, studies the exit from a domain contained in the basin of an attracting equilibrium.

I.3. <u>Theorem</u> - *Let 0 be an asymptotically stable equilibrium of Equation (1,0) and let $D \subset W^{1,2}$ be a bounded connected open neighborhood of 0, the closure of which admits a δ_o-neighborhood D_{δ_o} contained in the basin of 0. Let us suppose also that $\overline{D}$ and $\overline{D}_\delta$ are strictly contracted by the flow of the nonperturbed system. Let us suppose, moreover, that there exists a unique point $\varphi_o \in \partial D$ minimizing the quasipotential $V(0,\varphi)$ on ∂D.*

Then, for any $\delta > 0$ and all $\varphi \in D$ we have:

$$\lim_{\varepsilon \to 0} \mathbb{P}_\varphi \{ \| X^\varepsilon_{\tau^\varepsilon} - \varphi_o \|_{1,2} < \delta \} = 1$$

where $\tau_\varepsilon = \inf\{t > 0 : X^\varepsilon_t \in \partial D\}$.

Before proving the above theorem, we will study the extremals of the action functional.

II. The action functional

We will restrict ourselves to random perturbations of equations of the following type:

$$(2.1) \qquad \dot{x}(t) = f(x_t)$$

where $f: C \to R^n$ is given by

$$f(\varphi) = F(\varphi(0), \varphi(-1)) - \int_{-1}^{0} a(-\theta) g(\varphi(\theta)) d\theta.$$

We suppose that $F: R^n \times R^n \to R^n$ and $g: R^n \to R^n$ are bounded C^1 functions with bounded derivatives and that $a: [0,1] \to R$ is of class C^2.

The perturbed equation is

(2.3) $\dot{X}^{\varepsilon}(t) = f(X^{\varepsilon}_t) + \varepsilon\,\dot{w}(t)$

The corresponding action functional is

$$S_{0T}(\gamma) = S(\gamma) = \frac{1}{2}\int_0^T \left|\dot{\gamma}(t) - F(\gamma(t),\gamma(t-1)) + \int_{-1}^{0} a(-\theta)g(\gamma(t(\theta))d\theta\right|^2 dt.$$

We want to minimize S on the following class:

$$M = \{\gamma \in C[-1,T] \cap W^{1,2}[0,T] : \gamma_0 = \varphi,\ \gamma_T = \bar{\varphi}\}$$

where φ and $\bar{\varphi}$ are given functions.

It is clear that we have to assume $T \geq 1$.

II.1. <u>Proposition</u> - *A necessary condition for $\gamma \in M$ to minimize S on M is that γ satisfies the following Euler-Legrange equation (2.4):*

$$\frac{d}{dt}H(\gamma)(t) + \left[D_1F(\gamma(t),\gamma(t-1))\right]^* H(\gamma)(t) + \left[D_2F(\gamma(t+1),\gamma(t))\right]^* H(\gamma)(t+1)$$
$$- \left[g'(\gamma(t))\right]^* \int_{-1}^{0} a(-\theta)H(\gamma)(t-\theta)d\theta = 0$$

where

$$H(\gamma)(t) = \dot{\gamma}(t) - F(\gamma(t),\gamma(t-1)) + \int_{-1}^{0} a(-\theta)g(\gamma(t+\theta))d\theta$$

(the star indicates the matrix transposition).

Recall that $\gamma_o = \varphi$ and $\gamma_T = \bar{\varphi}$ and observe that the equation above which is retarded and advanced in time, is, as matter of fact, a second order integro-differential equation. In spite of the fact that we do not know "a priori" that γ has first and second derivatives in $[0,T-1]$, we remark that the function $H(\gamma)(t)$ is absolutely continuous in $[0,T-1]$ and $\frac{d}{dt} H(\gamma(\cdot)) \in L_2[0,T-1]$. Later on we will show that γ is C^2 in $[0,T-1]$.

<u>Proof</u>. Let $h \in C^1[0,T-1]$ such that $h(0) = h(T-1) = 0$ and put $h(\theta) = 0$, $\theta \in [-1,0]$ and $h(t) = 0$, $t \in [T-1,T]$. We know that if $\gamma \in \mathcal{H}$ is a local minimum for V then $\frac{d}{d\lambda} S(\gamma+\lambda h)\Big|_{\lambda=0} = 0$ for all h as above.

Let us make explicit this condition:

$$\frac{d}{d\lambda} S(\gamma+\lambda h)\Big|_{\lambda=0} = \int_0^T H^*(\gamma)(t)\Big[\dot{h}(t) - D_1F(\gamma(t),\gamma(t-1))h(t) -$$

$$- D_2F(\gamma(t),\gamma(t-1))h(t-1) + \int_{-1}^{0} a(-\theta)g'(\gamma(t+\theta))h(t-\theta)d\theta\Big]dt.$$

Since $h_o = h_T = 0$ and after using integration by parts and inversion of the two integrals we obtain the following expression:

$$\frac{dS}{d\lambda}(\gamma+\lambda h)\Big|_{\lambda=0} = \int_0^{T-1}\Big\{H^*(\gamma)(t) + \int_0^t \Big[H^*(\gamma)(s)D_1F(\gamma(s),\gamma(s-1)) +$$

$$+ H^*(\gamma)(s+1)D_2F(\gamma(s+1),\gamma(s)) - \int_{-1}^{0} a(-\theta)H^*(\gamma)(s-\theta)d\theta.g'(\gamma(s))\Big]ds\Big\}\dot{h}(t)dt$$

Since h is arbitrary, the result follows from Du-Bois-Reymond's lemma (see [A]) which says that if $\Phi:[a,b] \to R^n$ is continuous and if $\int_a^b \Phi^*(t)\dot{h}(t)dt = 0$ for all C^1 functions $h:[a,b] \to R^n$ such that $h(a) = h(b) = 0$, then Φ is constant on $[a,b]$.

We have then, for all $t \in [0,T-1]$:

$$H^*(\gamma)(t) + \int_0^t \Big[H^*(\gamma)(s)D_1F(\gamma(s),\gamma(s-1)) + H^*(\gamma)(s+1)D_2F(\gamma(s+1),\gamma(s))$$

$$- \int_{-1}^0 a(-\theta)H^*(\gamma)(s-\theta)d\theta . g'(\gamma(s))\Big]ds = \text{constant}.$$

Since the integrand belongs to $L_2[0,T-1]$ it follows that $H^*(\gamma)(t)$ is absolutely continuous on $[0,T-1]$. If we transpose the expression obtained after computing the derivate of the last equality we get the following Euler-Lagrange equation which holds for γ, almost everywhere in $[0,T-1]$:

$$\frac{d}{dt} H(\gamma)(t) + \Big[D_1F(\gamma(t),\gamma(t-1))\Big]^* H(\gamma)(t) + \Big[D_2F(\gamma(t+1),\gamma(t))\Big]^* H(\gamma)(t+1)$$

$$- \Big[g'(\gamma(t))\Big]^* \int_{-1}^0 a(-\theta)H(\gamma)(t-\theta)d\theta = 0 \text{ as claimed.}$$

■

Let us now consider the existence of an absolute minimum for S in the class M, which will imply the existence of solution for the equation (2.4) with boundary condition $\gamma_o = \varphi$ and $\gamma_T = \bar{\varphi}$.

Let $\mu = \inf\{S(\gamma): \gamma \in M\} \geq 0$ and $(\gamma_m)_{m\in N}$ a minimizing sequence of elements in M, that is, $S(\gamma_m) \to \mu$ as $m \to \infty$. Without loss of generality we may assume $S(\gamma_m) \leq S(\gamma_1)$.

II.2. Proposition - *There exists a subsequence, also denoted by $(\gamma_m)_{m\in N}$, which converges uniformly in $[0,T-1]$ to a function $\bar\gamma \in M$ such that $S(\bar\gamma) = \mu$.*

Proof. The idea is to show that M is compact in $C[0,T]$ and that S is lower semi-continuous on M.

We start observing that there exist constants α and β such that $\alpha > 0$ and

$$\frac{1}{2}\left|\dot\gamma_m(t)-F(\gamma_m(t),\gamma_m(t-1)) + \int_{-1}^{0} a(-\theta)g(\gamma_m(t+\theta))d\theta\right|^2 \geq \alpha\left|\dot\gamma_m(t)\right|^2 - \beta.$$

This implies that

$$-\beta T + \alpha\int_0^T\left|\dot\gamma_m(t)\right|^2 dt \leq S(\gamma_m) \leq A$$

and then

$$\|\dot\gamma_m\|_{L_2} \leq B = \left(\frac{A+\beta T}{\alpha}\right)^{1/2} \quad \text{for all } m.$$

By Cauchy-Schwarz inequality we have:

$$\left|\gamma_m(t)-\gamma_m(s)\right| \leq \left|\int_s^t\left|\dot\gamma_m(s)\right|\cdot 1\, ds\right| \leq \left|\int_s^t\left|\dot\gamma_m(s)\right|^2 ds\right|^{1/2}|t-s|^{1/2}$$

and we are able to conclude that $(\gamma_m)_{m\in N}$ is an equicontinuous sequence in $C[0,T-1]$. Again, by the same inequality we obtain:

$$|\gamma_m(t)| \le |\varphi(0)| + \int_0^t |\dot{\gamma}_m(s)|\cdot 1\, ds \le |\varphi(0)| + B.T^{1/2}$$

and (γ_m) is a bounded sequence in $C[0,T]$. By the Theorem of Ascoli and Arzela, there exists a subsequence of (γ_m) which converges uniformely to a function $\bar{\gamma} \in C[0,T-1]$; extend $\bar{\gamma}$ to $[-1,T]$ by $\bar{\gamma}_o = \varphi$ and $\bar{\gamma}_T = \bar{\varphi}$. To show that $\bar{\gamma}$ is absolutely continuous in $[0,T-1]$ we consider a chain of inequalities obtained by using the Cauchy-Schwarz inequality for integrals and sums. For any numbers $s_i, t_i \in [0,T-1]$, $i=1,\ldots,k$, such that $s_1 < t_1 < s_2 < t_2 < \ldots < s_k < t_k$ we have

$$\sum_{i=1}^{k} |\gamma_m(t_i)-\gamma_m(s_i)| \le \sum_{i=1}^{k} |t_i-s_i|^{1/2}\left(\int_{s_i}^{t_i} |\dot{\gamma}_m(s)|^2 ds\right)^{1/2} \le$$

$$\le \left(\sum_{i=1}^{k} |t_i - s_i|\right)^{1/2}\left(\sum_{i=1}^{k} \int_{s_i}^{t_i} |\dot{\gamma}_m(s)|^2 ds\right)^{1/2} \le$$

$$\le B\left(\sum_{i=1}^{k} |t_i-s_i|\right)^{1/2} .$$

Passing to the limit as $m \to \infty$ we obtain

$$\sum_{i=1}^{k} |\bar{\gamma}(t_i) - \bar{\gamma}(s_i)| \le B\left(\sum_{i=1}^{k} |t_i-s_i|\right)^{1/2} ,$$

so $\bar{\gamma}$ is absolutely continuous on $[0,T-1]$.

To prove $\dot{\bar{\gamma}} \in L_2[0,T-1]$ take $t \in (0,T-1)$ and choose $h>0$ small enough such that $[t,t+h] \subset [0,T-1]$.

Using again Cauchy-Schwarz inequality we have

$$\left| \frac{\gamma_m(t+h)-\gamma_m(t)}{h} \right|^2 \leq \frac{1}{h} \int_0^h |\dot{\gamma}_m(t+u)|^2 \, du$$

If we integrate between 0 and (T-1-h) and by inversion in the order of integration one obtains

$$\int_0^{T-1-h} \left| \frac{\gamma_m(t+h)-\gamma_m(t)}{h} \right|^2 dt \leq \int_0^h \left[\int_0^{T-1-h} |\dot{\gamma}_m(t+u)|^2 \right] du$$

and since the last term is bounded by B^2, we pass to the limit as $m \to \infty$ and see that, for all $\tau < T-1$ and $h > 0$ sufficiently small, we have

$$\int_0^{\tau} \left| \frac{\bar{\gamma}(t+h)-\bar{\gamma}(t)}{h} \right|^2 dt \leq B^2.$$

Since $\bar{\gamma}$ is absolutely continuous in $[0,T-1]$, we use Fatou's theorem and obtain

$$\int_0^{\tau} |\dot{\bar{\gamma}}(t)|^2 \, dt \leq B^2.$$

But $\tau < T-1$ arbitrary implies $\dot{\bar{\gamma}} \in L_2[0,T-1]$ and $\|\dot{\bar{\gamma}}\|_{L_2} \leq B$.

Let us show now that $\dot{\gamma}_m$ converges weakly to $\dot{\bar{\gamma}}$ on $[0,T-1]$, in the sense that, for each $\psi \in L_2\,[0,T-1]$, we have $\lim_{m\to\infty} I_m = 0$, where

$$I_m = \left| \int_0^{T-1} (\gamma_m(t) - \dot{\bar{\gamma}}(t))^* \psi(t)dt \right|.$$

Given $\varepsilon > 0$ and $\psi \in L_2\ [0,T-1]$, we can find a polynomial $p(t)$ such that

$$\int_0^{T-1} |p(t) - \psi(t)|^2\, dt < \varepsilon^2.$$

Choose m_o such that, for $m \geq m_o$, we have

$$\sup_{0\leq t\leq T-1} |\gamma_m(t) - \bar{\gamma}(t)| \leq \frac{\varepsilon}{\|\dot{p}\|_1}$$

where $\|\dot{p}\|_1 = \int_0^{T-1} |\dot{p}(t)|dt.$

Then

$$I_m \leq \left|\int_0^{T-1} (\gamma_m(t)-\bar{\gamma}(t))^* p(t)dt\right| + \left|\int_0^{T-1} (\gamma_m(t)-\bar{\gamma}(t))^* (\psi(t)-p(t))dt\right|.$$

Using integration by parts and Cauchy-Schwarz inequality we get:

$$I_m \leq \left|\int_0^{T-1} (\gamma_m(t)-\bar{\gamma}(t))^* \dot{p}(t)dt\right| + \|\dot{\gamma}_m - \dot{\bar{\gamma}}\|_{L_2} \|\psi-p\|_{L_2}$$

so

$$I_m \leq \left(\sup_{0\leq t\leq T-1} |\gamma_m(t) - \bar{\gamma}(t)|\right) \int_0^{T-1} |\dot{p}(t)|dt + 2B\varepsilon$$

and finally, $I_m \leq (1+2B)\varepsilon$ which shows that $\lim_{m\to\infty} I_m = 0$.

To conclude, let us prove that the absolute minimum μ of S is achieved in $\bar{\gamma}$.

Let us introduce the following abreviations:

$$f_m = F(\gamma_m(t), \gamma_m(t-1)) - \int_{-1}^{0} a(-\theta) g(\gamma_m(t+\theta)) d\theta$$

$$\bar{f} = F(\bar{\gamma}(t), \bar{\gamma}(t-1)) - \int_{-1}^{0} a(-\theta) g(\bar{\gamma}(t+\theta)) d\theta.$$

We already know that f_m converges to $\bar{f}$ uniformly on $[0,T]$.

We have:

$$S(\gamma_m) = S(\bar{\gamma}) + \frac{1}{2}\int_0^T (|\dot{\gamma}_m - f_m|^2 - |\dot{\bar{\gamma}} - f_m|^2) dt$$

$$+ \frac{1}{2}\int_0^T (|\dot{\bar{\gamma}} - f_m|^2 - |\bar{\gamma} - \bar{f}|^2) dt$$

$$= S(\bar{\gamma}) + \frac{1}{2}\int_0^T |\dot{\gamma}_m - \dot{\bar{\gamma}}|^2 dt + \int_0^T (\dot{\bar{\gamma}} - f_m)^*(\dot{\gamma}_m - \dot{\bar{\gamma}}) dt$$

$$+ \frac{1}{2}\int_0^T (f_m - \bar{f} - 2\dot{\bar{\gamma}})^*(f_m - \bar{f}) dt$$

Therefore

$$S(\gamma_m) \geq S(\bar{\gamma}) - \left|\int_0^T (\dot{\bar{\gamma}} - f_m)^*(\dot{\gamma}_m - \dot{\bar{\gamma}}) dt\right|$$

$$- \frac{1}{2}\left|\int_0^T (f_m - \bar{f} - 2\dot{\bar{\gamma}})^*(f_m - \bar{f}) dt\right|$$

Since these last integrals tend to zero as m goes to $+\infty$, we get $S(\bar{\gamma}) = \mu$.

III. Proof of Theorem I.2

In this section, following Freidlin and Wentzell's approach [F-W], we will decompose the proof of Theorem I.2 into a series of lemmas.

III.1. Lemma - *Let F be a compact subset of C and T, δ positive numbers. Then, there exist* $\varepsilon_o > 0$ *and* $\beta > 0$ *such that*

$$P\left\{ \sup_{0 \le t \le T} \|X_t^\varepsilon(\varphi) - x_t(\varphi)\| \ge \delta \right\} \le \exp\{-\varepsilon^{-2}\beta\}$$

for any $\varphi \in F$ *and any* $\varepsilon > 0$, $\varepsilon < \varepsilon_o$.

Proof: Let $\varphi \in F$ fixed and define:

$$G(\varphi) = \{\gamma \in C[-1,T]: \gamma_o = \varphi,\ \rho_T(\gamma, x(\varphi)) \ge \delta\}.$$

Recall that $\rho_T(\gamma, x(\varphi)) = \sup_{-1 \le t \le T} |\gamma(t) - x(t,\varphi)|$.

Since $G(\varphi)$ is closed and F is compact we see that

$$\bigcup_{\varphi \in F} G(\varphi)$$

is closed in C.

We know that the infimum d of $S(\gamma)$ for $\gamma \in \bigcup_{\varphi \in F} G(\varphi)$ is achieved.

Since $S(\gamma)$ vanishes only on the solution of Equation (1,0), it is clear that $d > 0$.

For each $d' < d$, let us consider the set

$$\Phi_\varphi(d') = \{\gamma \in C[-1,T]: \gamma_0 = \varphi \text{ and } S(\gamma) \leq d'\}.$$

It is compact and disjoint of $\bigcup_{\varphi\in F} G(\varphi)$; therefore, the distance between $\Phi_\varphi(d')$ and $\bigcup_{\varphi\in F} G(\varphi)$ is a positive number δ'.

Now, for any $\gamma_1 > 0$, we have for any ε and $\varphi \in F$:

$$\begin{aligned} \mathbb{P}\{\rho_T(X^\varepsilon(\varphi),x(\varphi)) \geq \delta\} &= \mathbb{P}\{X^\varepsilon(\varphi) \in G(\varphi)\} \\ &\leq \mathbb{P}\{\rho_T(X^\varepsilon(\varphi),\Phi_\varphi(d')) \geq \delta'\} \\ &\leq \exp\{-\varepsilon^{-2}(d' - \gamma_1)\}. \end{aligned}$$

Taking $\beta = d' - \gamma_1$, this proves the lemma.

III.2. <u>Lemma</u> - *Let us suppose that O is an asymptotically stable equilibrium of Equation (1,0). Let D be a bounded connected open set, whose closure $\bar{D}$ is contained in the basin of attraction of O. Moreover, let us suppose that the flow ϕ_t of Equation (1,0) satisfies:*

$$\phi_t(\bar{D}) \subset D$$

for all $t > 0$.

Then for any fixed $\alpha > 0$ such that the ball $B_\alpha(0)$ is contained in D, we have:

(a) *There exist positive constants* a *and* T_o *such that*

$$S(\gamma) > a(T - T_o)$$

for any $T > T_o$ *and any* $\gamma \in C[-1,T]$ *satisfying* $\gamma_t \in \overline{D}\backslash B(0,\alpha)$ *for* $t \in [0,T]$.

(b) *There exist positive constants* c *and* T_o *such that, for any* $\varepsilon > 0$ *small enough and any* $\varphi \in \overline{D}\backslash B_\alpha(0)$ *we have*

$$\mathbb{P}_\varphi\{\zeta_\alpha > T\} \leq \exp\{-\varepsilon^{-2}(T-T_o).c\}$$

for any $T > T_o$, *where* $\zeta_\alpha = \inf\{t: X_t^\varepsilon \notin D \setminus B_\alpha(0)\}$.

Proof: (a) We take $\alpha' < \alpha$ such that any solution of Equation (1,0) which starts in $B_{\alpha'}(0)$ never leaves $B_\alpha(0)$.

For a given $\varphi \in \overline{D}$ let $T(\alpha,\varphi)$ be the first instant t at which we have $x_t(\varphi) \in \overline{B}_{\alpha'}(0)$. It is clear that $T(\alpha,\varphi) < \infty$. Let us see that $T(\alpha,\varphi)$ is upper semicontinuous with respect to φ: given $\varepsilon>0$ we have $x_{T(\alpha,\varphi)+\varepsilon}(\varphi) \in B_{\alpha'}(0)$; therefore we can take $r > 0$ small enough in order to guarantee that $B_r(x_{T(\alpha,\varphi)+\varepsilon}) \subset B_{\alpha'}(0)$; using the continuity of the solution with respect to the initial conditions, we can find a positive number δ such that $|x_t(\overline{\varphi})-x_t(\varphi)| < r$ for all $\overline{\varphi} \in B_\delta(0)$ and all $t \in [0,T(\alpha,\varphi) + \varepsilon]$; therefore, $x_{T(\alpha,\varphi)+\varepsilon}(\overline{\varphi}) \in B(x_{T(\alpha,\varphi)+\varepsilon}(\varphi),r)$ so that $T(\alpha,\overline{\varphi}) \leq T(\alpha,\varphi) + \varepsilon$, which proves the upper semicontinuity of $T(\alpha,\varphi)$ with respect to φ.

Let D_1 be the closure of the image of $\overline{D}$ by the flow of Equation (1.0) after the time 1; since D_1 is compact, there exists

$$T_o = \max\{T(\alpha,\varphi)\colon \varphi \in D_1\} < \infty.$$

It is clear that any solution of Equation (1.1) which starts at $\varphi \in \bar{D}$ is in $B_{\alpha'}(0)$ after the time T_o.

We consider the closed subset C_{T_o} of $C[-1,T_o]$ given by $C_{T_o} = \{\gamma\colon[-1,T_o] \to R^n\colon \gamma_t \in \bar{D} \setminus B_\alpha(0) \text{ for all } t \in [0,T_o]\}$. The action functional S_{oT_o} assumes on C_{T_o} its infimum which is a positive number: in fact, if not, we would have a solution of Eq. (1.0) in C_{T_o}, which contradicts the definition of T_o. Therefore, there exists a constants $A > 0$ such that $S_{oT_o}(\gamma) \geq A$ for all $\gamma \in C_{T_o}$.

Let $T > T_o$, let $[T/T_o]$ be the integer part of T/T_o, and let $\gamma\colon [-1,T] \to R^n$ such that $\gamma_t \in \bar{D} \setminus B_\alpha(0)$ for all $t \in [0,T]$. Then,

$$S_{oT}(\gamma) = \frac{1}{2}\left\{\int_0^{T_o} + \dots + \int_{\left(\left[\frac{T}{T_o}\right]-1\right)T_o}^{\left[\frac{T}{T_o}\right]T_o} + \int_{\left[\frac{T}{T_o}\right]T_o}^{T}\right\} |\dot{\gamma}(t)-f(\gamma_t)|^2 dt$$

$$\geq A + \dots + A = A.\left[\frac{T}{T_o}\right] \geq \frac{A}{T_o}(T - T_o)$$

$$= a(T - T_o)$$

where $a = \frac{A}{T_o}$, which proves Lemma III.2.(a).

(b) We can assume that $\delta < \frac{\alpha}{2}$. Let $A = A(\delta)$ and $T_o = T_o(\delta)$ the constants obtained in the proof of part (a) above for the set

$D = \bar{D}_\delta$ and the neighborhood $B_{\alpha/2}(0)$. Therefore, if $\gamma:[-1,T_o(\delta)] \to \mathbb{R}^n$ satisfies $\gamma_t \in \bar{D}_\delta \setminus B_{\alpha/2}(0)$ then $S_{oT_o(\delta)} > A_\delta$.

For $\varphi \in D$, the functions of the set

$$\Phi_\varphi(A_\delta) = \{\gamma: [-1,T_o(\delta)] \to \mathbb{R}^n: \gamma_o = \varphi,\ S_{oT_o(\delta)}(\gamma) \le A_\delta\}$$

reach $B_{\alpha/2}(0)$ or leave D_δ during the interval of time $[0,T_o(\delta)]$.

Suppose $\zeta_\alpha > T_o(\delta)$.

From the above assertion, we have

$$\rho_{oT_o(\delta)}\ (X^\varepsilon, \Phi_\varphi(A_\delta)) \ge \delta$$

because $X_t^\varepsilon \in D \setminus B_\alpha(0)$ for all t in $[0,T_o(\delta)]$.

Hence, for any sufficiently small $\varepsilon > 0$ and any $\varphi \in D$ we have the set inclusion

$$\{\omega \in \Omega: \zeta_\alpha > T_o(\delta)\} \subset \{\omega \in \Omega: \rho_{oT_o(\delta)}(X^\varepsilon, \Phi_\varphi(A_\delta)) \ge \delta\}$$

so that, for any $\beta > 0$,

$$P_\varphi\{\zeta_\alpha > T_o(\delta)\} \le P_\varphi\{\rho_{oT_o(\delta)}(X^\varepsilon, \Phi_\varphi(A_\delta)) \ge \delta\}$$

$$\le \exp\{-\varepsilon^{-2}(A_\delta - \beta)\}.$$

Now,

$$\mathbb{P}_\varphi\{\zeta_\alpha > (n+1)T_o\} = \int_{\{\zeta_\alpha > nT_o\}} \mathbb{P}_{X^\varepsilon_{nT_o}}\{\zeta_\alpha > T_o\}$$

$$\leq \int_{\{\zeta_\alpha > nT_o\}} \sup_{\psi\in D} \mathbb{P}_\psi\{\zeta_\alpha > T_o\}$$

$$\leq \int_{\{\zeta_\alpha > nT_o\}} \exp\{-\varepsilon^{-2}(A_\delta - \beta)\}$$

$$= \mathbb{P}_\varphi\{\zeta_\alpha > nT_o\}.\exp\{-\varepsilon^{-2}(A_\delta - \beta)\}$$

and by induction

$$\mathbb{P}_\varphi\{\zeta_\alpha > T\} \leq \mathbb{P}_\varphi\{\zeta_\alpha > [T/T_o]T_o\}$$

$$\leq (\mathbb{P}_\varphi\{\zeta_\alpha > T_o\})^{[T/T_o]}$$

$$\leq \left(\sup_{\psi\in D} \mathbb{P}_\psi\{\zeta_\alpha > T_o\}\right)^{[T/T_o]}$$

$$\leq \left(\exp\{-\varepsilon^{-2}(A_\delta - \beta)\}\right)^{[T/T_o]}$$

$$\leq \exp\{-\varepsilon^{-2}(T - T_o).c\}$$

where $c = \dfrac{A_\delta - \beta}{T_o}$, which concludes the proof of Lemma III.2.(b).

III.3. <u>Lemma</u> - (a) *There exists a positive constant* L *such that, for any* $\bar{\varphi} \in W^{1,2}$ *in a sufficiently small neighbourhood of* O *in* $W^{1,2}$ *, we can find a* $T > 0$ *and a path* $\gamma \in W^{1,2}[-1,T]$

satisfying $\gamma_o = 0$, $\gamma_T = \bar{\varphi}$ *and* $S_{oT}(\gamma) \leq L\|\bar{\varphi}\|_{W^{1,2}}$.

(b) *Given* $\gamma \in W^{1,2}[-1,T]$ *satisfying* $\gamma_o = 0$ *and* $\gamma_T = \bar{\varphi}$ *given in* $W^{1,2}$, *there exists a constant* $K > 0$ *such that for any* $\bar{\varphi}_1 \in W^{1,2}$ *sufficiently near of* $\bar{\varphi}$ *in the space* $W^{1,2}$, *we can find a path* $\tilde{\gamma} \in W^{1,2}[-1,T]$ *verifying the conditions:* $\tilde{\gamma}_o = 0$, $\tilde{\gamma}_T = \bar{\varphi}_1$ *and*

$$S_{oT}(\tilde{\gamma}) \leq S_{oT}(\gamma) + K\|\bar{\varphi}_1 - \bar{\varphi}\|_{W^{1,2}} .$$

<u>Proof</u>: (a) We take $T > 1$ arbitrary. Let $\gamma: [-1,T] \to R^n$ be defined by $\gamma(t) = 0$ for $t \in [-1,0]$, $\gamma(t) = t\bar{\varphi}(-1)$ for $t \in [0,T-1]$ and $\gamma(t) = \bar{\varphi}(t-T)$ for $t \in [T-1,T]$. Then, $\|\gamma_t\|_\infty \leq \|\bar{\varphi}\|_\infty$ for any $t \in [0,T]$.

Now,

$$S_{oT}(\gamma) = \frac{1}{2}\int_0^{T-1} \left|\frac{\bar{\varphi}(-1)}{T-1} - f(\gamma_t)\right|^2 dt$$

$$+ \frac{1}{2}\int_{T-1}^{T} |\bar{\varphi}(t-T) - f(\gamma_t)|^2 dt$$

$$\leq \frac{1}{T-1}|\bar{\varphi}(-1)|^2 + \int_0^T |f(\gamma_t)|^2 dt + |\dot{\bar{\varphi}}|^2_{L_2} .$$

Since, $f(0) = 0$ and $|f(\gamma_t)| \leq |f|_1\|\gamma_t\|_\infty \leq |f|_1\|\bar{\varphi}\|_\infty$ and $\|\bar{\varphi}\|_\infty \leq |\bar{\varphi}(-1)| + \|\dot{\bar{\varphi}}\|_{L_2}$, we can write:

$$S_{oT}(\gamma) \leq \left(\frac{1}{T-1} + 2T|f|_1^2\right)|\bar{\varphi}(-1)|^2 + (1+2T|f|_1^2)|\dot{\bar{\varphi}}|^2_{L_2}$$

and, since we can suppose $\|\varphi\|_{W^{1,2}} < 1$,

$$S_{oT}(\gamma) \le L\|\varphi\|_{W^{1,2}}$$

where $L = \max\left\{\frac{1}{T-1} + 2T\|f\|_1^2 \,,\, 1 + 2T\|f\|_1^2\right\}$; this finishes the proof.

(b) Let $\tilde{\gamma}$ be defined by: $\tilde{\gamma}(t) = 0$ for $t \in [-1,0]$,

$$\tilde{\gamma}(t) = \gamma(t) + \frac{\bar{\varphi}_1(-1)-\bar{\varphi}(-1)}{T-1} \cdot t \quad \text{for } t \in [0,T-1], \quad \text{and}$$

$$\tilde{\gamma}(t) = \bar{\varphi}_1(t-T) \quad \text{for} \quad t \in [T-1,T].$$

Then, $\|\tilde{\gamma}_t - \gamma_t\|_\infty \le \|\bar{\varphi}_1 - \bar{\varphi}\|_\infty \le |\varphi_1(-1) - \bar{\varphi}(-1)| + \|\dot{\bar{\varphi}}_1 - \dot{\bar{\varphi}}\|_{L_2}$ and

$$S_{oT}(\tilde{\gamma}) - S_{oT}(\gamma) =$$

$$= \frac{1}{2}\int_0^T \langle \dot{\tilde{\gamma}}(t)-\dot{\gamma}(t)+f(\gamma_t)-f(\tilde{\gamma}_t), \dot{\tilde{\gamma}}(t)+\dot{\gamma}(t)- f(\tilde{\gamma}_t) - f(\gamma_t) \rangle \, dt$$

$$\le \frac{1}{2}\left(\|\dot{\tilde{\gamma}}-\dot{\gamma}\|_{L_2}+\|f(\tilde{\gamma}_t)-f(\gamma_t)\|_{L_2}\right)\left(\|\dot{\tilde{\gamma}}\|_{L_2}+\|\dot{\gamma}\|_{L_2}+\|f(\tilde{\gamma}_t)\|_2+\|f(\gamma_t)\|_2\right)$$

We have the estimates:

(i) $\|\dot{\tilde{\gamma}}-\dot{\gamma}\|_{L_2} \le K_1\|\bar{\varphi}_1-\bar{\varphi}\|_{W^{1,2}}$ where $K_1 = \max\{1,\frac{1}{T-1}\}$;

(ii) $\|f(\tilde{\gamma}_t)-f(\gamma_t)\|_{L_2} = \left(\int_0^T |f(\tilde{\gamma}_t) - f(\gamma_t)|^2 dt\right)^{1/2} \le$

$$\left(\int_0^T |f|_1^2 \; |\tilde{\gamma}_t-\gamma_t|_\infty^2 \, dt\right)^{1/2} \le K_2\|\bar{\varphi}_1 - \bar{\varphi}\|_{W^{1,2}} ;$$

where $K_2 = \|f\|_1 \cdot \left(2 + \frac{1}{T-1}\right)$.

(iii) $$\|\dot{\tilde{\gamma}}\|_{L_2} \leq \|\dot{\gamma}\|_{L_2} + \|\dot{\tilde{\gamma}}-\dot{\gamma}\|_{L_2} \leq \|\dot{\gamma}\|_{L_2} + K_1\|\bar{\varphi}_1-\bar{\varphi}\|_{W^{1,2}} .$$

(iv) $$\|f(\tilde{\gamma}_t)\|_{L_2} \leq \|f(\gamma_t)\|_{L_2} + K_2\|\bar{\varphi}_1-\bar{\varphi}\|_{L_2} .$$

From these estimates and supposing $\|\bar{\varphi}_1-\bar{\varphi}\|_{W^{1,2}} < 1$, we get the final estimate

$$S_{oT}(\tilde{\gamma}) - S_{oT}(\gamma) \leq K\|\bar{\varphi}_1 - \bar{\varphi}\|_{W^{1,2}}$$

where $K = \frac{1}{2}(K_1 + K_2).(K_1 + K_2 + 2\|\dot{\gamma}\|_{L_2} + 2\|f(\gamma_t)\|_{L_2})$.

Suppose we have the conditions of Lemma III.2 and suppose also that the quasipotential $V(\varphi)$ verifies the condition:

there exists a unique $\bar{\varphi}_o \in \partial D$ such that $V(\bar{\varphi}_o)=\min\{V(\bar{\varphi}):\bar{\varphi}\in\partial D\}$.

For any given $\delta > 0$, define d as the positive number

$$d = \min\{V(\bar{\varphi}): \bar{\varphi} \in \partial D, \|\bar{\varphi} - \bar{\varphi}_o\|_{W^{1,2}} \geq \delta\} - V(\bar{\varphi}_o).$$

We choose $\mu > 0$, $\mu < \dfrac{d}{5 \max\{K,L\}}$, where K and L are the constants given by Lemma II.3, such that $B\mu(0) \subset D$. Let us denote by S^r the sphere with center 0 and radius r.

Following F-W, we consider the following sequence of Markov times: $\tau_o = 0$, $\sigma_o = \inf\{t > \tau_o: X_t^\varepsilon \in S^\mu\}$, $\tau_1 = \inf\{t > \sigma_o: X_t^\varepsilon \in \partial D \cup S^{\mu/2}\},\ldots$, $\sigma_n = \inf\{t > \tau_n: X_t^\varepsilon \in S^\mu\}$, $\tau_{n+1} = \inf\{t > \sigma_n: X_t^\varepsilon \in \partial D \cup S^{\mu/2}\}$.

III.4. Lemma - *For any given $\varepsilon>0$ sufficiently small, we have, for all $\varphi \in S^{\mu/2}$:*

(a) $$P_\varphi\{X^\varepsilon_{\tau_1} \in \partial D\} \geq \exp\{-\varepsilon^{-2}(V(\bar\varphi_o) + 0{,}45d)\}.$$

(b) $$P_\varphi\{X^\varepsilon_{\tau_1} \in \partial D \backslash B_\delta(\bar\varphi_o)\} \leq \exp\{-\varepsilon^{-2}(V(\bar\varphi_o) + 0{,}55d)\}.$$

Proof: (a) We take $\bar\varphi_1$ out of $\bar D$ such that $\|\bar\varphi_1 - \bar\varphi\|_{W^{1,2}} \leq \frac{\mu}{4}$ and choose initially, using the fact that $\bar\varphi_o$ is a minimum point of V on ∂D, a path $\gamma^{(1)} \in W^{1,2}[-1,T_1] \to R^n$ such that $\gamma^{(1)}(t) = 0$ for $t \in [-1,0]$, $\gamma^{(1)}_{T_1} = \bar\varphi_o$ and $S_{oT_1}(\gamma^{(1)}) \leq V(\bar\varphi_o)+0{,}05d$.

Using Lemma III.3 (b), we can find a path $\tilde\gamma^{(1)} \in W^{1,2}[-1,T_1]\to R^n$ such that $\tilde\gamma^{(1)}(t) = 0$ for $t \in [-1,0]$, $\tilde\gamma^{(1)}_{T_1} = \bar\varphi_1$ and

$$S_{oT_1}(\tilde\gamma^{(1)}) \leq V(\bar\varphi_o) + 0{,}1\ d.$$

Let $t_1 = \max\{t > 0: \tilde\gamma^{(1)}(t) \in S^\mu\}$ and let $\psi_1 = \tilde\gamma^{(1)}_{t_1}$. Denote by $\tilde\gamma^{(2)} \in W^{1,2}[-1,T_1-t_1]$ the path given by $\tilde\gamma^{(2)}(t) = \psi_1(t)$ for $t \in [-1,0]$ and $\tilde\gamma^{(2)}(t) = \tilde\gamma^{(1)}(t-t_1)$ for $t \in [0,T_2]$, where $T_2=T_1-t_1$.

Then, $S_{oT_2}(\tilde\gamma^{(2)}) \leq V(\bar\varphi_o) + 0{,}1\ d.$

By Lemma III.3.(a), we can find a path $\tilde\gamma^{(3)} \in W^{1,2}[-1,T_3]$ such that $\tilde\gamma^{(3)}_o = 0$, $\tilde\gamma^{(3)}_{T_3} = \psi_1$ and $S_{oT_3}(\tilde\gamma^{(3)}) \leq 0{,}2\ d$; by the same argument, for each $\varphi \in S^{\mu/2}$, we can find a path $\tilde\gamma^{(4)} \in W^{1,2}[-1,T_4]$ such that $\tilde\gamma^{(4)}_o = 0$, $\tilde\gamma^{(4)}_{T_4} = \tilde\varphi$ and $S_{oT_4}(\tilde\gamma^{(4)}) \leq 0{,}1\ d$, where $\tilde\varphi$ is defined by $\tilde\varphi(\theta) = \varphi(-1-\theta)$, $\theta \in [-1,0]$.

We construct now the path $\gamma^\varphi \in W^{1,2}[-1,T]$ where $T = T_1+T_2+T_3$ by $\gamma^\varphi(t) = \tilde\gamma^{(4)}(T_4 - 1 - t)$ for $t \in [-1,0]$, $\gamma^\varphi(t) = \gamma^{(3)}(t)$ for

$t \in [0,T_3]$, $\gamma^\varphi(t) = \gamma^{(2)}(t-T_3)$ for $t \in [T_3,T_3+T_2]$ and finally, $\gamma^\varphi(t) = \gamma^{(1)}(t-T_3-T_2)$ for $t \in [T_2+T_3,T]$. Then, $\gamma_o^\varphi = \varphi$, $\gamma^\varphi_{T_\varphi} = \bar\varphi_1$ and

$$S_{oT}(\gamma^\varphi) \le V(\bar\varphi_o) + 0,4\ d.$$

We choose now $\delta' < \min\{ \frac{\mu}{4}, \mathrm{dist}(\bar\varphi_1, \partial D)\}$. We know that [F-V] there exists an $\varepsilon > 0$ such that for any $\varphi \in S^{\mu/2}$ we have

$$P_\varphi \left\{ \sup_{t\in[-1,T]} |X^\varepsilon(t)-\gamma^\varphi(t)| < \delta' \right\} \ge \exp\left\{-\varepsilon^{-2}(V(\bar\varphi_o)+0,45d)\right\}$$

Since

$$\left\{\omega \in \Omega\colon X^\varepsilon_{\tau_1} \in \partial D\right\} \supset \left\{\omega \in \Omega\colon \sup_{t\in[-1,T]} |X^\varepsilon(t) - \gamma^\varphi(t)| < \delta'\right\}$$

it results

$$P_\varphi\left\{X^\varepsilon_\tau \in \partial D\right\} \ge \exp\left\{-\varepsilon^{-2}(V(\bar\varphi_o) + 0,45d)\right\}$$

which concludes the proof.

(b) As in [F-V] define $\tau(S^{\mu/2} \cup \partial D) = \inf\{t>0\colon X^\varepsilon_t \in S^{\mu/2} \cup \partial D\}$.

Then, $\tau(S^{\mu/2} \cup \partial D) \le \tau_1$ and

$$X^\varepsilon(\tau_1,0,\varphi) = X^\varepsilon(\tau(S^{\mu/2} \cup \partial D),0,X^\varepsilon(\sigma_o,0,\varphi)).$$

Now, for any $\varphi \in S^{\mu/2}$ there exists $\hat\varphi \in S^\mu$ such that

$$\left\{\omega \in \Omega: X^{\varepsilon}(\tau_1, 0, \varphi) \in \partial D \backslash B_{\delta}(\bar{\varphi}_o)\right\} =$$

$$= \left\{\omega \in \Omega: X^{\varepsilon}(\tau(S^{\mu/2} \cup \partial D), 0, \hat{\varphi}) \in \partial D \backslash B_{\delta}(\bar{\varphi}_o)\right\} ;$$

then, $$P_{\varphi}\left\{X^{\varepsilon}_{\tau_1} \in \partial D \backslash B_{\delta}(\bar{\varphi}_o)\right\} \leq \sup_{\hat{\varphi} \in S^{\mu}} P_{\hat{\varphi}}\left\{X^{\varepsilon}_{\tau(S^{\mu} \cup \partial D)} \in \partial D \backslash B_{\delta}(\bar{\varphi}_o)\right\} .$$

We majorize this last quantity. First, note that for any T we have: the set

$$\left\{\omega \in \Omega: X^{\varepsilon}_{\tau(S^{\mu/2} \cup \partial D)} \in \partial D \backslash B_{\delta}(\bar{\varphi}_o)\right\}$$

is contained in the union $\Big\{\omega \in \Omega: \tau(S^{\mu/2} \cup \partial D) > T$ and

$$X^{\varepsilon}_{\tau(S^{\mu/2} \cup \partial D)} \in \partial D \backslash B_{\delta}(\bar{\varphi}_o)\Big\} \cup \Big\{\omega \in \Omega: \tau(S^{\mu/2} \cup \partial D) \leq T \text{ and}$$

$$X^{\varepsilon}_{\tau(S^{\mu/2} \cup \partial D)} \in \partial D \backslash B_{\delta}(\bar{\varphi}_o)\Big\} ;$$

therefore

$$P_{\hat{\varphi}}\left\{X^{\varepsilon}_{\tau(S^{\mu/2} \cup \partial D)} \in \partial D \backslash B_{\delta}(\bar{\varphi}_o)\right\} \leq P_{\hat{\varphi}}\left\{\tau(S^{\mu/2} \cup \partial D) > T\right\} +$$

$$P_{\hat{\varphi}}\left\{\tau(S^{\mu/2} \cup \theta D) \leq T \text{ and } X^{\varepsilon}_{\tau(S^{\mu/2} \cup \partial D)} \in \partial D \backslash B_{\delta}(\bar{\varphi}_o)\right\} .$$

By Lemma III.3.(b), these exist constants $c > 0$ and T_o such that for any $\varepsilon > 0$ small enough and any $\hat{\varphi} \in S^{\mu}$ we have

$$P_{\hat{\varphi}}\left\{\zeta_\delta > T\right\} \leq \exp\left\{-\varepsilon^{-2}.c\right\}$$

where $\zeta_\delta = \inf\left\{t\colon X_t^\varepsilon \notin D\backslash B_\delta(0)\right\}$; since $\tau(S^{\mu/2} \cup \partial D) \leq \zeta_\delta$ we have also

$$\left\{\omega \in \Omega\colon \tau(S^{\mu/2} \cup \partial D) > T\right\} \subset \left\{\omega \in \Omega\colon \zeta_\delta > T\right\}$$

and

$$P_{\hat{\varphi}}\left\{\tau(S^{\mu/2} \cup \partial D) > T\right\} \leq \exp\left\{-\varepsilon^{-2}c\right\}.$$

Making $c = V(\overline{\varphi}_o) + d$ we can write

$$P_{\hat{\varphi}}\left\{\tau(S^{\mu/2} \cup \partial D) > T\right\} \leq \exp\left\{-\varepsilon^{-2}(V(\overline{\varphi}_o) + d)\right\}.$$

Let now K be the closure of the $\mu/2$ - neighbourhood of $\partial D\backslash B_\delta(\overline{\varphi}_o)$. We prove that no path $\gamma \in W^{1,2}[-1,T]$ such that $\gamma_o = \hat{\varphi} \in S^\mu$ and action $S_{oT}(\gamma) \leq V(\overline{\varphi}_o) + 0,65d$ reaches K. In fact, suppose by contradiction that there exists $t_1 \in [0,T]$, such that $\gamma_{t_1} \in K$. Then, $S_{ot_1}(\gamma) \leq V(\overline{\varphi}_o) + 0,65d$. By Lemma III.3.(a), we can construct a path $\gamma^{(1)} \in W^{1,2}[0,T_1]$ such that $\gamma_o^{(1)} = 0$, $\gamma_{T_1}^{(1)} = \hat{\varphi}$ and $S_{oT_1}(\gamma^{(1)}) \leq L\|\hat{\varphi}\|_{W^{1,2}} \leq 0,2d$.

Let $\overline{\gamma}$ be the path given by $\overline{\gamma}(t)=0$ for $t\in[-1,0]$, $\overline{\gamma}(t)=\gamma^{(1)}(t)$ for $t\in[0,T_1]$ and $\overline{\gamma}(t)=\gamma(t-T_1)$ for $t \in[T_1,T_1+t_1]$. Then, $\overline{\gamma}_o = 0$, $\overline{\gamma}_{t_1+T_1} = \gamma_{t_1} \in K$ and

$$S_{o(T_1+t_1)}(\overline{\gamma}) \leq V(\overline{\varphi}_o) + 0,85d.$$

By Lemma III.3.(b), we can find a path $\tilde{\gamma} \in W^{1,2}[0,T_1+t_1]$ such that $\tilde{\gamma}_0 = 0$, $\tilde{\gamma}_{t_1+T_1} = \hat{\psi} \in \partial D\backslash B_\delta(\bar{\varphi}_0)$ (for $\hat{\psi}$ in the open ball $B_{\mu/2}(\gamma_{t_1})$) and such that $S_{0(T_1+t_1)}(\tilde{\gamma}) \leq S_{0(T_1+t_1)}(\bar{\gamma}) + K\|\psi-\gamma_{t_1}\|_{W^{1,2}}$ $= V(\bar{\varphi}_0) + 0{,}85d + K\|\psi-\gamma_{t_1}\| \leq V(\bar{\varphi}_0) + 0{,}95d$.

Now, this implies that $V(\hat{\psi}) \leq V(\bar{\varphi}_0) + 0{,}95d$ which is not possible by the definition of the number d since $\hat{\psi} \in \partial D\backslash B_\delta(\bar{\varphi}_0)$.

Therefore, all the paths γ in the set $\bigcup_{\hat{\varphi}\in S^\mu} \Phi_{\hat{\varphi}}(V(\varphi_0) + 0{,}65d)$ satisfy

$$\mathrm{dist}(\gamma_t\ ,\ \partial D\backslash B_\delta(\bar{\varphi}_0)) \geq \frac{\mu}{2}$$

and then

$$P_{\hat{\varphi}}\left\{\tau(S^{\mu/2} \cup \partial D) \leq T, X^\varepsilon_{\tau(S^{\mu/2}\cup\partial D)} \in \partial D\backslash B_\delta(\bar{\varphi}_0)\right\} \leq$$

$$P_{\hat{\varphi}}\left\{\mathrm{dist}(X^\varepsilon, \Phi_{\hat{\varphi}}(V(\bar{\varphi}_0) + 0{,}65d)) \geq \frac{\mu}{2}\right\}.$$

This last quantity is, by Theorem 1,1, less or equal to $\exp\{-\varepsilon^{-2}(V(\bar{\varphi}_0) + 0{,}65d - 0{,}05d)\}$ if ε is small enough.

Collecting the estimates above we can assert that, for any $\hat{\varphi} \in S^\mu$, we have:

$$P_{\hat{\varphi}}\left\{X^\varepsilon_{\tau(S^{\mu/2}\cup\partial D)} \in \partial D\backslash B_\delta(\bar{\varphi}_0)\right\} \leq \exp\left\{-\varepsilon^{-2}(V(\bar{\varphi}_0)+d)\right\}$$

$$+ \exp\left\{-\varepsilon^{-2}(V(\bar{\varphi}_0) + 0{,}6d)\right\} \leq \exp\left\{-\varepsilon^{-2}(V(\bar{\varphi}_0) + 0{,}55d)\right\},$$

which finishes the proof.

Let us proof now Theorem I.2.

It follows from Lemma III.4 that for any $\varphi \in S^{\mu/2}$ and $\varepsilon > 0$ small enough:

$$(*) \qquad \frac{P_\varphi\left\{X_{\tau_1} \in \partial D \setminus B_\delta(\bar{\varphi}_o)\right\}}{P_\varphi\left\{X_{\tau_1} \in \partial D\right\}} \leq \frac{\exp\{-\varepsilon^{-2}(V(\bar{\varphi}_o) + 0{,}55d)\}}{\exp\{-\varepsilon^{-2}(V(\bar{\varphi}_o) + 0{,}45d)\}} .$$

Let ν be the minimum integer n for which $Z_n \in \partial D$, where $Z_n = X^\varepsilon_{\tau_n}$. Then, for any $\in S^{\mu/2}$:

$$P_\varphi\left\{\|X^\varepsilon_{\tau^\varepsilon} - \bar{\varphi}_o\| \geq \delta\right\} = P_\varphi\left\{Z_\nu \in \partial D \setminus B_\delta(\bar{\varphi}_o)\right\} ,$$

and, since the set $\{\omega \in \Omega: Z_{\nu(\omega)}(\omega) \in \partial D\setminus B_\delta(\bar{\varphi}_o)\}$ is equal to the union $\bigcup_{n=1}^{\infty} \{\omega \in \Omega: \nu(\omega)=n, Z_n \in \partial D\setminus B_\delta(\bar{\varphi}_o)\}$, we can write

$$P_\varphi\{Z_\nu \in \partial D\setminus B_\delta(\bar{\varphi}_o)\} = \sum_{n=1}^{\infty} P_\varphi\{\nu=n, Z_n \in \partial D\setminus B_\delta(\bar{\varphi}_o)\}$$

$$= \sum_{n=1}^{\infty} \int_{A_n} P_{Z_{n-1}} \{Z_1 \in \partial D\setminus B_\delta(\bar{\varphi}_o)\} ,$$

where $A_n = \{\omega \in \Omega: Z_1 \in S^{\mu/2}, \ldots, Z_{n-1} \in S^{\mu/2}\}$.

From the inequality (*), we have

$$P_\varphi\{Z_\nu \in \partial D\setminus B_\delta(\bar{\varphi}_o)\} \leq \sum_{n=1}^{\infty} \int_{A_n} P_{Z_{n-1}}\{Z_1 \in \partial D\}.\exp\{-\varepsilon^{-2}.0{,}1d\}$$

$$= \sum_{n=1}^{\infty} P_\varphi\{\nu=n\}.\exp\{-\varepsilon^{-2}0,1d\}$$

$$= \exp\{-\varepsilon^{-2}0,1.d\}.$$

It follows that, for $\varepsilon \to 0$, we have

$$P_\varphi\left\{\|X^\varepsilon_{\tau^\varepsilon} - \bar{\varphi}_0\| \geq \delta\right\} \to 0$$

for any $\varphi \in S^{\mu/2}$. Now, for any $\varphi \in D$:

$$P_\varphi\left\{\|X^\varepsilon_{\tau^\varepsilon} - \bar{\varphi}_0\| \geq \delta\right\} = P_\varphi\left\{\|X^\varepsilon_{\tau^\varepsilon} - \bar{\varphi}_0\| \geq \delta, \tau(S^{\mu/2} \cup \partial D)=\tau^\varepsilon\right\} +$$

$$P_\varphi\left\{\|X^\varepsilon_{\tau^\varepsilon} - \bar{\varphi}_0\| \geq \delta, \tau(S^{\mu/2} \cup \partial D) < \tau^\varepsilon\right\}$$

Therefore, the probability $P_\varphi\left\{\|X^\varepsilon_{\tau^\varepsilon} - \bar{\varphi}_0\| \geq \delta\right\}$ is less than

$$P_\varphi\left\{X^\varepsilon_{\tau(S^{\mu/2}\cup\partial D)} \in \partial D\right\} + P_\varphi\left\{\|X^\varepsilon_{\tau^\varepsilon} - \bar{\varphi}_0\| \geq \delta, X^\varepsilon_{\tau(S^{\mu/2}\cup\partial D)} \in S^{\mu/2}\right\}.$$

The first probability of this sum, by Lemma III.1, tends to zero as $\varepsilon \to 0$, for any $\varphi \in S^{\mu/2}$. Also,

$$P_\varphi\left\{\|X^\varepsilon_{\tau^\varepsilon} - \bar{\varphi}_0\| \geq \delta, X^\varepsilon_{\tau(S^{\mu/2}\cup\partial D)} \in S^{\mu/2}\right\} =$$

$$= \int_A P_{X^\varepsilon_{\tau(S^{\mu/2}\cup\partial D)}}\left\{\|X^\varepsilon_{\tau^\varepsilon} - \bar{\varphi}_0\| \geq \delta\right\}$$

where $$A = \left\{ X^{\varepsilon}_{\tau(S^{\mu/2} \cup \partial D)} \in S^{\mu/2} \right\}$$

and this last integral is $\leq \exp\{-\varepsilon^{-2}.0,1d\}$, which also goes to zero as $\varepsilon \to 0$.

This finishes the proof of the Theorem.

IV. Example

Let us apply the foregoing results to the problem of exit from a domain attracted to 0, in the case of the dynamical system defined by the scalar linear retarded differential difference equation:

(4.0) $$\dot{x}(t) = -x(t-b).$$

We know [Ha] that the condition $0 < b < \frac{\pi}{2}$ is a necessary and sufficient condition to ensure that 0 is an asymptotically stable equilibrium of Eq. (4.0); in fact, this condition is equivalent to assume that all roots of the characteristic equation

$$\lambda + e^{-\lambda b} = 0$$

verify $\mathrm{Re}\lambda < 0$.

The action functional corresponding to equation (4.0) is given by

(4.1) $$S_T(\gamma) = \frac{1}{2} \int_{-T}^{b} \left(\dot{\gamma}(t) + \gamma(t-b)\right)^2 dt$$

and the Euler-Lagrange equations for the extremals of S are given by

$$(4.2.1) \qquad \dot{\gamma}(t) + \gamma(t-b) = H(t) \ , \ t \in (-T,b)$$

$$(4.2.2) \qquad \dot{H}(t) - H(t+b) = 0 \ , \ t \in (-T,0]$$

We will compute the quasipotential of equation (4.1) with respect to the origin.

As noted before we may suppose that $T = \infty$.

Equation (4.2.2) is an advanced difference-differential equation, which becomes a retarded equation by performing the change of independent variable $t \mapsto -t$. Therefore, by [B-T], given $\psi \in L_2[-b,0]$, and $\xi \in \mathbb{R}$, we can solve (4.2.2) to find a unique function $H\colon (-\infty,b] \to \mathbb{R}$ which satisfies: 1º) $H(b+\theta) = \psi(\theta)$ for almost all $\theta \in [-b,0]$; 2º) $H(0) = \xi$; 3º) H is absolutely continuous on $(-\infty,0]$, and, 4º) for almost all t in $(-\infty,0]$, $\dot{H}(t)=H(t+b)$.

With $H = H(t,\xi,\psi)$ so determined, we solve Eq. (4.2.1) in $(-\infty,0]$ with initial condition $\gamma_{-\infty}=0$. We get a function $\gamma\colon (-\infty,+b]\to\mathbb{R}$ which is absolutely continuous on $(-\infty,0]$.

Of course, γ depends upon ψ and ξ ; the relations

$$(4.3) \qquad \gamma_b = \varphi \ , \ \varphi \in W^{1,2}$$

allow us determine ψ and ξ uniquely in function of φ. In fact, from the variation of constants formula [Ha], we have

$$(4.4) \qquad \gamma(t) = \int_{-\infty}^{t} X(t-s)H(s,\psi,\zeta)ds \ , \ t \in (-\infty,b] \ ,$$

where X is the fundamental solution:

$$\dot{X}(t) = -X(t-b) \ , \ t > 0$$
$$X(0) = 1$$
$$X(t) = 0, \ t < 0 \ .$$

Given $\psi \in L_2$, let $J(t)$ be the solution of $\dot{J}(t) = -J(t-b)$, $t > 0$, $J(0) = 0$, $J_o = \tilde{\psi}$, where $\tilde{\psi}$ is defined by $\tilde{\psi}(\theta) = \psi(-b-\theta)$ for all $\theta \in [-b,0]$.

It is easy to see that

$$J(t) = \int_{-b}^{0} X(t+u)\psi(u)du \ , \ t > 0$$

and that

$$H(t,\psi,\zeta) = X(-t)\zeta + J(-t) \ , \ t \in (-\infty,0).$$

By equation (4.2.1) we have for $t \in [-b,0]$:

$$\psi(t-b) = \dot{\varphi}(t-b) + \int_{-\infty}^{t-b} X(t-b-S)\left[X(-S)\zeta + \int_{-b}^{0} X(-S+u)\psi(u)du\right]dt$$

or

$$(4.5) \quad \psi(\theta) = \dot{\varphi}(\theta) + \int_{-\theta}^{\infty} X(t+\theta)X(t)dt\zeta + \int_{-b}^{0}\left[\int_{-\theta}^{\infty} X(t+\theta)X(t+u)dt\right]\psi(u)du$$

Define the function $a(\theta)$, $\theta \in [-b,0]$, by

$$a(\theta) = \int_0^\infty X(t-\theta)X(t)dt.$$

Then, we can write equation (4.4) as

(4.6) $$\psi(\theta) = \dot{\varphi}(\theta) + a(\theta)\xi + \int_{-b}^{0} a(-|\theta-u|)\psi(u)du$$

Letting $t = 0$ in equation (4.4), we get

(4.7) $$\varphi(-b) = a(0)\xi + \int_{-b}^{0} a(u)\psi(u)du.$$

Since $a(0) > 0$, we can solve Eq. (4.7) for ξ so that equation (4.6) can be written as

(4.8) $$\psi(\theta) = \dot{\varphi}(\theta) + \frac{a(\theta)}{a(0)}\varphi(-b) + \int_{-b}^{0} K(\theta,u)\psi(u)du$$

where $K(\theta,u) = a(-|\theta-u|) - \dfrac{a(\theta)a(u)}{a(0)}$.

We now compute the function $a(\theta)$.

It follows from the definition that

$$\dot{a}(\theta) = -\int_0^\infty \dot{X}(t-\theta)X(t)dt = \int_0^\infty X(t-\theta-b)X(t)dt = a(-b-\theta).$$

Therefore, $\ddot{a}(\theta) = -a(\theta)$ and there exist constante c_1 and c_2 such that

$$a(\theta) = c_1 \cos\theta + c_2 \operatorname{sen}\theta.$$

From the equation $\dot{a}(\theta) = a(-b-\theta)$ it follows that

$$a(\theta) = \frac{a(0)}{\cos\beta}\cos(\theta-\beta)$$

where $\beta = \frac{\pi}{4} - \frac{b}{2}$.

Now, $a(-b) = \int_0^\infty X(t+b)X(t)dt = \frac{1}{2}$ so that

$$a(0) = \frac{1 + \text{sen } b}{2 \cos b} \quad \text{and finally}$$

$$a(\theta) = \frac{1 + \text{sen } b}{2 \cos b \cos \beta}\cos(\theta-\beta).$$

Also, the kernel $K(\theta,u)$ has the expression

$$K(\theta,u) = -\frac{\cos(\theta+b)\,\text{sen } u}{\cos b} \quad \text{for} \quad -b \le \theta \le u \le 0$$

$$K(\theta,u) = -\frac{\cos(u+b)\,\text{sen } \theta}{\cos b} \quad \text{for} \quad -b \le u \le \theta \le 0.$$

Equation (4.8) now becomes

(4.9) $$\psi(\theta) = \dot{\varphi}(\theta) - \frac{\cos(\theta-\beta)}{\cos \beta}\varphi(-b) - \int_{-b}^{\theta} \frac{\text{sen } \theta}{\cos b}\cos(u+b)\psi(u)du - \int_{\theta}^{0} \frac{\cos(\theta+b)}{\cos b}\text{sen } u\, \psi(u)du.$$

We prove now that 1 is not an eigenvalue of the operator k defined by

$$(k\,\psi)(\theta) = \int_{-b}^{0} K(\theta,u)\psi(u)du.$$

Suppose that $k\psi = \psi$. Then,

$$\psi(0) = 0 \quad \text{and} \quad \dot{\psi}(-b) = 0 \quad \text{and} \quad \ddot{\psi} + 2\psi = 0.$$

Hence, $\psi(\theta) = c_1 \cos \sqrt{2}\,\theta + c_2 \operatorname{sen} \sqrt{2}\,\theta$.

The condition $\psi(0) = 0$ implies $c_1 = 0$ and the condition $\dot{\psi}(-b) = 0$ implies that $c_2 = 0$. Hence, $\psi \equiv 0$.

Since the operator k is compact, and $1 \notin \sigma(k)$ equation (4.9) has a unique solution ψ for each $\varphi \in W^{1,2}$.

We now compute the quasipotential relative to the origin.

From what has been proved above, the quasipotential $V(\varphi)$ is given by

$$V(\varphi) = \frac{1}{2}\int_{-\infty}^{b} |\dot{\gamma}(t) + \gamma(t-b)|^2 dt$$

where $\gamma(t) = x(-t)$, where $x(t)$ is the solution of $\dot{x}(t) = -x(t-b)$, $t > 0$, $x_o = \tilde{\varphi}$. In fact, $\gamma_b = \varphi$ and $\gamma_{-\infty} = 0$ and γ satisfies the variational equations (4.2.1) and (4.2.2).

We have $V(\varphi) = \frac{1}{2}\int_{-b}^{\infty} |\dot{x}(t) - x(t+b)|^2 dt$,

$$V(\varphi) = \frac{1}{2}\int_{-b}^{\infty} |\dot{x}(t)|^2 dt - \int_{-b}^{\infty} \dot{x}(t)x(t+b)dt + \frac{1}{2}\int_{-b}^{\infty} |x(t+b)|^2 dt$$

$$= \frac{1}{2}\int_{-b}^{0} |\dot{\varphi}(\theta)|^2 dt + \frac{1}{2}\int_{0}^{\infty} |x(t-b)|^2 dt$$

$$- \left[x(t)x(t+b)\Big|_{-b}^{\infty} - \int_{-b}^{\infty} x(t)\dot{x}(t+b)dt\right] + \frac{1}{2}\int_{-b}^{\infty} |x(t+b)|^2 dt$$

$$= \frac{1}{2}\int_{-b}^{0} |\varphi(\theta)|^2 d\theta - \frac{1}{2}\int_{-b}^{0} |\varphi(\theta)|^2 d\theta + \varphi(0)\varphi(-b).$$

From the Cauchy-Schwarz inequality one can prove that, if $b \in (0,1/2)$, then

$$V(\varphi) \geq \left(\frac{1}{2} - b\right)\|\varphi\|_{1,2}^2$$

which shows that $\sqrt{V}$ is a norm on $W^{1,2}$, equivalent to the natural norm.

If we take D as the open ball with respect to the norm $\sqrt{V}$, with center c and radius R, where c and R are positive constants, $c < R$, then, theorem I.2 can be applied to the pertubed system

$$\dot{x}(t) = - x(t-b) + \varepsilon\, \dot{w}(t).$$

In fact, the minimum of $V(\varphi)$ on $\partial D = \{\varphi \in W^{1,2}: V(\varphi-c) = R^2\}$ is achieved only at the point $\varphi = R + c$.

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