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Strongly stratifying ideals, Morita contexts and Hochschild homology ☆

Claude Cibils [a,*], Marcelo Lanzilotta [b], Eduardo N. Marcos [c], Andrea Solotar [d]

[a] *Institut Montpelliérain Alexander Grothendieck, CNRS, Univ. Montpellier, France*
[b] *Instituto de Matemática y Estadística "Rafael Laguardia", Facultad de Ingeniería, Universidad de la República, Uruguay*
[c] *Departamento de Matemática, IME-USP, Universidade de São Paulo, Brazil*
[d] *IMAS-CONICET y Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Argentina*

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We consider stratifying ideals of finite dimensional algebras in relation with Morita contexts. A Morita context is an algebra built on a data consisting of two algebras, two bimodules and two morphisms. For a strongly stratifying Morita context - or equivalently for a strongly stratifying ideal - we show that Han's conjecture holds if and only if it holds for the diagonal subalgebra. The main tool is the Jacobi-Zariski long exact sequence. One of the main consequences is that Han's conjecture holds for an algebra admitting a strongly (co-)stratifying chain whose steps verify Han's conjecture.

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* Corresponding author.

E-mail addresses: Claude.Cibils@umontpellier.fr (C. Cibils), marclan@fing.edu.uy (M. Lanzilotta), enmarcos@ime.usp.br (E.N. Marcos), asolotar@dm.uba.ar (A. Solotar).

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Homology
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Stratifying
Morita context

If Han's conjecture is true for local algebras and an algebra Λ admits a primitive strongly (co-)stratifying chain, then Han's conjecture holds for Λ.

1. Introduction

In this paper we consider finite dimensional associative algebras over a field k, which we call *algebras* for short. We study Morita contexts in relation with Han's conjecture, more details are given below. The intention of this paper is to reduce the analysis of Han's conjecture on an algebra to an easier algebra where Hochschild homology is more manageable.

A Morita context [8,31] is a matrix algebra built on two "diagonal" algebras A and B, two bimodules M and N and two bimodule maps α and β verifying natural conditions equivalent to associativity of the product of the matrix algebra - see Definition 2.1.

One of the main results that this paper relies on is that algebras with a distinguished idempotent are essentially the same that Morita contexts, see for instance [14,24]. For later use, we start Section 2 by recalling this well known fact in a categorical framework.

Stratifying ideals are defined by E. Cline, B. Parshall and L. Scott in [22] and further studied in [5,26,27]. We show that for Morita contexts the definition of a stratifying ideal of an algebra generated by an idempotent translates into the conditions $\mathsf{Tor}_n^A(M,N) = 0$ for $n > 0$ and β injective. See [36, Proposition 4.1].

Recollements are introduced by A. A. Beilinson, J. Bernstein and P. Deligne in [9]. Let $\Lambda e \Lambda$ be a stratifying ideal of an algebra Λ, where e is an idempotent of Λ. This gives us a recollement of the unbounded derived category $D(\Lambda)$ of complexes of Λ-modules relative to the unbounded derived categories $D(\Lambda/\Lambda e\Lambda)$ and $D(e\Lambda e)$. See [22,26].

In Section 2 we also introduce strongly stratifying ideals generated by an idempotent. For Morita contexts, this corresponds to $\mathsf{Tor}_n^A(M,N) = 0$ for all n, and $\mathsf{Tor}_n^B(N,M) = 0$ for $n > 0$.

The Hochschild homology $HH_*(\Lambda)$ of an algebra Λ (see [28], [40], [41]) is called *finite* if $HH_*(\Lambda) = 0$ for $* > N$ for some N. A main purpose of this paper is to study under which circumstances the Hochschild homology of a Morita context being finite implies the Hochschild homology of its diagonal algebra is also finite. Our motivation is to attack Han's conjecture [25], which states that if an algebra has finite Hochschild homology, then it should have finite global dimension. For results in this direction, see for instance [7,10–13,17,19,21,38,39]. Note that if Han's conjecture is true, then the following dichotomy holds: either $HH_*(\Lambda)$ is infinite or $HH_*(\Lambda) = 0$ for $* > 0$. See [25].

Previous results by L. Angeleri Hügel, S. Koenig, Q. Liu and D. Yang in [4] provide a motivation for approaching Han's conjecture through recollements. Roughly, the finiteness of the projective dimension is preserved for an idempotent which gives a stratifying ideal, through the recollement of the derived category. Moreover B. Keller in [33] and

Y. Han in [26] showed that in the same situation there is a long exact sequence relating the Hochschild homologies of the algebras involved.

In Section 3 we adjust and extend the Jacobi-Zariski long nearly exact sequence obtained in [18]. The additional hypothesis for fitting Theorem 4.2 of [18] is given in (3.4). This sequence links Hochschild homology of Λ to the relative one with respect to a subalgebra introduced by G. Hochschild in [29]. With this adjustement, we confirm our previous results in [18] and the earlier Jacobi-Zariski long exact sequence of A. Kaygun in [32].

Let Λ be an algebra with a distinguished idempotent e satisfying that $\Lambda e\Lambda$ is a strongly stratifying ideal, and let $f = 1 - e$. Using the previously described tools, in Section 4 we obtain the key result of this paper, that is Theorem 4.6: if Λ has finite Hochschild homology, then the same holds for $e\Lambda e$ and $f\Lambda f$. In Section 5, we prove our main result: Λ verifies Han's conjecture if and only if its subalgebra $e\Lambda e \times f\Lambda f$ does.

We next consider algebras admitting a strongly stratifying chain, that is those algebras with an ordered complete system of orthogonal idempotents such that the successive quotients of the induced filtration by ideals are strongly stratifying in the corresponding algebra. We obtain the following interesting consequence of our key result Theorem 4.6. Let $\mathcal{C}$ be a class of algebras verifying Han's conjecture which is closed by taking quotients. If an algebra Λ admits a strongly stratifying chain $\{e_1, \ldots, e_n\}$ such that all the algebras $e_i\Lambda e_i$ belong to $\mathcal{C}$, then Han's conjecture is true for Λ.

To avoid classes of algebras closed by taking quotients, we filter instead an algebra Λ by algebras $f\Lambda f$ where the f's are partial decreasing sums of a complete system of orthogonal idempotents $\{e_1, \ldots, e_n\}$. We consider strongly co-stratifying chains, that is the f's provide ideals which are strongly stratifying in the next algebra. We infer from Theorem 4.6 another main result: if an algebra Λ admits a strongly co-stratifying chain $\{e_1, \ldots, e_n\}$ such that all the algebras $e_i\Lambda e_i$ verify Han's conjecture, then Han's conjecture is true for Λ.

In particular if Han's conjecture is true for local algebras and if an algebra admits a primitive strongly (co-)stratifying chain, then Han's conjecture is true for this algebra.

Those algebras admitting a strongly stratifying or co-stratifying chain will be compared with standardly stratified algebras (see for instance [1,2,37,42]) in a forthcoming paper.

In the last section, assuming *ad-hoc* projectivity conditions on the bimodules of a Morita context, we give patterns to produce examples.

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2. Stratifying and strongly stratifying Morita contexts

Let k be a field. As mentioned, a finite dimensional associative k-algebra is called an *algebra* throughout this paper. We first recall the definition of Morita context and show that it is the same that an algebra with a distinguished idempotent e. Then we specialize to the case where the two-sided ideal generated by e is stratifying, in order to obtain a stratifying Morita context.

As E.L. Green and C. Psaroudakis pointed out in [24], Morita contexts have been introduced by H. Bass [8] in 1962, and considered by P. M. Cohn [23] in 1996.

Definition 2.1. A *Morita context* is a matrix algebra $\begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ where A and B are algebras, M and N are finite dimensional $B-A$ and $A-B$-bimodules respectively, and α and β are A and B-bimodule maps respectively

$$\alpha : N \otimes_B M \to A \text{ and } \beta : M \otimes_A N \to B$$

which verify "associativity" conditions

$$\alpha(n \otimes m)n' = n\beta(m \otimes n') \text{ and } \beta(m \otimes n)m' = m\alpha(n \otimes m'). \tag{2.1}$$

The product of the Morita context is the matrix product using the above, namely

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix}\begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' + \alpha(n \otimes m') & an' + nb' \\ ma' + bm' & \beta(m \otimes n') + bb' \end{pmatrix}.$$

This product is associative if and only if the "associativity" conditions on α and β hold.

For completeness, we recall the well known fact that Morita contexts are in bijection with k-categories with an ordered pair of objects and morphisms are k-vector spaces. Indeed, starting with a Morita context, the associated category has A (resp. B) as endomorphism algebra of the first (resp. second) object; morphisms from the first (resp. second) object to the second (resp. first) object are M (resp. N); finally compositions of those morphisms are given by α and β, their "associativity" conditions ensure that the composition is associative. Conversely, given a k-category with an ordered pair of objects, the Morita context has the algebras of endomorphisms of the objects on its diagonal. On the antidiagonal, the bimodules are the morphisms between the objects - which are indeed bimodules over the previous algebras of endomorphisms. The maps α and β are provided by the composition of the category.

Definition 2.2. The objects of the category $\mathsf{Morita.Contexts}$ are the Morita contexts. A morphism of Morita contexts

$$\begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta} \to \begin{pmatrix} A' & N' \\ M' & B' \end{pmatrix}_{\alpha',\beta'}$$

is a quadruple of maps

$$(A \xrightarrow{\varphi} A', B \xrightarrow{\psi} B', M \xrightarrow{f} M', N \xrightarrow{g} N'),$$

where φ and ψ are algebra maps - they provide M' and N' with respective structures of $B-A$ and $A-B$-bimodule. Moreover f and g are respectively $B-A$ and $A-B$-bimodule maps. In addition, these maps verify the following conditions:

$$\varphi(\alpha(n \otimes m)) = \alpha'(g(n) \otimes f(m)) \text{ and } \psi(\beta(m \otimes n)) = \beta'(f(m) \otimes g(n)).$$

The direct sum of the four maps of a morphism is an algebra map.

Note that a morphism between Morita contexts is equivalent to a functor between the corresponding categories with an ordered pair of objects, which respects the ordered pairs.

On the other hand the category of k-algebras with an idempotent is as follows.

Definition 2.3. The objects of the category Algebras.Idempotent are pairs (Λ, e) where Λ is a k-algebra and e is a *distinguished* idempotent of Λ. A morphism $\varphi : (\Lambda, e) \to (\Lambda', e')$ is a morphism of algebras $\varphi : \Lambda \to \Lambda'$ such that $\varphi(e) = e'$.

The following result is well known.

Theorem 2.4. *The categories* Morita.Contexts *and* Algebras.Idempotent *are isomorphic.*

Proof. Let $\begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ be a Morita context. The associated object in Algebras.Idempotent is the Morita context with distinguished idempotent $\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$. Starting from a morphism of Morita contexts, that is a quadruple of appropriate maps, their direct sum clearly preserves the distinguished idempotents.

Conversely, let (Λ, e) be an object in Algebras.Idempotent and consider the idempotent $f = 1 - e$. Then $\begin{pmatrix} e\Lambda e & e\Lambda f \\ f\Lambda e & f\Lambda f \end{pmatrix}_{\alpha,\beta}$ is a Morita context where

$$\alpha : e\Lambda f \otimes_{f\Lambda f} f\Lambda e \to e\Lambda e \text{ and } \beta : f\Lambda e \otimes_{e\Lambda e} e\Lambda f \to f\Lambda f$$

are given by the product of Λ. Observe that a morphism $\varphi : (\Lambda, e) \to (\Lambda', e')$ also verifies $\varphi(f) = f'$, where $f' = 1 - e'$. Therefore, a morphism of algebras with distinguished idempotents provides a morphism of the corresponding Morita contexts. These functors are mutual inverses. □

Remark 2.5. The previous Theorem 2.4 generalizes to an algebra with a finite complete set of orthogonal idempotents (non necessarily primitive), see for instance [16,20].

In 1996 E. Cline, B. Parshall and L. Scott [22] considered a stratifying ideal generated by an idempotent of an algebra, we next recall its definition.

Definition 2.6. [22] Let Λ be an algebra and $e \in \Lambda$ an idempotent. The ideal $\Lambda e\Lambda$ is stratifying if

1. $\mathsf{Tor}_n^{e\Lambda e}(\Lambda e, e\Lambda) = 0$ for $n > 0$,
2. The surjection given by the product $\Lambda e \otimes_{e\Lambda e} e\Lambda \to \Lambda e\Lambda$ is injective.

Remark 2.7. ([22], [26, Example 1, p.537]) Let Λ be an algebra and let $D(\Lambda)$ denote the unbounded derived category of complexes of left Λ-modules. Let $e \in \Lambda$ be an idempotent such that $\Lambda e\Lambda$ is a stratifying ideal. Then $D(\Lambda)$ admits a recollement relative to $D(\Lambda/\Lambda e\Lambda)$ and $D(e\Lambda e)$, which is interpreted as a short exact sequence of triangulated categories, as introduced by A. A. Beilinson, J. Bernstein and P. Deligne [9] in 1982.

In 2009 S. Koenig and H. Nagase [34, p. 888] showed that an idempotent $e \in \Lambda$ gives a stratifying ideal if and only if the canonical surjection $\Lambda \to \Lambda/\Lambda e\Lambda$ induces isomorphisms

$$\mathsf{Ext}^*_{\Lambda/\Lambda e\Lambda}(X, Y) \to \mathsf{Ext}^*_\Lambda(X, Y)$$

for all $\Lambda/\Lambda e\Lambda$-modules X and Y.

It is worth noting that the above is precisely the definition of a *strong* idempotent ideal given in 1992 by M. Auslander, M.I. Platzeck and G. Todorov, see [6, p. 669]. They are not to be confused with "strongly stratifying" ideals that we will consider later.

The following definition will allow to consider stratifying ideals in the framework of Morita contexts.

Definition 2.8. A Morita context $\begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ is *stratifying* if

1. $\mathsf{Tor}_n^A(M, N) = 0$ for $n > 0$,
2. β is injective.

Both stratifying definitions agree through the identification of Theorem 2.4 between algebras with distinguished idempotents and Morita contexts:

Theorem 2.9. *[36, Proposition 4.1] Let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ be a Morita context and let e be the idempotent $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The Morita context is stratifying if and only the ideal $\Lambda e\Lambda$ is stratifying.*

Proof. Let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ and $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We have

$$\Lambda e = A \oplus M \qquad e\Lambda = A \oplus N$$

respectively as right and left A-modules because $e\Lambda e = A$. Therefore

$$\mathsf{Tor}_n^{e\Lambda e}(\Lambda e, e\Lambda) = \mathsf{Tor}_n^A(A, A) \oplus \mathsf{Tor}_n^A(A, N) \oplus \mathsf{Tor}_n^A(M, A) \oplus \mathsf{Tor}_n^A(M, N).$$

The three first direct summands on the right side of the equality are 0 for $n > 0$. This shows the equivalence on the Tor conditions.

Then note that

$$\Lambda e\Lambda = \begin{pmatrix} A & N \\ M & \mathsf{Im}\beta \end{pmatrix}.$$

Moreover the morphism given by the product $\Lambda e \otimes_{e\Lambda e} e\Lambda \to \Lambda e\Lambda$ decomposes diagonally as the direct sum of four morphisms

$$A \otimes_A A \to A, \quad A \otimes_A N \to N, \quad M \otimes_A A \to M \quad \text{and} \quad M \otimes_A N \to \mathsf{Im}\beta.$$

The first three are clearly isomorphisms, while the last one provides the equivalence on the injectivity conditions. □

Remark 2.10. From the proof of the previous Theorem 2.9 we have

$$\Lambda/\Lambda e\Lambda = B/\mathsf{Im}\beta.$$

We will consider strongly stratifying ideals as follows.

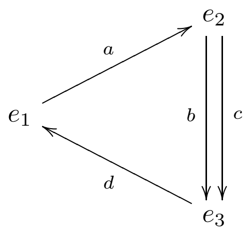
Definition 2.11. Let (Λ, e) be an algebra with a distinguished idempotent e, and let $f = 1 - e$. The ideal $\Lambda e\Lambda$ is *strongly stratifying* if

1. $\mathsf{Tor}_n^{e\Lambda e}(\Lambda e, e\Lambda) = 0$ for $n > 0$,
2. $f\Lambda e \otimes_{e\Lambda e} e\Lambda f = 0$,
3. $\mathsf{Tor}_n^{f\Lambda f}(\Lambda f, f\Lambda) = 0$ for $n > 0$.

Remark 2.12.

- In Proposition 2.17 we will prove that if $\Lambda e\Lambda$ is strongly stratifying, then $\Lambda e\Lambda$ is indeed stratifying.
- The last requirement of Definition 2.11 for $\Lambda e\Lambda$ to be a strongly stratifying ideal coincides with the first requirement of the definition for $\Lambda f\Lambda$ to be a stratifying ideal (see Definition 2.6).

Example 2.13. We consider the example of [35, Example 4.4] and [5, Example 2.3]. Let Λ be the following bound quiver algebra



$$ba = 0, \ ad = 0, \ dc = 0.$$

As proved in [35], the idempotent $e = e_2 + e_3$ provides a stratifying ideal $\Lambda e \Lambda$. We assert that $\Lambda e \Lambda$ is a strongly stratifying ideal.

The algebra $A = e\Lambda e$ is the Kronecker algebra $\begin{array}{c} e_2 \\ {\scriptstyle b}\downarrow\downarrow{\scriptstyle c} \\ e_3 \end{array}$.

Let $f = e_1 = 1 - e$. Consider the right A-module $M = f\Lambda e$ and the left A-module $N = e\Lambda f$. Their associated quiver representations are

$$\begin{array}{c} Me_2 = k\{db\} \\ {\scriptstyle 1}\uparrow\uparrow{\scriptstyle 0} \\ Me_3 = k\{d\} \end{array} \qquad\qquad \begin{array}{c} e_2N = k\{a\} \\ {\scriptstyle 0}\downarrow\downarrow{\scriptstyle 1} \\ e_3N = k\{ca\} \end{array}$$

We assert that $M \otimes_A N = 0$. Indeed,

$$db \otimes a = d \otimes ba = d \otimes 0 = 0, \quad db \otimes ca = dbe_2 \otimes ca = db \otimes e_2ca = db \otimes 0 = 0$$
$$d \otimes a = de_3 \otimes a = d \otimes e_3a = d \otimes 0 = 0, \quad d \otimes ca = dc \otimes a = 0 \otimes a = 0.$$

Moreover $B = f\Lambda f = k$, thus the last requirement of Definition 2.11 holds. For the sake of completeness we next check that $\mathsf{Tor}_n^A(M, N) = 0$ for $n > 0$, after [35]. Consider the projective left A-modules

$$P_1 = \begin{array}{c} 0 \\ \downarrow\downarrow \\ k\{e_3\} \end{array} \qquad\qquad P_0 = \begin{array}{c} k\{e_1\} \\ {\scriptstyle \binom{1}{0}}\downarrow\downarrow{\scriptstyle \binom{0}{1}} \\ k\{b\} \oplus k\{c\} \end{array}$$

and the projective resolution of N

$$0 \to P_1 \to P_0 \to N \to 0$$

where the map $P_1 \to P_0$ sends e_3 to b. It is straightforward to verify that $M \otimes_A P_1 = k\{d \otimes e_3\}$ and $M \otimes_A P_0 = k\{d \otimes b\}$. Hence the complex computing $\mathsf{Tor}_n^A(M, N)$ for $n \geq 0$ is

$$0 \to M \otimes_A P_1 \to M \otimes_A P_0 \to 0$$

which is exact. Note that this gives another proof that $M \otimes_A N = 0$.

Definition 2.14. A Morita context $\begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ is *strongly stratifying* if

1. $\mathsf{Tor}_n^A(M, N) = 0$ for $n > 0$,
2. $M \otimes_A N = 0$,
3. $\mathsf{Tor}_n^B(N, M) = 0$ for $n > 0$.

Remark 2.15. The morphism β of a strongly stratifying Morita context is $\beta : 0 \to B$ which is of course injective. Therefore strongly stratifying Morita contexts are indeed stratifying.

Proposition 2.16. *Let Λ be an algebra with a distinguished idempotent e, and let $f = 1-e$. The associated Morita context $\begin{pmatrix} e\Lambda e & e\Lambda f \\ f\Lambda e & f\Lambda f \end{pmatrix}_{\alpha,\beta}$ is strongly stratifying if and only if the ideal $\Lambda e\Lambda$ is strongly stratifying.*

Proof. We have

$$\begin{aligned}\mathsf{Tor}_n^{e\Lambda e}(\Lambda e, e\Lambda) =& \mathsf{Tor}_n^{e\Lambda e}(e\Lambda e, e\Lambda e) \ \oplus \ \mathsf{Tor}_n^{e\Lambda e}(e\Lambda e, e\Lambda f) \oplus \\ & \mathsf{Tor}_n^{e\Lambda e}(f\Lambda e, e\Lambda e) \ \oplus \ \mathsf{Tor}_n^{e\Lambda e}(f\Lambda e, e\Lambda f).\end{aligned}$$

Of course $e\Lambda e$ is a projective left $e\Lambda e$-module, it is also projective as a right $e\Lambda e$-module. Thus for $n > 0$

$$\mathsf{Tor}_n^{e\Lambda e}(\Lambda e, e\Lambda) = \mathsf{Tor}_n^{e\Lambda e}(f\Lambda e, e\Lambda f).$$

Moreover for $n = 0$ the second condition in both definitions is $f\Lambda e \otimes_{e\Lambda e} e\Lambda f = 0$.

Analogously, for $n > 0$ we have

$$\mathsf{Tor}_n^{f\Lambda f}(\Lambda f, f\Lambda) = \mathsf{Tor}_n^{f\Lambda f}(e\Lambda f, f\Lambda e). \quad \square$$

Proposition 2.17. *Let Λ be an algebra and $e \in \Lambda$ an idempotent. If the ideal $\Lambda e\Lambda$ is strongly stratifying, then it is stratifying.*

Proof. By Proposition 2.16, the Morita context is strongly stratifying. Thus the Morita context is stratifying by Remark 2.15. Finally the ideal $\Lambda e\Lambda$ is stratifying by Theorem 2.9. $\square$

3. Jacobi-Zariski long nearly exact sequence and bounded extensions of algebras

We begin this section with a brief account of relative Hochschild homology with respect to a subalgebra, and not with respect to an ideal, as it is often considered. Next, we will provide the adjusted and extended version of the Jacobi-Zariski long near exact sequence of in [18], then we confirm the results of [19].

Let $C \subset \Lambda$ be an extension of algebras, that is C is a subalgebra of Λ. The relative projective Λ-modules are direct summands of $\Lambda \otimes_C V$, where V is any left C-module. Note that if $C = k$, the relative projectives are the usual Λ-projective modules.

Recall that a relative projective resolution of a Λ-module U requires the existence of a C-contracting homotopy, by [29, p. 250] it always exits. For a right Λ-module U', this leads to well defined vector spaces $\mathsf{Tor}_*^{\Lambda|C}(U', U)$.

Let C^{e} be the enveloping algebra $C \otimes_k C^{\mathsf{op}}$ of an algebra, and consider the extension of algebras $C^{\mathsf{e}} \subset \Lambda^{\mathsf{e}}$. The relative Hochschild homology $H_*(\Lambda|C, X)$ of a Λ-bimodule X is defined as $\mathsf{Tor}_*^{\Lambda^{\mathsf{e}}|C^{\mathsf{e}}}(X, \Lambda)$. Note that G. Hochschild considered the extension $C \otimes \Lambda^{\mathsf{op}} \subset \Lambda^{\mathsf{e}}$ in [29], but the Tor vector spaces obtained this way are the same, see for instance [17].

If $C = k$, relative Hochschild homology is Hochschild homology as defined in [28], which is denoted $H_*(\Lambda, X)$. If $X = \Lambda$ the usual notation is $HH_*(\Lambda)$.

The setting of [18] is as follows.

Definition 3.1. A sequence of positively graded chain complexes of vector spaces

$$0 \to C_* \xrightarrow{\iota} D_* \xrightarrow{\kappa} E_* \to 0 \tag{3.1}$$

with ι injective, κ surjective and $\kappa\iota = 0$ is called *short nearly exact*. The middle quotient complex $(\mathsf{Ker}\kappa/\mathsf{Im}\iota)_*$ is called the *gap complex*.

Remark 3.2. Consider the double complex with columns at $p = 0, 1$ and 2 given by the short nearly exact sequence (3.1) after the usual change of signs. Recall that the horizontal maps go from right to left, since the double complex is of homological type.

By filtering the double complex by rows we have a spectral sequence. At page 1 the only possible non zero column is column 1, which is the gap complex $(Ker\kappa/Im\iota)_*$. Therefore the spectral sequence converges to $H_*\left(\mathsf{Ker}\kappa/\mathsf{Im}\iota\right)$.

Definition 3.3. A *long nearly exact sequence* is a complex of vector spaces

$$\begin{aligned} \dots &\xrightarrow{\delta} U_m \xrightarrow{I} V_m \xrightarrow{K} W_m \xrightarrow{\delta} U_{m-1} \xrightarrow{I} V_{m-1} \to \dots \\ &\xrightarrow{\delta} U_n \xrightarrow{I} V_n \xrightarrow{K} W_n \end{aligned}$$

ending at some n which is exact at W_m for all $m > n$ and at all U_m. The graded vector space $(KerK/ImI)_*$ is called the *gap* of the long nearly exact sequence.

Next we adjust and we extend Theorem 4.2 of [18]. Part a) is new, while part b) requires the additional hypothesis (3.4) which is missing in [18]. It is worth noting that Jonathan Lindell wrote to us raising a question about the results in [18], just after we realized that Theorem 4.2 of [18] needs this hypothesis.

Theorem 3.4. *Consider a short nearly exact sequence of positively graded chain complexes*

$$0 \to C_* \xrightarrow{\iota} D_* \xrightarrow{\kappa} E_* \to 0. \tag{3.2}$$

a) *If $H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota) = 0$ for $* >> 0$, then there is long exact sequence*

$$\begin{aligned} \ldots \xrightarrow{\delta} H_m(C_*) \xrightarrow{I} H_m(D_*) \xrightarrow{K} H_m(E_*) \xrightarrow{\delta} H_{m-1}(C_*) \xrightarrow{I} \ldots \\ \cdots \xrightarrow{\delta} H_n(C_*) \xrightarrow{I} H_n(D_*) \xrightarrow{K} H_n(E_*) \end{aligned} \tag{3.3}$$

ending at some n.

b) *Let I and K be the maps induced in homology by ι and κ respectively. If*

$$\mathsf{dim}_k(\mathsf{Ker}K/\mathsf{Im}I)_* = \mathsf{dim}_k H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota) \text{ for } * >> 0 \tag{3.4}$$

then the sequence (3.3) exists and it is a long nearly exact sequence.

Proof. We now filter the double complex of Remark 3.2 by columns. At page 1 the vertical differentials are 0. The horizontal ones are

$$0 \leftarrow H_m(E_*) \xleftarrow{K} H_m(D_*) \xleftarrow{I} H_m(C_*) \leftarrow 0$$

At page 2 we have the following:

- column 0 is $(\mathsf{Coker}K)_*$,
- column 1 is $(\mathsf{Ker}K/\mathsf{Im}I)_*$,
- column 2 is $(\mathsf{Ker}I)_*$

and all the other columns are zero.

At page 2, the differentials are zero except possibly

$$(E^2_{0,q+1} = \mathsf{Coker}K) \xleftarrow{d_2} (E^2_{2,q} = \mathsf{Ker}I)$$

see for instance [40, p. 122]. Hence at page 3 we have that:

- column 0 is $(\mathsf{Coker}d_2)_*$,
- column 1 is $(\mathsf{Ker}K/\mathsf{Im}I)_*$,
- column 2 is $(\mathsf{Ker}d_2)_*$,

all the other columns are zero and the differentials are zero. Hence these vector spaces remain the same in the following pages. Consequently the spectral sequence converges to

$$(\mathsf{Coker} d_2)_{*+1} \oplus (\mathsf{Ker} K/\mathsf{Im} I)_* \oplus (\mathsf{Ker} d_2)_*.$$

Both filtrations converge to the same limit, hence Remark 3.2 gives

$$\mathsf{dim}_k(\mathsf{Ker} d_2)_* + \mathsf{dim}_k(\mathsf{Ker} K/\mathsf{Im} I)_* + \mathsf{dim}_k(\mathsf{Coker} d_2)_{*+1} = \mathsf{dim}_k H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota). \tag{3.5}$$

For a) we assume that $H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota) = 0$ for $* >> 0$, hence

$$\mathsf{dim}_k(\mathsf{Ker} d_2)_* + \mathsf{dim}_k(\mathsf{Ker} K/\mathsf{Im} I)_* + \mathsf{dim}_k(\mathsf{Coker} d_2)_{*+1} = 0 \text{ for } * >> 0.$$

In particular we have $(\mathsf{Ker} K/\mathsf{Im} I)_* = 0$ for $* >> 0$. In other words the sequence is exact at $H_m(D_*)$ for $m >> 0$.

Moreover, in high enough degrees we have

$$\mathsf{dim}_k(\mathsf{Ker} d_2)_* = 0 = \mathsf{dim}_k(\mathsf{Coker} d_2)_{*+1}$$

which means that d_2 is invertible in high enough degrees.

Let us consider the canonical maps $p : H_m(E_*) \twoheadrightarrow \mathsf{Coker} K$ and $q : \mathsf{Ker} I \hookrightarrow H_{m-1}(C_*)$. Define

$$\delta = q\left(d_2^{-1}\right)p.$$

We do have $\mathsf{Ker}\delta = \mathsf{Im} K$ and $\mathsf{Im}\delta = \mathsf{Ker} I$.

For b), the hypothesis (3.4) implies that d_2 is invertible in high enough degrees. The previous construction provides δ, which gives a long nearly exact sequence which gap is $H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota)$. □

In the following, we confirm the results in [18] after Theorem 3.4.

Let $C \subset \Lambda$ be an extension of algebras. Let X be a Λ-bimodule. By [18, Theorem 3.3] there is a *fundamental nearly exact sequence* for $* > 1$

$$0 \to \check{C}_*(C, X) \xrightarrow{\iota} \check{C}_*(\Lambda, X) \xrightarrow{\kappa} \check{C}_*(\Lambda|C, X) \to 0 \tag{3.6}$$

where the positively graded complexes are defined in [18, Section 2]. The homology of these complexes gives respectively $H_*(C, X)$, $H_*(\Lambda, X)$ and $H_*(\Lambda|C, X)$.

We denote by I and K the maps in Hochschild homology induced by ι and κ respectively. In this context, we reformulate below Theorem 4.4 of [18], which is actually a particular case of the above Theorem 3.4.

Theorem 3.5. *Let $C \subset \Lambda$ be an extension of algebras and let X be a Λ-bimodule. With the above notations, we have*

a) *If $H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota) = 0$ for $* >> 0$, then there is a Jacobi-Zariski sequence*

$$\dots \xrightarrow{\delta} H_m(C,X) \xrightarrow{I} H_m(\Lambda,X) \xrightarrow{K} H_m(\Lambda|C,X) \xrightarrow{\delta} H_{m-1}(C,X) \xrightarrow{I} \dots$$
$$\xrightarrow{\delta} H_n(C,X) \xrightarrow{I} H_n(\Lambda,X) \xrightarrow{K} H_n(\Lambda|C,X) \qquad \text{(JZ)}$$

which is long exact and ends for some n.

b) *If $\mathsf{dim}_k(\mathsf{Ker}K/\mathsf{Im}I)_* = \mathsf{dim}_k H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota)$ for $* >> 0$, then the sequence (JZ) exists and it is a long nearly exact sequence.*

The name Jacobi-Zariski is given after [3, p. 61], [30], see also [18].

Next we approximate the homology of the gap $H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota)$ of the fundamental sequence (3.6). This way we reconsider Theorem 5.1 of [18].

Theorem 3.6. *With the above notations, if $\mathsf{Tor}_*^C(\Lambda/C, (\Lambda/C)^{\otimes_C n}) = 0$ for $* > 0$ and for all n, then there is a spectral sequence converging to $H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota)$ in large enough degrees. Its terms at page 1 are*

$$E^1_{p,q} = \mathsf{Tor}_q^{C^e}(X, (\Lambda/C)^{\otimes_C p}) \quad \textit{for } p,q > 0$$

and 0 anywhere else.

Proof. Theorem 5.1 of [18] intends to approximate the gap of the Jacobi-Zariski sequence (JZ). The proof there focuses on the homology of the gap complex. This focus is now our aim.

Thus the proof of [18, p. 1645, Theorem 5.1] is relevant avoiding its first three lines. □

In the following, we confirm that the previous tools provide an alternative proof of the results of A. Kaygun in [32] as in [18, Theorem 6.2].

Theorem 3.7. *Let $C \subset \Lambda$ be an extension of k-algebras such that Λ/C is a flat C-bimodule, and let X be a Λ-bimodule. There is a Jacobi-Zariski long exact sequence*

$$\dots \xrightarrow{\delta} H_m(C,X) \xrightarrow{I} H_m(\Lambda,X) \xrightarrow{K} H_m(\Lambda|C,X) \xrightarrow{\delta} H_{m-1}(C,X) \xrightarrow{I} \dots$$
$$\xrightarrow{\delta} H_n(C,X) \xrightarrow{I} H_n(\Lambda,X) \xrightarrow{K} H_n(\Lambda|C,X)$$

ending at some n.

Proof. Λ/C is flat as a left and as a right C-module (see for instance the first part of the proof of [18, Lemma 6.1]), hence $\mathsf{Tor}_*^C(\Lambda/C, (\Lambda/C)^{\otimes_C n}) = 0$ for $* > 0$ and for all n.

Now by Theorem 3.6, there is a spectral sequence converging to the homology of the gap of the fundamental sequence (3.6) in large enough degrees. The first page of this spectral sequence is

$$E^1_{p,q} = \mathsf{Tor}_q^{C^{\mathsf{e}}}(X, (\Lambda/C)^{\otimes_C p}) \quad \text{for } p, q > 0$$

and 0 elsewhere. For $p > 0$ we have that the C^{e}-module $(\Lambda/C)^{\otimes_C p}$ is flat, see for instance [18, Lemma 6.1]. If $q > 0$, then $\mathsf{Tor}_q^{C^{\mathsf{e}}}(X, (\Lambda/C)^{\otimes_C p}) = 0$. Consequently the first page of the spectral sequence is 0, so the homology of the gap of the fundamental sequence is 0. Then by Theorem 3.4 a), there exists a long Jacobi-Zariski exact sequence as stated. □

We recall from [18,19] that an extension of algebras $C \subset \Lambda$ is *left (respectively right) bounded* if

- Λ/C is projective as a left (respectively right) C-module,
- Λ/C is tensor nilpotent as a C-bimodule,
- Λ/C is of finite projective dimension as a C-bimodule.

We confirm now [18, Theorem 6.5] - see also [19, Theorem 2.9], by means of the previous results. We underline that we consider an extension of algebras $C \subset \Lambda$ which is not necessarily split, namely it may not exist a two sided ideal I of Λ such that $\Lambda = C \oplus I$.

Theorem 3.8. *With the above notations, assume that the extension is left or right bounded. Then there is a Jacobi-Zariski long exact sequence*

$$\dots \xrightarrow{\delta} H_m(C, X) \xrightarrow{I} H_m(\Lambda, X) \xrightarrow{K} H_m(\Lambda|C, X) \xrightarrow{\delta} H_{m-1}(C, X) \xrightarrow{I} \dots$$
$$\xrightarrow{\delta} H_n(C, X) \xrightarrow{I} H_n(\Lambda, X) \xrightarrow{K} H_n(\Lambda|C, X)$$

ending at some n.

Proof. We have that $\mathsf{Tor}_*^{C}(\Lambda/C, (\Lambda/C)^{\otimes_C n}) = 0$ for $* > 0$ and for all n. Hence by Theorem 3.6 there is a spectral sequence converging to $H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota)$ in large enough degrees. At page 1 we have $E^1_{p,q} = \mathsf{Tor}_q^{C^{\mathsf{e}}}(X, (\Lambda/C)^{\otimes_C p})$ for $p, q > 0$ and 0 otherwise. Let u be the projective dimension of the C-bimodule Λ/C. Then $(\Lambda/C)^{\otimes_C p}$ is of projective dimension at most pu, see [15, Chapter IX, Proposition 2.6].

Let v be such that $(\Lambda/C)^{\otimes_C v} = 0$. Note that if $p \geq v$ or $q > pu$, then $E^1_{p,q} = 0$. Therefore if $p + q \geq v(u + 1)$, then $E^1_{p,q} = 0$. That is the terms of the spectral sequence vanish at page 1 for high enough total degrees. Hence $H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota) = 0$ for $* >> 0$. Then Theorem 3.4 part a) provides the Jacobi-Zariski long exact sequence. □

Remark 3.9. Consider a Morita context $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ and let $C = A \times B$ be its diagonal subalgebra.

- It may happen that $(\Lambda, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})$ is strongly stratifying but $C \subset \Lambda$ is not a bounded extension. Nevertheless in this case Λ/C is tensor nilpotent, as we will see in the next section.
- If the extension $C \subset \Lambda$ is bounded, then $M \otimes_A N = 0$ if and only if the Morita context is strongly stratifying.

4. Hochschild homology of strongly stratifying Morita contexts

The main purpose of this section is to prove that if a strongly stratifying Morita context $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ has finite Hochschild homology, then the same holds for its diagonal subalgebra $C = A \times B$ and consequently for each diagonal algebra A and B.

We first recall some easy to show facts that we will use.

(F1) A left C-module X is the direct sum of a left A-module ${}_aX$ and a left B-module ${}_bX$, where ${}_aX = (1,0)X$ and ${}_bX = (0,1)X$.
Conversely, if ${}_aX$ and ${}_bX$ are left A and B-modules respectively, then ${}_aX \oplus {}_bX$ is a left C-module. Note that B and A annihilate respectively ${}_aX$, and ${}_bX = 0$.

(F2) In particular a left A-module U becomes a left $A \times B$-module through $U \oplus 0$, with $BU = 0$.

(F3) Let U (resp. V) be a right A (resp. left B)-module, viewed as a right (resp. left) C-module. We have $U \otimes_C V = 0$.

(F4) Let Y (resp. X) be a right (resp. left) C-module and $Y = Y_a \oplus Y_b$ (resp. $X = {}_aX \oplus {}_bX$) be the decomposition as above. We have

$$\mathsf{Tor}_*^C(Y_a \oplus Y_b, {}_aX \oplus {}_bX) = \mathsf{Tor}_*^A(Y_a, {}_aX) \oplus \mathsf{Tor}_*^B(Y_b, {}_bX).$$

Indeed a left C-projective resolution of ${}_aX \oplus {}_bX$ is given by the direct sum of a left A-projective resolution of ${}_aX$ and a left B-projective resolution of ${}_bX$. Using (F3) we infer the result.

(F5) As $C = A \times B$,

$$C^{\mathsf{e}} = A^{\mathsf{e}} \times (A \otimes B^{\mathsf{op}}) \times (B \otimes A^{\mathsf{op}}) \times B^{\mathsf{e}}.$$

Let X be a C-bimodule. We have

$$X = {}_aX_a \oplus {}_aX_b \ \oplus {}_bX_a \oplus {}_bX_b$$

where ${}_aX_a$, ${}_aX_b$, ${}_bX_a$ and ${}_bX_b$ are respectively an A-bimodule, an $A-B$-bimodule, a $B-A$-bimodule and a B-bimodule.

(F6) Let Y and X be C-bimodules decomposed as above. After (F4) we have

$$\mathsf{Tor}_*^{C^{\mathsf{e}}}(Y, X) = \mathsf{Tor}_*^{A^{\mathsf{e}}}({}_aY_a, {}_aX_a) \oplus \mathsf{Tor}_*^{A\otimes B^{\mathsf{op}}}({}_bY_a, {}_aX_b) \oplus$$

$$\mathsf{Tor}_*^{B\otimes A^{\mathrm{op}}}({}_aY_b, {}_bX_a) \oplus \mathsf{Tor}_*^{B^{\mathrm{e}}}({}_bY_b, {}_bX_b).$$

(F7) As mentioned the $B-A$-bimodule M is viewed as a C-bimodule by extending the actions by zero, that is $AM = MB = 0$. Analogously, N is a C-bimodule. This way

$$\Lambda/C = M \oplus N$$

as C-bimodules.

Next we show that the hypotheses of Theorem 3.6 hold for a strongly stratifying Morita context.

Proposition 4.1. *Let Λ be a strongly stratifying Morita context $\begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$, and let $C = A \times B$ as a subalgebra of the Morita context.*

We have that $\mathsf{Tor}_*^C(\Lambda/C, (\Lambda/C)^{\otimes_C n}) = 0$ for $* > 0$ and for all n.

Proof. As noted in (F7), $\Lambda/C = M \oplus N$ as C-bimodules. To compute

$$(M \oplus N)^{\otimes_C 2}$$

note that $M \otimes_C M = 0 = N \otimes_C N$ by (F3). Analogously, $N \otimes_C M = N \otimes_B M$ and $M \otimes_C N = M \otimes_A N$; the latter is 0 since the Morita context is strongly stratifying. Finally

$$(M \oplus N)^{\otimes_C 2} = N \otimes_B M. \tag{4.1}$$

Moreover

$$(M \oplus N)^{\otimes_C 3} = M \otimes_A N \otimes_B M = 0. \tag{4.2}$$

For $n \geq 3$, we infer $(M \oplus N)^{\otimes_C n} = 0$ and $\mathsf{Tor}_*^C(\Lambda/C, (\Lambda/C)^{\otimes_C n}) = 0$.

For $n = 2$ we have

$$\begin{aligned} \mathsf{Tor}_*^C(M \oplus N, (M \oplus N)^{\otimes_C 2}) &= \mathsf{Tor}_*^C(M \oplus N, N \otimes_B M) \\ &= \mathsf{Tor}_*^A(M, N \otimes_B M) \\ &\stackrel{\star}{=} \mathsf{Tor}_*^B(M \otimes_A N, M) \\ &= 0 \end{aligned}$$

The equality $\stackrel{\star}{=}$ is ensured by [15, Theorem 2.8, p.167] in case the following takes place

$$\mathsf{Tor}_n^A(M, N) = 0 = \mathsf{Tor}_n^B(N, M) \quad \text{for } n > 0.$$

Indeed, this holds since the Morita context is strongly stratifying.

For $n = 1$ we have

$$\begin{aligned}\mathsf{Tor}_*^C(\Lambda/C, (\Lambda/C)) &= \mathsf{Tor}_*^C(M \oplus N, M \oplus N) \\ &= \mathsf{Tor}_*^A(M, N) \oplus \mathsf{Tor}_*^B(N, M)\end{aligned}$$

according to (F4). Now $\mathsf{Tor}_*^A(M, N) = 0$ for $* > 0$ since the Morita context is stratifying. Moreover $\mathsf{Tor}_*^B(N, M) = 0$ for $* > 0$ since it is strongly stratifying. $\square$

We will now show that the terms at the first page of the spectral sequence of Theorem 3.6 for $X = \Lambda$ vanish.

Lemma 4.2. *Let* $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ *be a strongly stratifying Morita context. For* $n \geq 0$

$$\mathsf{Tor}_n^{A \otimes B^{\mathsf{op}}}(M, N) = 0 = \mathsf{Tor}_n^{B \otimes A^{\mathsf{op}}}(N, M).$$

Proof. We make use of the "associativity formula" of H. Cartan and S. Eilenberg [15, p. 347, (5a)], namely there is a spectral sequence

$$H_q(B, \mathsf{Tor}_p^A(M, N)) \Rightarrow \mathsf{Tor}_n^{A \otimes B^{\mathsf{op}}}(M, N).$$

Since the Morita context is strongly stratifying, $\mathsf{Tor}_p^A(M, N) = 0$ for $p \geq 0$. Hence $H_q(B, \mathsf{Tor}_p^A(M, N)) = 0$ for all p and q, and $\mathsf{Tor}_n^{A \otimes B^{\mathsf{op}}}(M, N) = 0$ for all n.

Given an algebra D, a right D-module X and a left D-module Y, it is well known that for all n

$$\mathsf{Tor}_n^D(X, Y) = \mathsf{Tor}_n^{D^{\mathsf{op}}}(Y, X).$$

Hence

$$\mathsf{Tor}_q^{B \otimes A^{\mathsf{op}}}(N, M) = \mathsf{Tor}_q^{A \otimes B^{\mathsf{op}}}(M, N) = 0. \quad \square$$

Proposition 4.3. *Let* Λ *be a strongly stratifying Morita context* $\begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$, *and let* $C = A \times B$. *We have*

$$\mathsf{Tor}_q^{C^{\mathsf{e}}}(\Lambda, (\Lambda/C)^{\otimes_C p}) = 0 \quad \textit{for } p, q > 0.$$

Proof. By (4.2) we have $(\Lambda/C)^{\otimes_C p} = 0$ for $p \geq 3$, thus $\mathsf{Tor}_q^{C^{\mathsf{e}}}(\Lambda, (\Lambda/C)^{\otimes_C p}) = 0$ for $p \geq 3$.

The decomposition of (F5) of Λ as a C-bimodule is

$$\Lambda = A \oplus N \oplus M \oplus B.$$

For $p = 1$, according to (F6) we have

$$\mathsf{Tor}_q^{C^e}(A \oplus N \oplus M \oplus B,\ M \oplus N) = \mathsf{Tor}_q^{B \otimes A^{op}}(N, M) \oplus \mathsf{Tor}_q^{A \otimes B^{op}}(M, N).$$

Both summands vanish by the previous Lemma 4.2.

For $p = 2$, we know by (4.1) that $(M \oplus N)^{\otimes_C 2} = N \otimes_B M$. Hence

$$\mathsf{Tor}_q^{C^e}(A \oplus N \oplus M \oplus B,\ (M \oplus N)^{\otimes_C 2}) = \mathsf{Tor}_q^{A^e}(A, N \otimes_B M).$$

We show next that the hypotheses of [15, p.347 (4a)] hold: firstly note that $\mathsf{Tor}_n^A(A, N) = 0$ for $n > 0$. Secondly, since the Morita context is strongly stratifying, $\mathsf{Tor}_n^B(N, M) = 0$ for $n > 0$. Therefore there is an isomorphism

$$\mathsf{Tor}_q^{A^e}(A, N \otimes_B M) \simeq \mathsf{Tor}_q^{A^{op} \otimes B}(A \otimes_A N, M).$$

The latter is $\mathsf{Tor}_q^{A^{op} \otimes B}(N, M)$, which is zero by the previous Lemma 4.2. □

Theorem 4.4. *Let Λ be an algebra with a distinguished idempotent e, such that $\Lambda e \Lambda$ is a strongly stratifying ideal and let $C = e\Lambda e \times f\Lambda f$, where $f = 1 - e$.*

There exists a Jacobi-Zariski long exact sequence

$$\dots \xrightarrow{\delta} H_m(C, \Lambda) \xrightarrow{I} H_m(\Lambda, \Lambda) \xrightarrow{K} H_m(\Lambda|C, \Lambda) \xrightarrow{\delta} H_{m-1}(C, \Lambda) \xrightarrow{I} \dots$$
$$\xrightarrow{\delta} H_n(C, \Lambda) \xrightarrow{I} H_n(\Lambda, \Lambda) \xrightarrow{K} H_n(\Lambda|C, \Lambda)$$

ending at some n.

Proof. Consider the corresponding strongly stratifying Morita context $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ where $A = e\Lambda e$ and $B = f\Lambda f$, thus $C = A \times B$. We will show that we can use part a) of Theorem 3.5, that is we assert $H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota) = 0$ for $* >> 0$. Indeed, the spectral sequence of Theorem 3.6 is available if we prove that $\mathsf{Tor}_*^C(\Lambda/C, (\Lambda/C)^{\otimes_C n}) = 0$ for $* > 0$ and for all n. This follows from Proposition 4.1.

Moreover the first page of this spectral sequence is

$$E^1_{p,q} = \mathsf{Tor}_q^{C^e}(\Lambda, (\Lambda/C)^{\otimes_C p}) \quad \text{for } p, q > 0$$

and 0 anywhere else, see Theorem 3.6 for $X = \Lambda$.

This first page vanishes, due to Proposition 4.3. We have proved that

$$H_*(\mathsf{Ker}\kappa/\mathsf{Im}\iota) = 0 \text{ for } * >> 0$$

and of Theorem 3.5 a) provides the existence of the Jacobi-Zariski long exact sequence. □

Lemma 4.5. *In the situation of Theorem 4.4, let X be a Λ-bimodule. We have that $H_m(\Lambda|C, X) = 0$ for $m \geq 3$.*

Proof. By [18, Corollary 2.4] we have that $H_m(\Lambda|C, X)$ is the homology of the following chain complex

$$\cdots \to X \otimes_{C^e} (\Lambda/C)^{\otimes_C m} \to \cdots \to X \otimes_{C^e} \Lambda/C \to X_C \to 0$$

where $X_C = \Lambda \otimes_{C^e} C = X/\langle cx - xc\rangle = H_0(C, X)$. On the other hand, (4.2) ensures that $(\Lambda/C)^{\otimes_C m} = 0$ for $m \geq 3$. $\square$

Theorem 4.6. *Let Λ be an algebra with a distinguished idempotent e such that $\Lambda e\Lambda$ is a strongly stratifying ideal, and let $f = 1 - e$. If $HH_*(\Lambda)$ is finite, then $HH_*(e\Lambda e)$ and $HH_*(f\Lambda f)$ are finite.*

Proof. Consider the corresponding strongly stratifying Morita context
$\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ where $A = e\Lambda e$ and $B = f\Lambda f$ and $C = A \times B$. The Lemma 4.5 and the Jacobi-Zariski long exact sequence of Theorem 4.4 provide $H_*(C, \Lambda) = 0$ for $* >> 0$. Moreover

$$\mathsf{Tor}_*^{C^e}(C, \Lambda) = \mathsf{Tor}_*^{C^e}(A \oplus B,\ A \oplus N \oplus M \oplus B).$$

By (F6) the latter is

$$\mathsf{Tor}^{A^e}(A, A) \oplus \mathsf{Tor}^{B^e}(B, B) = HH_*(A) \oplus HH_*(B).$$

We infer that $HH_*(A)$ and $HH_*(B)$ are finite. $\square$

5. Han's conjecture

We recall Han's conjecture [25] for an algebra Λ: if $HH_*(\Lambda)$ is finite, then Λ has finite global dimension.

Theorem 5.1. *Let Λ be an algebra with a distinguished idempotent e such that $\Lambda e\Lambda$ is a strongly stratifying ideal, and let $f = 1 - e$. The algebra Λ verifies Han's conjecture if and only if $e\Lambda e \times f\Lambda f$ does.*

Proof. Consider the corresponding strongly stratifying Morita context

$$\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta},$$

where $A = e\Lambda e$ and $B = f\Lambda f$. Assume that $A \times B$ satisfies Han's conjecture and let us prove that this is also the case for Λ. So suppose $HH_*(\Lambda)$ is finite. By Theorem 4.6, we have that $HH_*(A)$ and $HH_*(B)$ are finite. It is well known that $HH_*(A \times B) = HH_*(A) \oplus HH_*(B)$, then $H_*(A \times B)$ is finite and thus $A \times B$ has finite global dimension. Hence A and B have finite global dimension.

Note that by Remark 2.15 we have $\beta : 0 \to B$, then $\mathsf{Im}\beta = 0$ and thus $B/\mathsf{Im}\beta = B$. Therefore $\Lambda/\Lambda e\Lambda = B$, see Remark 2.10. Since the ideal $\Lambda e\Lambda$ is strongly stratifying, it is stratifying. Hence there is a recollement of $D(\Lambda)$ relative to $D(\Lambda/\Lambda e\Lambda)$ and $D(e\Lambda e)$, that is relative to $D(B)$ and $D(A)$.

According to L. Angeleri Hügel, S. Koenig, Q. Liu and D. Yang in [4, Theorem I, p. 17] (see also [26, Proposition 4, p. 541]), since A and B have finite global dimension, Λ has finite global dimension.

Next we show the other implication. Assume that Λ satisfies Han's conjecture, our aim is to show that $A \times B$ also does. Let's suppose that $HH_*(A \times B)$ is finite. Since $HH_*(A \times B) = HH_*(A) \oplus HH_*(B)$ we infer that $HH_*(A)$ and $HH_*(B)$ are finite.

We have that $\Lambda e\Lambda$ is a strongly stratifying ideal, hence it is stratifying and there is a recollement. According to [26, Corollary 2, p. 543] after B. Keller [33], there is a long exact sequence in Hochschild homology

$$\cdots \to HH_{n+1}(\Lambda/\Lambda e\Lambda) \to HH_n(e\Lambda e) \to HH_n(\Lambda) \to HH_n(\Lambda/\Lambda e\Lambda) \to \cdots$$

that is for the Morita context

$$\cdots \to HH_{n+1}(B) \to HH_n(A) \to HH_n(\Lambda) \to HH_n(B) \to \cdots .$$

We infer that $HH_*(\Lambda)$ is finite. Since Λ verifies Han's conjecture, Λ is of finite global dimension.

Using again the above cited result in [4,26] we infer that A and B, and thus $A \times B$, have finite global dimension. □

Theorem 5.1 will also be useful for considering algebras filtered by ideals which successive quotients provide strongly stratifying ideals. The following result shows that Definition 5.3 below makes sense.

Lemma 5.2. *Let Λ be an algebra and let $u, v \in \Lambda$ be orthogonal idempotents. We have*

$$\frac{\Lambda(u+v)\Lambda}{\Lambda u\Lambda} = \frac{\Lambda}{\Lambda u\Lambda}\overline{v}\frac{\Lambda}{\Lambda u\Lambda}$$

where $\overline{v}$ is the class of v in $\Lambda/\Lambda u\Lambda$.

Proof. First note that since u and v are orthogonal idempotents we have

$$\Lambda(u+v)\Lambda = \Lambda u\Lambda + \Lambda v\Lambda,$$

consequently

$$\frac{\Lambda(u+v)\Lambda}{\Lambda u\Lambda} = \frac{\Lambda u\Lambda + \Lambda v\Lambda}{\Lambda u\Lambda} = \frac{\Lambda v\Lambda}{\Lambda u\Lambda \cap \Lambda v\Lambda}.$$

Next consider the composition $\Lambda v\Lambda \hookrightarrow \Lambda \twoheadrightarrow \Lambda/\Lambda u\Lambda$. Its image is $\frac{\Lambda}{\Lambda u\Lambda}\overline{v}\frac{\Lambda}{\Lambda u\Lambda}$ and its kernel is $\Lambda u\Lambda \cap \Lambda v\Lambda$. $\square$

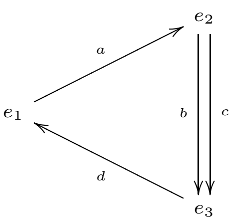
Definition 5.3. Let Λ be an algebra. A *strongly stratifying n-chain* is an ordered complete system of orthogonal idempotents $\{e_1, \ldots, e_n\}$ of Λ such that the filtration by ideals

$$0 \subset \Lambda e_1\Lambda \subset \Lambda(e_1+e_2)\Lambda \subset \cdots \subset \Lambda(e_1+e_2+\cdots+e_{n-1})\Lambda \subset \Lambda$$

verifies that for $1 \leq i \leq n$ the quotient $\Lambda(e_1+\cdots+e_i)\Lambda/\Lambda(e_1+\cdots+e_{i-1})\Lambda$ is a strongly stratifying ideal of $\Lambda/\Lambda(e_1+\cdots+e_{i-1})\Lambda$.

Remark 5.4.

- The bound quiver algebra Λ of Example 2.13



$ba = 0, \ ad = 0, \ dc = 0.$

  admits a strongly stratifying 2-chain $\{e_2+e_3, e_1\}$.
- For $1 \leq i \leq n$, let $\Lambda_i = \Lambda/\Lambda(e_1+\cdots+e_i)\Lambda$. According to Lemma 5.2

$$\frac{\Lambda(e_1+\cdots+e_i)\Lambda}{\Lambda(e_1+\cdots+e_{i-1})\Lambda} = \Lambda_{i-1}\overline{e_i}\Lambda_{i-1}$$

  where $\overline{e_i}$ denotes the class of e_i in Λ_{i-1}.

Definition 5.5. Let $\mathcal{C}$ be a class of algebras. A *$\mathcal{C}$-strongly stratifying n-chain* of an algebra Λ is a strongly stratifying n-chain $\{e_1, \ldots, e_n\}$ of Λ such that for $1 \leq i \leq n$ the algebra $e_i\Lambda e_i$ belongs to $\mathcal{C}$.

We will need the following lemma.

Lemma 5.6. *Let $\mathcal{C}$ be a class of algebras which is closed by taking quotients. Let Λ be an algebra admitting a $\mathcal{C}$-strongly stratifying n-chain $\{e_1, \ldots, e_n\}$ for $n > 1$. The algebra $\Lambda/\Lambda e_1\Lambda$ admits a $\mathcal{C}$-strongly stratifying $n-1$-chain $\{\overline{e_2}, \overline{e_3}, \ldots, \overline{e_n}\}$.*

Proof. We have the following quotient filtration of $\Lambda/\Lambda e_1\Lambda$

$$0 \subset \frac{\Lambda(e_1+e_2)\Lambda}{\Lambda e_1\Lambda} \subset \cdots \subset \frac{\Lambda(e_1+e_2+\cdots+e_{n-1})\Lambda}{\Lambda e_1\Lambda} \subset \frac{\Lambda}{\Lambda e_1\Lambda}.$$

Using Lemma 5.2, the ideals of this filtration are as follows

$$\frac{\Lambda(e_1+\cdots+e_i)\Lambda}{\Lambda e_1\Lambda} = \frac{\Lambda(e_2+\cdots+e_i)\Lambda}{\Lambda e_1\Lambda \cap \Lambda(e_2+\cdots+e_i)\Lambda} = \frac{\Lambda}{\Lambda e_1\Lambda}(\overline{e_2}+\cdots+\overline{e_i})\frac{\Lambda}{\Lambda e_1\Lambda}.$$

Hence the quotient filtration is indeed the one corresponding to the complete system of orthogonal idempotents $\{\overline{e_2},\ldots,\overline{e_i}\}$ of $\Lambda/\Lambda e_1\Lambda$, namely

$$0 \subset \frac{\Lambda}{\Lambda e_1\Lambda}\overline{e_2}\frac{\Lambda}{\Lambda e_1\Lambda} \subset \cdots \subset \frac{\Lambda}{\Lambda e_1\Lambda}(\overline{e_2}+\cdots+\overline{e_n})\frac{\Lambda}{\Lambda e_1\Lambda} \subset \frac{\Lambda}{\Lambda e_1\Lambda}.$$

To verify that the successive quotients of this filtration of $\Lambda/\Lambda e_1\Lambda$ are strongly stratifying in the corresponding quotient of $\Lambda/\Lambda e_1\Lambda$, note that

$$\frac{\Lambda(e_1+\cdots+e_i)\Lambda/\Lambda e_1\Lambda}{\Lambda(e_1+\cdots+e_{i-1})\Lambda/\Lambda e_1\Lambda} = \frac{\Lambda(e_1+\cdots+e_i)\Lambda}{\Lambda(e_1+\cdots+e_{i-1})\Lambda}$$

and

$$\frac{\Lambda/\Lambda e_1\Lambda}{\Lambda(e_1+\cdots+e_{i-1})\Lambda/\Lambda e_1\Lambda} = \frac{\Lambda}{\Lambda(e_1+\cdots+e_{i-1})\Lambda}.$$

Finally observe that

$$\overline{e_i}\frac{\Lambda}{\Lambda e_1\Lambda}\overline{e_i} = \frac{e_i\Lambda e_i}{\Lambda e_1\Lambda \cap e_i\Lambda e_i}.$$

Since $e_i\Lambda e_i$ is in $\mathcal{C}$ which is closed by taking quotients, we infer that $\overline{e_i}\frac{\Lambda}{\Lambda e_1\Lambda}\overline{e_i}$ also belongs to $\mathcal{C}$. This way $\{\overline{e_2},\ldots,\overline{e_n}\}$ is a $\mathcal{C}$-strongly stratifying $n-1$-chain of $\Lambda/\Lambda e_1\Lambda$. □

Theorem 5.7. *Let $\mathcal{C}$ be a class of algebras which is closed by taking quotients, and assume that Han's conjecture holds for all algebras in $\mathcal{C}$. Let Λ be an algebra which admits a $\mathcal{C}$-strongly stratifying n-chain for some $n>0$. Then Λ verifies Han's conjecture.*

Proof. By induction, let Λ be an algebra admitting a $\mathcal{C}$-strongly stratifying n-chain $\{e_1,\ldots,e_n\}$. If $n=1$, then $e_1=1$ and the algebra $\Lambda = 1\Lambda 1$ is in $\mathcal{C}$. By hypothesis, Λ verifies Han's conjecture.

Let $n>1$ and consider the algebra $\Lambda/\Lambda e_1\Lambda$ which admits a $\mathcal{C}$-strongly stratifying $n-1$-chain by Lemma 5.6. Hence Han's conjecture holds for it.

The ideal $\Lambda e_1\Lambda$ is strongly stratifying in Λ. By Theorem 5.1, Han's conjecture holds for Λ if and only if it holds for $e_1\Lambda e_1 \times (1-e_1)\Lambda(1-e_1)$. To verify the latter, suppose that

$HH_*(e_1\Lambda e_1 \times (1-e_1)\Lambda(1-e_1))$ is finite. So $HH_*(e_1\Lambda e_1)$ is finite. But $e_1\Lambda e_1$ belongs to $\mathcal{C}$, thus by hypothesis it verifies Han's conjecture. Then $e_1\Lambda e_1$ has finite global dimension.

On the other hand we also have that $HH_*((1-e_1)\Lambda(1-e_1))$ is finite. Consider the Morita context given by e_1. By Remark 2.10

$$\frac{\Lambda}{\Lambda e_1 \Lambda} = \frac{(1-e_1)\Lambda(1-e_1)}{\mathsf{Im}\beta}.$$

Since $\Lambda e_1 \Lambda$ is strongly stratifying, we have that $\beta = 0$. Consequently

$$\Lambda/\Lambda e_1\Lambda = (1-e_1)\Lambda(1-e_1)$$

and $HH_*(\Lambda/\Lambda e_1\Lambda)$ is finite. Han's conjecture holds for $\Lambda/\Lambda e_1\Lambda$ by the inductive hypothesis - see above. Then $\Lambda/\Lambda e_1\Lambda = (1-e_1)\Lambda(1-e_1)$ has finite global dimension.

We have established before that $e_1\Lambda e_1$ has finite global dimension. We infer that $e_1\Lambda e_1 \times (1-e_1)\Lambda(1-e_1)$ has finite global dimension, that is this algebra verifies Han's conjecture as needed. □

Corollary 5.8. *Assume that Han's conjecture holds for local algebras. If an algebra Λ admits a strongly stratifying chain $\{e_1, \dots, e_n\}$ with e_i primitive for all i, then Han's conjecture is true for Λ.*

In order to avoid classes of algebras closed by taking quotients, instead of filtering by ideals of an algebra Λ, below we filter Λ by algebras $f\Lambda f$, where f is a partial sum of a complete system of orthogonal idempotents.

The following lemma can be easily proved.

Lemma 5.9. *Let Λ be an algebra with an ordered complete set of orthogonal idempotents $\{e_1, \dots, e_n\}$. Consider the following idempotents*

$$f_0 = e_1 + \dots + e_n,\ f_1 = e_2 + \dots + e_n,\ \dots\ , f_i = e_{i+1} + \dots + e_n,\ \dots,\ f_{n-1} = e_n.$$

For $0 \leq i \leq n-1$, consider the algebra $f_i\Lambda f_i$ with unit f_i.

- *For $j > i$ we have $e_j = f_ie_j = e_jf_i = f_ie_jf_i$, therefore $e_j \in f_i\Lambda f_i$ and $e_j(f_i\Lambda f_i)e_j = e_j\Lambda e_j$,*
- *$f_i\Lambda f_i$ has a complete set of orthogonal idempotents $\{e_{i+1}, \dots, e_n\}$ and $f_{i+1} = f_i - e_{i+1}$,*
- *For $j \geq i$ we have $f_if_j = f_jf_i = f_j$, therefore $f_j\Lambda f_j = f_j(f_i\Lambda f_i)f_j$,*
- *$0 \subset f_{n-1}\Lambda f_{n-1} \subset \dots \subset f_i\Lambda f_i \subset \dots \subset f_1\Lambda f_1 \subset f_0\Lambda f_0 = \Lambda$.*

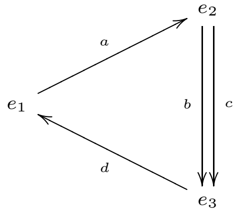
We keep the notations of Lemma 5.9 in the sequel.

Definition 5.10. Let Λ be an algebra. A *strongly co-stratifying n-chain* of Λ is an ordered complete set of orthogonal idempotents $\{e_1, \ldots, e_n\} \subset \Lambda$ such that the ideals provided by the following idempotents

$$f_1 \in \Lambda,\ f_2 \in f_1\Lambda f_1,\ \ldots\ ,\ f_{i+1} \in f_i\Lambda f_i,\ \ldots\ ,\ f_{n-1} \in f_{n-2}\Lambda f_{n-2}$$

are strongly stratifying in their respective algebras.

Example 5.11. The bound quiver algebra Λ of Example 2.13



$$ba = 0,\ ad = 0,\ dc = 0.$$

admits a strongly co-stratifying 3-chain $\{e_1, e_2, e_3\}$. Indeed, consider the filtration

$$0 \subset e_3\Lambda e_3 \subset (e_2 + e_3)\Lambda(e_2 + e_3) \subset \Lambda.$$

We know that the idempotent $e_2 + e_3$ provides a strongly stratifying ideal in Λ. Moreover e_3 gives trivially a strongly stratifying ideal of the Kronecker algebra $(e_2 + e_3)\Lambda(e_2 + e_3)$ since the corresponding Morita context is $\begin{pmatrix} k & k \oplus k \\ 0 & k \end{pmatrix}$.

Definition 5.12. An *$\mathcal{H}$-strongly co-stratifying n-chain* of Λ is a strongly co-stratifying n-chain $\{e_1, \ldots, e_n\}$ such that $e_i\Lambda e_i$ verifies Han's conjecture for all i.

Theorem 5.13. *Let Λ be an algebra which admits an $\mathcal{H}$-strongly co-stratifying n-chain. Then Λ verifies Han's conjecture.*

Proof. By induction, let Λ be an algebra which admits an $\mathcal{H}$-strongly co-stratifying n-chain $\{e_1, \ldots, e_n\}$. If $n = 1$, then $e_1 = 1$ and $\Lambda = e_1\Lambda e_1$ verifies Han's conjecture.

For $n > 1$, recall that $f_1 = e_2 + \cdots + e_n = 1 - e_1$. Since $\Lambda f_1\Lambda$ is a strongly stratifying ideal of Λ, by Theorem 5.1 we have that Han's conjecture holds for Λ if and only if it holds for $f_1\Lambda f_1 \times e_1\Lambda e_1$. To verify the latter, suppose that $HH_*(f_1\Lambda f_1 \times e_1\Lambda e_1)$ is finite, then $HH_*(f_1\Lambda f_1)$ and $HH_*(e_1\Lambda e_1)$ are finite. We have that $e_1\Lambda e_1$ verifies Han's conjecture thus $e_1\Lambda e_1$ is of finite global dimension.

On the other hand we assert that $f_1\Lambda f_1$ admits a $\mathcal{H}$-strongly co-stratifying $(n-1)$-chain $\{e_2, \ldots, e_n\}$. First by Lemma 5.9, for $j \geq 2$ we have $f_j(f_1\Lambda f_1)f_j = f_j\Lambda f_j$. Thus the ideal provided by f_{j+1} in $f_j(f_1\Lambda f_1)f_j$ is strongly stratifying since $\{e_1, \ldots, e_n\}$ is a strongly co-stratifying n-chain of Λ. Second, by Lemma 5.9 for $j \geq 2$, we have $e_j(f_1\Lambda f_1)e_j = e_j\Lambda e_j$ and the latter verifies Han's conjecture.

Therefore the inductive hypothesis ensures that $f_1\Lambda f_1$ verifies Han's conjecture, hence $f_1\Lambda f_1$ is of finite global dimension. We infer that $f_1\Lambda f_1 \times e_1\Lambda e_1$ is of finite global dimension, that is Han's conjecture is true for $f_1\Lambda f_1 \times e_1\Lambda e_1$. □

Corollary 5.14. *Assume that Han's conjecture holds for local algebras. If an algebra Λ admits a co-stratifying chain consisting of primitive idempotents, then Han's conjecture is true for Λ.*

Remark 5.15. As quoted in the Introduction, a comparison between algebras admitting a strongly stratifying or co-stratifying chain with algebras which are standardly stratified will be considered in a forthcoming paper. For the convenience of the reader, we recall the definition of standardly stratified algebras (see for instance [1,2,37,42]).

With the same notations as in Lemma 5.9, recall that $f_i = e_{i+1} + \cdots + e_n$, and let $f_n = 0$. Consider the set Δ of *standard left Λ-modules* $\Delta_i = \Lambda e_i/\Lambda f_i \Lambda e_i$ for $i = 1, \ldots, n$. As mentioned in [1] the module Δ_i is the largest quotient of Λe_i such that its composition factors are not isomorphic to $(\Lambda/r)e_j$ for $j > i$, where r is the radical of Λ.

The algebra Λ is *standardly stratified* if it admits a filtration by left submodules which successive quotients belong to Δ, up to isomorphism.

6. Patterns for examples of strongly stratifying Morita contexts

In the following we provide patterns for obtaining families of strongly stratifying Morita contexts, through assuming projectivity hypothesis for M and/or N.

Remark 6.1.

- In Example 2.13 from [35, Example 4.4], [5, Example 2.3], neither M is projective as a right A-module, nor N is projective as a left A-module.
- In [21] Morita contexts with $\alpha = \beta = 0$ and M and N projective bimodules are considered. In what follows, in general $\alpha \neq 0$. In Proposition 6.7, N is any bimodule and M is a projective bimodule. In Proposition 6.8, M and N are left projective modules.
- We emphasize that our results for a strongly stratifying Morita context do not depend on the morphism α, while $\beta = 0$ since its source vector space vanishes. In other words changing α to α' provides in general different Morita contexts, nevertheless the Morita context remains strongly stratifying and the results of this paper still apply.

Lemma 6.2. *Let Λ be a Morita context $\begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ with $\beta = 0$. The associativity conditions (2.1) are equivalent to*

$$(\mathsf{Im}\alpha)N = 0 = M(\mathsf{Im}\alpha).$$

Proposition 6.3. *Let A and B be algebras, a, a' be idempotents in A and b, b' be idempotents in B. Let*

$$M = Bb \otimes aA \text{ and } N = Aa' \otimes b'B.$$

Let $\alpha : N \otimes_B M \to A$ be a morphism of A-bimodules. There is a strongly stratifying Morita context $\begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ if and only if $aAa' = 0$.

Proof. Note that M and N are projective bimodules, so they are left and right projective. Thus both $\mathsf{Tor}_n^A(M, N) = 0$ and $\mathsf{Tor}_n^B(N, M) = 0$ are zero for $n > 0$. Also

$$M \otimes_A N = Bb \otimes aA \otimes_A Aa' \otimes b'B = Bb \otimes aAa' \otimes b'B.$$

If the Morita context is strongly stratifying then $M \otimes_A N = 0$. We infer $aAa' = 0$. Conversely, if $aAa' = 0$, then $M \otimes_A N = 0$. Note that $N \otimes_B M = Aa' \otimes b'Bb \otimes aA$, so $\mathsf{Im}\alpha \subset Aa'AaA$. Consequently

$$(\mathsf{Im}\alpha)N \subset Aa'AaAa' \otimes b'B = 0 \text{ and } M(\mathsf{Im}\alpha) \subset Bb \otimes aAa'AaA = 0.$$

The associativity conditions of Lemma 6.2 are satisfied, thus there is a Morita context. □

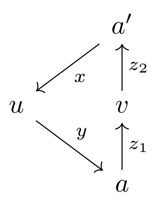
Remark 6.4. Under the hypothesis of Proposition 6.3

$$\begin{aligned}\mathsf{dim}_k\mathsf{Hom}_{A-A}(N \otimes_B M, A) =& \mathsf{dim}_k\mathsf{Hom}_{A-A}(Aa' \otimes b'Bb \otimes aA, A) \\ =& \mathsf{dim}_k(a'Aa)\mathsf{dim}_k(b'Bb).\end{aligned}$$

Hence it is possible to choose $\alpha \neq 0$ if and only if $a'Aa \neq 0$ and $b'Bb \neq 0$.

In the following we provide an example for Proposition 6.3, keeping the same notations.

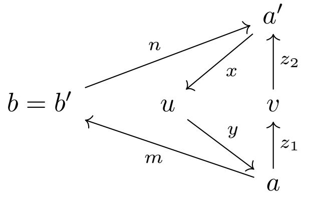
Example 6.5. Let A be the algebra of the quiver



with the relation $yx = 0$. Let $B = k$, with $b = b' = 1$. We have $M = aA$ and $N = Aa'$. Moreover, $aAa' = 0$. The projective A-bimodule $N \otimes_B M$ is $Aa' \otimes aA$. We have

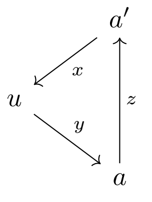
$$\mathsf{Hom}_{A-A}(Aa' \otimes aA, A) = a'Aa = k\{z_2 z_1\}.$$

A non-zero α is determined by $\alpha(a' \otimes a) = z_2 z_1$. We denote by m and n the generators a and a' of M and N respectively. The strongly stratifying Morita context has a presentation given by the quiver



and the relations $yx = 0$ and $nm = z_2 z_1$.

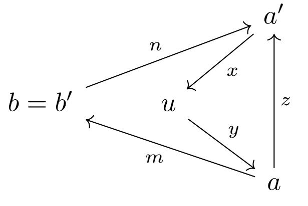
Example 6.6. Consider A the algebra of the quiver



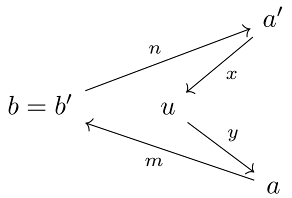
with the relation $yx = 0$. Let $B = k$, with $b = b' = 1$. We have $M = aA$ and $N = Aa'$. Moreover, $aAa' = 0$. The projective A-bimodule $N \otimes_B M$ is $Aa' \otimes aA$. We have

$$\mathsf{Hom}_{A-A}(Aa' \otimes aA, A) = a'Aa = kz.$$

A non-zero α is determined by $\alpha(a' \otimes a) = z$. We denote m and n the generators of M and N respectively. The strongly stratifying Morita context has a presentation given by the quiver



and the relations $yx = 0$ and $nm = z$. An admissible presentation of this Morita context is given by the quiver



and the admissible relation $yx = 0$.

In other words, this algebra is also a Morita context but relative to an algebra A' instead of A.

The projectivity requirements for M and N can be relaxed as follows.

Proposition 6.7. *Let A and B be algebras with respective idempotents a and b. Let $M = Bb \otimes aA$, and let N be any $A-B$-bimodule. Let $\alpha : N \otimes_B M \to A$ be an A-bimodule map. There is a strongly stratifying Morita context $\begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ if and only if $aN = 0$ and $a(\mathsf{Im}\alpha) = 0$.*

Proof. For $n > 0$ we have $\mathsf{Tor}_n^A(M, N) = 0$ and $\mathsf{Tor}_n^B(N, M) = 0$. Moreover $M \otimes_A N = Bb \otimes aN$, hence $M \otimes_A N = 0$ if and only if $aN = 0$. Also, $N \otimes_B M = Nb \otimes aA$, hence $\mathsf{Im}\alpha \subset Aa$.

If the Morita context is strongly stratifying then $M \otimes_A N = 0$, whence $aN = 0$. By Lemma 6.2, $M(\mathsf{Im}\alpha) = 0$, that is $Bb \otimes Aa(\mathsf{Im}\alpha) = 0$ which is equivalent to $a(\mathsf{Im}\alpha) = 0$.

For the converse, it remains to prove that $(\mathsf{Im}\alpha)N = 0$ in order to satisfy the conditions of Lemma 6.2. We have that $\mathsf{Im}\alpha \subset Aa$. Hence

$$\mathsf{Im}\alpha N \subset AaN = 0. \quad \square$$

Proposition 6.8. *Let A and B be algebras with respective idempotents a and b. Let $M' \neq 0$ be a right A-module and $M = Bb \otimes M'$. Let $N' \neq 0$ be a right B-module and $N = Aa \otimes N'$. Let $\alpha : N \otimes_B M \to A$ be an A-bimodule map. There is a strongly stratifying Morita context $\begin{pmatrix} A & N \\ M & B \end{pmatrix}_{\alpha,\beta}$ if and only if $M'a = 0$ and $(\mathsf{Im}\alpha)a = 0$.*

Proof. We have that

- $\mathsf{Tor}_n^A(M, N) = 0$ and $\mathsf{Tor}_n^B(N, M) = 0$ for $n > 0$,
- $M \otimes_A N = Bb \otimes M'a \otimes N'$, whence $M \otimes_A N = 0$ if and only if $M'a = 0$,
- $N \otimes_B M = Aa \otimes N'b \otimes M'$, whence $\mathsf{Im}\alpha \subset aA$.

If the Morita context is strongly stratifying then $M'a = 0$. By Lemma 6.2, $(\mathsf{Im}\alpha)N = 0$, that is $(\mathsf{Im}\alpha)Aa \otimes N' = 0$ which is equivalent to $(\mathsf{Im}\alpha)a = 0$.

For the converse, it remains to prove that $M(\mathsf{Im}\alpha) = 0$. We have that $(\mathsf{Im}\alpha) \subset aA$. Hence

$$M(\mathsf{Im}\alpha) = Bb \otimes M'(\mathsf{Im}\alpha) \subset Bb \otimes M'aA = 0. \quad \square$$

Data availability

No data was used for the research described in the article.

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