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A Glance at the Early History of

Group Rings

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A GLANCE AT THE EARLY HISTORY OF GROUP RINGS

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I. - INTRODUCTION - Group rings usually appear in courses
on Group Representation Theory as a means to gain a broader
view on the subject and connect it to the general theory
of Algebras and their representations (e.g. in H.Borner[1]
or C. W. Curtis and I. Reiner [8]). This may suggest the
misleading idea that it was precisely this point of view
that motivated the definition and study of group rings. In
fact, this is explicitly stated by several authors who at-
tribute the idea to E. Noether [23].

Though both topics are closely related and
Representation Theory was actually a motivation for much
of the work which was done in Group Rings, the historical
order of development was rather the inverse: interest on
the structure of group rings led to the discovery of some
of the earlier theorems on group representations.

This fact was pointed out in a most inter-
esting paper by T. Hawkins [14] but, perhaps due to the
fact that it was published in a journal devoted to the His
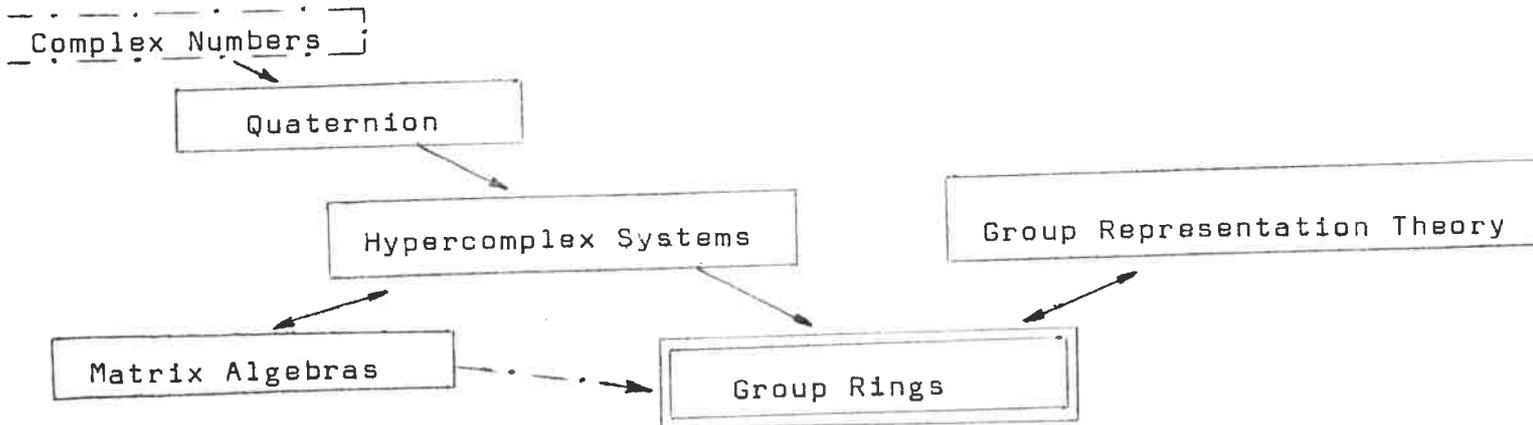tory of Science rather than to mathematics itself, it seems

to have remained unnoticed by those working on the subject. Recent books and surveys fail to credit either A.Cayley or T. Molien, and some still attribute to E.Noether the crea tion of the theory, omitting even the influence of R.Brauer on this process.

In this note, we do not intend to give a full account of what was done before Noether's paper, but rather describe the sucessive "births" of the theory. We have included first a section describing the mathematical circunstances in which the theory was born as an attempt to show that these ideas were a natural consequence of the mathematics of the times.

Our main sources were the paper by T.Hawkins [14], the works by A.Cayley [7], R.Brauer and E. Noether [5], E. Noether [23] and, for the more general aspects,the well-known treatise by M. Kline [19].

II. - "PRE-HISTORY" - As we shall try to show, the roots of the notion of group ring should be sought in the theory of hypercomplex systems and these, in turn,developed from the concepts of quaternions, around the second half of the past century. Of course, all these ideas are also closely related to the theory of matrices, which was created ap- proximately at the same time.

The reciprocal influences among these theo
ries can be visualized in the following diagram:

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 ._._._._._. ─┐
_Complex_Numbers_│
         ↓
   ┌─────────────────┐
   │  Quaternion     │
   └─────────────────┘
              ↓
        ┌──────────────────────┐          ┌────────────────────────────┐
        │ Hypercomplex Systems │          │ Group Representation Theory │
        └──────────────────────┘          └────────────────────────────┘
       ↙            ↘                  ↘          ↗
┌─────────────────┐      ┌───────────────────┐
│ Matrix Algebras │ ···→ │   Group Rings     │
└─────────────────┘      └───────────────────┘
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Complex numbers were introduced in the XVIth
century as a result of the work of Italian mathematicians
while studying equations of the third degree. A long contro
versy regarding their existence and meaning was raised,and
they gradually gained acceptance after a geometrical inter
pretation was given by Wessel, Argand and Gauss.

However, though better understood, a need
for an algebraic system in which the square of a "quantity"
would actually be equal to 1 was still felt.

Such a construction was given by Sir William
Rowan Hamilton (1805-1865) in 1837, when he published his
paper *"Conjugate functions and on algebra as a science of
pure time"* in the Transactions of the Royal Irish Academy.
There, he pointed out that an expresion such as 2+3i is not

a genuine sum in the sense that 2+3 is, and introduced the idea of ordered pair (a,b) to represent the complex number a+bi, developping the theory on that basis, as is now on use.

Hamilton was a man of many talents and had a special interest in physics. Thus, he was well aware that his "ordered pairs" gave an algebraic system that could be represented as plane vectors and it was clear to him that, if he could develop a similar system with "ordered triples", he would be able to deal with space vectors. Needless to say, this would mean an invaluable tool for the study of the physical world.

After several failures, he realized that he actually needed to work with "ordered quatruples" an de-fined a quaternion to be an element of the form

$$a + bi + cj + dk.$$

It was natural to define the sum of two such elements by adding corresponding coefficients.Since Hamilton implicitely assumed that the distributive law should hold, to define the product of quaternions, he only needed to decide how to multiply the symbols i,j,k among themselves. The rules he gave seem quite reasonable from a modern point of view, since they closely resemble vector products:

$$i^2 = j^2 = k^2 = -1$$

$$ij=k=-ji \; ; \quad jk=i=-kj \; ; \quad ki=j=-ik.$$

However, at his times, this ment a revolution, since it was the first algebraic system where multiplication was not commutative.

Though quaternions never had the importance that Hamilton expected them to have in Physics, this was to be a decisive discovery for the future developement of mathematics and, specially, of algebra. Comments on those consequences will be found in [32,chapter 32], [7,chapter 26] and [20].

From our present interest, the relevant feature is that, once quaternions were discovered, it was only natural to consider "algebraic quantities" of the same type, but of higher dimensions.

In this perspective, an *Hypercomplex System* H (a finite dimensional associative algebra over the field of real or complex numbers, in present-day terminology) was naturaly defined to be the set of all elements of the form:

$$x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$$

where $x_1, x_2, \ldots, x_n$ are real or complex numbers and $e_1, e_2, \ldots, e_n$ are symbols which were called the *units* of the hypercomplex system. As in the case of quaternions, the sum of two such elements is defined by adding corresponding coefficients and, to define the product, it suffices to decide how to multiply the units among themselves.

Since the product of two such units   must
be another element of H , it is possible to write it   in
the form:

$$e_i e_j = \sum_{k=1}^{n} a_k(i,j) \, e_k.$$

Thus, the multiplicative structure of   H
was determined   by giving the values of the coefficients
$a_k(i,j)$, which were called the *structural constants* of H.
Of course, this should be done in such a way as to ensure
that the associative law of multiplication holds   (though
sometimes non-associative systems were considered   as the
octonions, defined by A. Cayley shortly afterwords).

Though Hamilton himself began the work   on
hypercomplex systems in a paper in the Transactions of the
Royal Irish Academy in 1848 and considered *Biquaternions*
(i.e. quaternions with complex coefficients) in 1853,   it
was mainly his work on quaternions which raised the inter-
est on algebras.

It was precisely at this early stage, where
the theory was giving its first steps and most of the bas-
ic concepts were not yet formulated, that group rings were
implicitely considered for the first time, in a paper   by
Arthur Cayley (1821-1895).

III. - ABSTRACT GROUPS AND GROUP RINGS - As it is well-known, interest on permutations was first focused by the work of Joseph Louis Lagrange (1736-1813) on algebraic equations, followed by P. Ruffini (1765-1822) and N.H.Abel (1802-1829). E. Galois (1811-1832) was the first to consider permutation groups, using the term "group" with its actual sense in his classical work of 1830.

A. Cauchy (1789-1857) was a pioneer in understanding the relevance of permutation groups as an independent subject; he wrote a series of important papers in the period 1844-1846. Influenced by Cauchy's work, A. Cayley realized that the notion of group could be formulated in a more abstract setting.

In 1854 he published a paper entitled "On the theory of groups, as depending on the symbolic equation $\theta^n=1$" in the Philosophical Magazine. This paper is usually regarded as the first work on abstract groups (for example in [2], [19] and [21]) and it is also there that the construction of a group ring is given for the first time.

Since the terminology of set theory was not current in Cayley's times, he starts his paper trying to make clear that he is working with abstract symbols rather than concrete objects such as permutations or numbers. In this sense, he states: "... $\theta\phi$ *is of course different from* $\phi\theta$. *But the symbols* $\theta,\phi,\ldots$ *are in general such that*

$\Theta.\phi\chi=\Theta\phi.\chi$, &c, *so that* $\Theta\phi\chi$, $\Theta\phi\chi\omega$, &c. *have a definite signi_ fication independent of the particular mode of compounding the symbols ...*".

In the same direction, he also states: "*It is not necessary (even if this could be done) to attach any meaning to a symbol such as* $\Theta\pm\phi$, *or to the symbol* 0, *nor consequently to an equation such as* $\Theta=0$, *or* $\Theta\pm\phi=0$;..."As we shall see, this remark was the key to his construction of a group ring.

He proceeds to discuss some elementary prop- erties, introduces what we now call the Cayley table of an operation and remarks: "*The distinction between the theory of the symbolic equation* $\Theta^n=1$ *and that of the ordinary equation* $x^n=1$, *presents itself in the very simple case,* n=4".

By an analysis of the possible tables of op- eration he shows that there are two possible "*essentially distinct*" groups; in present-day terminology: The cyclic group of order 4 which "*is analogous to the system of roots of the ordinary equation* $x^4-1=0$"and the Klein four-group. He remarks that this second group is "*of frequent occurrence in analysis*" and "*it is only in account of their extreme sim_ plicity that they have not been expressly remarked*".

Then, he studies the possible groups of six elements showing again that there exists essentially two different cases and cites, as an example, the group of permutations

of three letters.

At the end of the paper, Cayley returns to the possibility of giving a meaning to the sum of two sym̲ bols:

"It is, I think, worth noticing, that if, instead of considering α, β, &c. as symbols of operation, we consider them as quantities (or, to use a more abstract term, 'cognitables') such as the quaternion imaginaries; the equations expressing the existence of the group are, in fact, the equations defining the meaning of the product of two complex quantities of the form

$$w + a\alpha + b\beta + ...".$$

He illustrates this remark by showing explicitly how to multiply two elements of the form $(w+a\alpha+b\beta+c\gamma+d\delta+e\varepsilon)$, where $\{1,\alpha,\beta,\gamma,\delta,\varepsilon\}$ is the non abelian group of order 6 he had just studied, and concludes with the remark: "It does not appear that there is in this sys̲ tem anything analogous to the modulus $w^2+x^2+y^2+z^2$, so important in the theory of quaternions".

It is then clear that Cayley actually gave the formal construction of a group ring in essentially the same way as we do today: namely, that of CS_3. However, not even the basic notions of the theory of rings and algebras were formulated at this time and this concept remained unnoticed. In this sense, it is perhaps correct to trace

the origin of the theory up to the work of Molien - which we briefly discuss in the next section - as is done, for example in [12].

IV. - STRUCTURE OF RINGS AND GROUP RINGS - We now return to Hypercomplex Systems, since it was their study that led naturally to group rings once more. We shall refer to Molien's work only briefly, since it was covered by T. Hawkins in [14] which we somehow follow in this section (for a general view on the history of algebras, see also [13]).

Soon after the introduction of this concept, many individual algebras were discovered and a need for a classification was felt. Benjamin Peirce (1809-1880) under took this task in a paper which was read in 1870 and pub lished in a lithographed form in 1871. There, 162 algebras of dimension less than or equal to 6 were determined. In doing so he introduced some of the important ideas of ring theory, such as the notion of idempotent and nilpotent el ement and the use of idempotents to obtain a decomposition of a given algebra. This paper was published definitively in the American Journal of Mathematics in 1881 (which was then being edited by J.J.Sylvester) with notes and addenda by his son, Charles Sanders Peirce. It might be worth no- ticing that in these addenda, the concept of regular rep- resentation and a new proof of Frobenius' theorem (showing

that the only finite dimensional associative division alge
bras over the reals are those of complex numbers and qua-
ternions) were given.

Following the work of S. Lie and W. Killing
in the study of Lie groups and algebras, E. Study and G.
Scheffers introduced some basic notions for the structure
theory of algebras in the period 1889-1898, though working
in the non-associative case.

Finally, T. Molien (1891-1941)in [22]   and
E.Cartan (1869 - 1951) in [6]   obtained   impor -
tant results regarding the structure of finite-dimensional
real and complex algebras, introducing the notions of simple
and semisimple algebras and characterizing simple algebras
as complete matrix algebras.

All this work culminated in [25]with   the
beautiful results by J.H.M.Wedderburn (1882-1848)describing
the structure of finite dimensional algebras over arbitrary
fields. In the proof of these, techniques related to idem-
potent elements, as suggested by the earlier work of B.
Peirce, were used.

Now, we briefly turn our attention to ma-
trices. The theory of determinants began in the   XVIIIth
Century, with the works of Mac Laurin (1729) and   Cramer
(1750), in connection with the resolution of linear systems
of equations, thus preceeding the explicit formulation of

the notion of matrix.

It was again Arthur Cayley who first defined matrices in a paper in the *Jour. für Math.* in 1855 as a convenient notation to express linear systems and quadratic forms. As such, only the product of matrices was of interest at first, and Cayley arrived to this concept by considering the effect of two successive linear transformations. In a subsequent paper in the *Phil.Trans. R. Soc. London*, in 1858 he defined addition and multiplication by scalars, studying the properties of these operations without explicit mention of connections with Hypercomplex Systems. Actually, it was in this last paper that Cayley announced the now called theorem of Cayley-Hamilton and stated that he had verified if for 3x3 matrices and that further proof was not neces - sary (1).

However, he did observe a connection with quaternions. He mentioned that if M and N are 2x2 matrices verifying $M^2 = N^2 = -I$ and $MN = -NM$ then, setting $L = MN$, the matrices M,N and L satisfy *"a system of relations precisely similar to that in the theory of quaternions"*.

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(1) Because of these papers, Cayley is generally regarded as the founder of matrix theory. For views contesting his importance in the creation and developement of the theory see [17] and [18].

This remark was to arise the interest of
J.J.Sylvester (1814-1897) on matrices. The results   in
one of C.S.Peirce's addenda to his father's paper   sug-
gested to Sylvester, in 1884 *"the method by which   a
matrix is robbed of its areal dimensions and represented
as a linear sum"*. He was refering, of course, to the now
well-known fact that a total nxn matrix algebra can   be
viewed as an n^2-dimensional vector space.By   defining
E_{ij} as a matrix with 1 in the i,j entry and 0 in   all
the others, a matrix $A=(a_{ij})$ can be written in the form
$A=\Sigma_{ij}\ a_{ij}\ E_{ij}$. Sylvester included a discussion on   how
2x2 and 3x3 matrices can be viewed as a linear combina-
tion of the corresponding $\{E_{ij}\}$ basis (he did not   use
this notation, though, which was introduced apparently
by E. Study in 1889).

It should be noticed that this establishes
a clear connection between matrix algebras and   hyper -
complex systems; matrices form such a system,with units
$\{E_{ij}\}$,whose structural constant are determined by   the
rules giving the product of these elements: $E_{ij}\ E_{kl}$ $=$
$\delta_{jk}\ E_{il}$.

Finally, let us see how this developement
led again, in a natural way, to group rings. The   work
on the subject was undertaken by Theodor Molien,   as   a
setting for applying some of his earlier ideas.

Molien, who was born in Riga, Latvia, studied at the University of Yurev in Estonia, where he submitted his thesis in 1892. These results were published in 1893 in the *Math. Annalen.*

To obtain a structure theory for hypercomplex systems, he considered such a system H with basis $\{e_i\}_{1 \leq i \leq n}$ and structural constants $a_k(i,j)$. Given two elements $x = \Sigma x_i e_i$, $y = \Sigma y_i e_i$, their product can be expressed in the form $xy = \Sigma c_k e_k$ where

$$c_k = \Sigma_{ij} \, a_k(i,j) \, x_i \, y_j.$$

His approach was to study the bilinear forms c_k, $1 \leq i \leq n$, which can be regarded as defining the product in H. He also considered the regular representation of the hypercomplex system; i.e., to each element $u \in H$ he assigned a linear function $R_u : H \to H$ defined by $R_u(x) = xu$.

Molien used this concept to obtain a necessary and sufficient condition for semisimplicity: he showed that H is semisimple if and only if the bilinear form $\psi(x,y) = \text{trace}(R_{xy})$ is non-singular.

Since the structural constants are particularly simple in the case where the basis $\{e_i\}$ forms a group, the attempts to apply his semisimplicity criterion naturally led Molien to consider group rings.

In this way, he was led to important results *"relating to the representability of a given discrete group in the form of a homogeneus linear substitution group"* in two papers which he published in 1897. He was able to conclude that the group ring was a direct sum of complete matrix algebra thus showing that *"a given substitution group can be decomposed into its irreducible components"* and proposed to study *"only the properties of irreducible groups"*. As a consequence, Molien obtained some of the basic theorems in group representation theory, including the orthogonality relations for characters.

Results in group representation theory were obtained independently by several other authors. G.Frobenius (1846-1917) studied group determinants and group characters around 1896 and presented his first paper on matrix representations in 1897 at a meeting of the Berlin Academy.W. Burnside (1852-1927) starded to publish his work on the subject in 1898 and H. Maschke (1853-1908) obtained his result on complete reducibility in 1898.

For more details on these discoveries, the reader should consult [14], [15] and [16].

Somehow, the approaches by Frobenius and Burnside became better known than that of Molien. However, it should be noted that Frobenius himself refers to Molien in [10] and [11]where he introduces Frobenius algebras as a generalization of group algebras.

V. - FINAL REMARKS - Group rings earned a definitive status after the connection between group representation theory and the structure theory of rings and algebras was widely recognized. This was mainly due to a most influencial paper by Emmy Nöther (1882-1935) which, as we mentioned in the introduction, is frequently quoted as the first work on the area.

This paper was of central importance in the development of the whole subject. In this regard, Bourbaki [2, p.156] states: "...*because of the importance of the ideas that are introduced and the lucidity of the exposition, it deserves to appear, together with Stenitz' memoir on commutative fields, as one of the pillars of modern linear algebra*". At this point, it might be interesting to recall that B.L.van der Waerden's famous book [24] is generally regarded as a landmark in the birth of modern algebra as it is undertood today. He also had a direct participation in Noether's paper. In fact, in her first footnote she states: "*This is a free elaboration of my winter semester course of 1927/28 done by B.L.van der Waerden. We wrote together this work to be edited. I should also aknowledge B.L. van der Waerden for a series of critical remarks*".

In the introduction to her paper, Noether recalls that hypercomplex systems had received an arithmetic treatment from Wedderburn and that representation theory was developed on an elementary basis by Burnside and Schur,

independently from hypercomplex systems, by considering
directly a given representation and working with theo-
rems on matrices. She proceeds, then, to present a uni
fied approach to the subject.

The article is divided in four chapters:
"Foundations of Group Theory", "Non commutative Ideal
Theory", " Modules and Representation Theory" and "Group
Representations and Hypercomplex Systems".

We do not intend to give a detail account
of the contents of this paper here. However,we would
like to mention that, in section §6, she defines an
Hypercomplex System as *"a ring R which is at the same
time a right module over a commutative field K (also
known in the specialized literature as an "algebra over
K") such that:*

*1) It is of finite rank (has a linearly
independent basis $u_1, \ldots, u_n$)*

*2) $ab.x=a.bx=ax.b$. This is expressed as
follows: K is commutatively linked to R.*

*3) The unit element ε of K is also the
identity operator: $a^\varepsilon = a$ for a in R".*

This section ends with the following re-
mark: *"An example of hypercomplex system is the group
ring of a finite group which is obtained when we take
the elements of a finite group as elements of a basis
u_i and we admit as a multiplication the group multipli
cation. In this case K is an arbitrary field."*   The

concept is used again in chapter IV where the connection between representations of group and hypercomplex Systems is established.

It should be noted that Richard Brauer (1901-1977) also played an important part in this stage of the theory, as is stated by W. Feit in [9]:

"In his years at Königsberg his mathemati_cal interests were centered on the theory of representations of groups and also in the structure of algebras.The intimate connection between these two subjects had only recently been recognized. This was at least partly due to his joint paper with E. Noether [5]".

Perhaps we should mention that [5] was actually published before [23]. There, the close relationship between splitting fields and maximal subfields of a simple algebra is studied. The paper starts with a reference to I. Schur's earlier results on the subject and gathers results, without proofs, that had been obtained independently by both authors.

As it is mentioned in the second footnote to this paper, E. Noether had already obtained her construction of representation theory based on the theory of ideals and modules. Further reference to this paper was done by R. Brauer in [4] where he again credits E.Noether for a new approach to the subject.

In section §1 of [5] group rings over a field P are explicitely defined as formed *by all " group numbers" (grouppenzahlen); i.e., all linear combinations of the elements of the group with coefficients in P, where group elements are considered as linearly independent and multiplication is defined through the group multiplica - tion and arithmetic rules.*

Since then, group rings have been an im- portant tool in representation theory and were used in other branches of mathematics such as Homology, Cohomology and Algebraic Topology. More recently, they have also been considered as interesting algebraic objects in their own right and have been the subject of active research.

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