RT-MAT 99-11

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Abril 1999

THE GROTHENDIECK GROUP OF THE CATEGORY OF MODULES OF FINITE PROJECTIVE DIMENSION OVER CERTAIN WEAKLY TRIANGULAR ALGEBRAS

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We dedicate this paper to Helmut Lenzig, in the year of his sixtieth birthday

ABSTRACT. In this paper we study the category of finitely generated modules of finite projective dimension over a class of weakly triangular algebras, which includes the algebras whose idempotent ideals have finite projective dimension. In particular, we prove that the relations given by the (relative) almost split sequences generate the group of all relations for the Grothendieck group of $\mathcal{P}^{<\infty}(\Lambda)$ if and only if $\mathcal{P}^{<\infty}(\Lambda)$ is of finite type. A similar statement is known to hold for the category of all finitely generated modules over an artin algebra, and was proven by C.M.Butler and M. Auslander ([B] and [A]).

We assume in this paper that the artin algebra Λ is weakly triangular, that is, that the nonisomorphic indecomposable projective Λ-modules $\mathbf{P}_1, \cdots, \mathbf{P}_n$ can be ordered so that $\mathrm{Hom}_\Lambda(\mathbf{P}_i, \mathbf{P}_j) = 0$ for $i > j$. Let $\mathbf{A}_i$ be the factor of the projective module $\mathbf{P}_i$ modulo the trace in $\mathbf{P}_i$ of all the other indecomposable projective. Then $\mathbf{A}_1, \cdots, \mathbf{A}_n$ play an important role in the study of the category $\mathcal{P}^{<\infty}(\Lambda)$ of finitely generated modules of finite projective dimension. It is known that the finitistic projective dimension of Λ is bounded by the largest of the projective dimensions of the $\mathbf{A}_i$'s ([**FS**, **CP**]). Moreover, if all the $\mathbf{A}_i$'s have finite projective dimension then $\mathcal{P}^{<\infty}(\Lambda)$ consists of the modules having a filtration with factors amongst the $\mathbf{A}_i$'s. In this case $\mathcal{P}^{<\infty}(\Lambda)$ is a functorially finite subcategory of the category modΛ of finitely generated Λ-modules. Being $\mathcal{P}^{<\infty}(\Lambda)$ closed under extensions it follows that it has relative almost split sequences (see [**AS**]).

These results were proven in [**CP**] for weakly triangular algebras under the assumption that some idempotent ideals of Λ (namely, the traces $\tau_{\mathbf{Q}_i}(\Lambda)$, where $\mathbf{Q}_i = \mathbf{P}_1 \oplus \cdots \oplus \mathbf{P}_i$ for all $i \leq n$) have finite projective dimension. This hypothesis implies that all the $\mathbf{A}_i$'s have finite projective dimension, since $\mathbf{A}_i = \mathbf{P}_i/\tau_{\hat{\mathbf{P}}_i}(\mathbf{P}_i) = \mathbf{P}_i/\tau_{\mathbf{Q}_{i-1}}(\mathbf{P}_i)$ the trace of a projective module in $\mathbf{P}_i$ is an idempotent ideal, for all i. The converse is also true, as it is proven in the first section.

Throughout the remainder of the paper, we assume that Λ satisfies these properties. In the second section we study further properties of $\mathcal{P}^{<\infty}(\Lambda)$ and notions relative to $\mathcal{P}^{<\infty}(\Lambda)$. The modules $\mathbf{A}_1, \cdots, \mathbf{A}_n$ are the simple objects in $\mathcal{P}^{<\infty}(\Lambda)$, and $\mathcal{P}^{<\infty}(\Lambda)$ inherits several properties of modΛ. For example, every module in $\mathcal{P}^{<\infty}(\Lambda)$ has a well defined relative socle, and the relative injective indecomposable modules in $\mathcal{P}^{<\infty}(\Lambda)$ have $\mathcal{P}^{<\infty}(\Lambda)$-simple socle, and coincide with the $\mathcal{P}^{<\infty}(\Lambda)$-injective envelopes of $\mathbf{A}_1, \cdots, \mathbf{A}_n$.

On the other hand, we can consider for every simple module $\mathbf{S}_i$ the $\mathcal{P}^{<\infty}(\Lambda)$-approximation of the injective envelope of $\mathbf{S}_i$. In general this module decomposes, but has, up to isomorphism, a unique indecomposable direct summand, which is precisely the $\mathcal{P}^{<\infty}(\Lambda)$-injective envelope of $\mathbf{A}_i$. The multiplicity of this summand is also described, and we show that it can be arbitrarily large.

In the last section we study the Grothendieck group of $\mathcal{P}^{<\infty}(\Lambda)$, finding again an analogy between $\mathcal{P}^{<\infty}(\Lambda)$ and mod Λ.

M. C. Butler proved in [**B**] that if Γ is an artin algebra of finite representation type then the relations given by the almost split sequences generate the group of all relations for the Grothendieck group of Γ, and M. Auslander proved that the converse also holds ([**A**]). We prove an analogous result for the relations defining the Grothendieck group of the subcategory $\mathcal{P}^{<\infty}(\Lambda)$. The methods we use are similar to those in [**A**]. In order to write down the precise statement, given below, we explain some notations that are introduced in this last section.

The statement concerns the defining relations of the Grothendieck group $\mathbf{K}_0(\mathcal{P}^{<\infty})$ as a quotient of the group $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ that we identify with the free abelian group having ind(Λ) as a basis. The image of a module M in $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ is denoted by $[M]$. If C is an indecomposable, non projective Λ-module, r_C denotes the element $[C] + [A] - [B]$ of $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ associated to the almost split sequence ending up at C. And a_i will denote the element $[\mathbf{P}_i] - [\tau_{\mathbf{Q}_j}(\mathbf{P}_i)] \in \mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$.

Theorem. *The relations given by the almost split sequences generate the* *defining relations of the Grothendieck group of* $\mathcal{P}^{<\infty}(\Lambda)$ *if and only if* $\mathcal{P}^{<\infty}(\Lambda)$ *is of finite type. Moreover, if* $\mathcal{P}^{<\infty}(\Lambda)$ *is of finite type then*

a) $\{a_i : i = 1, \cdots, n\} \bigcup \{r_C : C \in \mathrm{ind}\mathcal{P}^{<\infty}(\Lambda),$ *nonprojective* $\}$
is a free basis for the kernel of the map from $\mathrm{K}_0(\mathcal{P}^{<\infty}, 0)$ *onto* $\mathrm{K}_0(\mathcal{P}^{<\infty})$.

b) $\{r_C : C \in \mathrm{ind}\mathcal{P}^{<\infty}(\Lambda)$ *nonprojective* $\}$
is a free basis for $\mathrm{K}_0(\mathcal{P}^{<\infty})$.

1 Preliminaries.

We start this section introducing our assumptions, and some of the notations.

We will denote by R a commutative artinian ring, Λ an artin algebra over R and mod Λ the category of finitely generated left Λ-modules. For the sake of simplicity, we assume that Λ is basic and connected and $\mathbf{P}_1, \ldots, \mathbf{P}_n$ will be chosen representatives of the isoclasses of indecomposable projective modules. $\mathbf{S}_i = \mathrm{top}\, \mathbf{P}_i$, and $\mathbf{I}_i$ the injective envelope of $\mathbf{S}_i$.

In general, for a ring Γ (resp. for an additive category $\mathcal{C}$), indΓ (resp. ind$\mathcal{C}$) will denote the full subcategory of mod Γ (resp. $\mathcal{C}$) defined by a family of representatives of the isoclasses of indecomposable modules in mod Γ (resp. in $\mathcal{C}$).

If M and N are Λ-modules, $\tau_N(M)$ denotes the trace of N in M, that is, the submodule of M generated by the images of all morphisms from N to M. For each $j = 1, 2, ..., n$, $\hat{\mathbf{P}}_j$ will denote the direct sum of all the $\mathbf{P}_i$ for i different from j and $\mathbf{Q}_j$, the direct sum of all the $\mathbf{P}_i$ for $i = 1, ..., j$. Also, $\mathbf{A}_j$ will be the quotient $\mathbf{P}_j/\tau_{\hat{\mathbf{P}}_j}(\mathbf{P}_j) = \mathbf{P}_j/\tau_{\mathbf{Q}_{j-1}}(\mathbf{P}_j)$. There is a natural isomorphism $\mathbf{A}_j \simeq \Lambda/\tau_{\hat{\mathbf{P}}_j}(\Lambda)$, which induces a ring structure on $\mathbf{A}_j$.

Following [CP], we will assume also throughout the paper that Λ is *weakly triangular*, that is, that the projective modules $\mathbf{P}_1, \cdots, \mathbf{P}_n$ may be ordered in such a way that $\Lambda(\mathbf{P}_i, \mathbf{P}_j) = 0$ provided that $i > j$. Here we use the simplified notation $\Lambda(M, N)$ instead of $\mathrm{Hom}_\Lambda(M, N)$, as we will continue to do in the remainder of the paper. In this case, $\mathbf{A}_i = \mathbf{P}_i/\tau_{\mathbf{Q}_{i-1}}(\mathbf{P}_i)$. Whenever we consider a weakly triangular algebra we will assume that the indecomposable projective modules are ordered in this way and we will keep these notations.

When Λ is weakly triangular then the local algebra $\mathrm{End}_\Lambda(\mathbf{P}_i)^{op}$ is isomorphic to the factor $\mathbf{A}_i$, where, in general, Γ^{op} denotes the opposite of the ring Γ. We state this more precisely in the following proposition.

Proposition 1 . *Let* Λ *be a weakly triangular artin algebra. Then the morphism* $\Lambda \to \mathrm{End}_\Lambda(\mathbf{P}_i)^{op}$ *which associates to* $\lambda \in \Lambda$ *the right multiplication* $x \mapsto x\lambda$, *induces an algebra isomorphism from* $\mathbf{A}_i = \Lambda/\tau_{\hat{\mathbf{P}}_i}(\Lambda)$ *to* $\mathrm{End}_\Lambda(\mathbf{P}_i)^{op}$, *for all* $i = 1, \cdots, n$.

The following proposition shows that being weakly triangular is a symmetric property.

Proposition 2 . *If Λ is weakly triangular, then so is Λ^{op}, (for the opposite ordering). Moreover, if $i < j$ then $\Lambda(\mathbf{I}_i, \mathbf{I}_j) = 0$ and so $\mathbf{S}_j$ is not a composition factor of $\mathbf{I}_i$.*

PROOF. The first claim follows from the well known R-module isomorphisms

$$\Lambda^{op}(e_j\Lambda, e_i\Lambda) \cong e_i\Lambda e_j \cong \Lambda(\Lambda e_i, \Lambda e_j).$$

The second statement follows by duality (see **[FS]**, 2.2). □

We are going to prove (cf. Theorem 1, below) the following useful result. If Λ is weakly triangular then

$$\mathbf{A}_1, \cdots, \mathbf{A}_n \in \mathcal{P}^{<\infty}(\Lambda) \Leftrightarrow \tau_{\mathbf{Q}_i}(\Lambda) \in \mathcal{P}^{<\infty}(\Lambda)\ \forall i = 1, \cdots, n.$$

In order to prove some lemmas we will need, let us recall the following results of **[APT]**

Let us consider for a projective Λ-module Q the full subcategories $\mathbf{C}^Q_o$, $\mathbf{C}^Q_1$ and $\mathbf{C}^Q_\infty$ of mod Λ defined in the following way.

The modules in $\mathbf{C}^Q_o$ are those having their projective cover in add$Q$, the full subcategory of mod Λ consisting of direct summands of finite sums of Q. The modules in $\mathbf{C}^Q_1$ are those having a projective presentation in add$Q$, and, finally $\mathbf{C}^Q_\infty$ consists of the modules with a projective resolution in add$Q$.

Let Q be a projective Λ-module and $\Gamma = \mathrm{End}_\Lambda(Q)^{op}$. It is known that the functor $\Lambda(Q,\ ) : \mathrm{mod}\Lambda \to \mathrm{mod}\Gamma$ induces an equivalence of categories between $\mathbf{C}^Q_1$ and mod Γ. Moreover, it is proven in **[APT]** that this equivalence carries projective resolutions of modules in $\mathbf{C}^Q_\infty$ into projective resolutions of Γ -modules, proving in particular the following result.

Proposition 3 . [**APT**, Cor. 3.3 a)]. *Let Λ be an artin algebra, Q a projective Λ-module and $M \in \mathbf{C}^Q_\infty$. Then* $\mathrm{pd}_\Lambda M = \mathrm{pd}_\Gamma \Lambda(Q, M)$.

When the artin algebra Λ is weakly triangular we obtain the following corollary.

Corollary 1 . *Let Λ be weakly triangular, $\mathbf{Q}_i = \oplus_{j \le i} \mathbf{P}_j$, $\Gamma_i = \mathrm{End}_\Lambda(\mathbf{Q}_i)^{op}$. Let $M \in$ mod Λ be such that $\tau_{\mathbf{Q}_i}(M) = M$. Then proj $\dim_\Lambda M =$ proj $\dim_{\Gamma_i} \Lambda(\mathbf{Q}_i, M)$.*

PROOF. Since Λ is weakly triangular the subcategories $\mathbf{C}_o^{Q_i}$, $\mathbf{C}_1^{Q_i}$ and $\mathbf{C}_\infty^{Q_i}$ coincide. The Corollary follows by observing that $\tau_{Q_i}(M) = M$ if and only if $M \in \mathbf{C}_o^{Q_i}$. □

Lemma 1 . *Let us assume that Λ is weakly triangular and that $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$ have finite projective dimension. Then $\Gamma_i = \mathrm{End}_\Lambda(\mathbf{Q}_i)^{op}$ has the same properties for each $i \leq n$.*

PROOF. We get that Γ_i is weakly triangular from the fact that $\Lambda(\mathbf{Q}_i, -)$ defines an equivalence of categories between add$\mathbf{Q}_i$ and the category of projective Γ_i-modules.

We observe next that, if Q is a projective Λ-module and if $P, X \in \mathbf{C}_1^Q$, then

$$\tau_{\Lambda(Q,P)}\Lambda(Q,X) = \Lambda(Q, \tau_P X).$$

It follows then that

$$\mathbf{A}_{\Gamma_i,t} =: \Lambda(\mathbf{Q}_i, \mathbf{P}_t)/\tau_{\oplus_{j<t}\Lambda(\mathbf{Q}_i,\mathbf{P}_j)}\Lambda(\mathbf{Q}_i, \mathbf{P}_t) \cong \Lambda(\mathbf{Q}_i, \mathbf{A}_t).$$

On the other hand, our hypothesis and Corollary 1 imply that the Γ_i-projective dimension of $\Lambda(\mathbf{Q}_i, \mathbf{A}_t)$ is finite. □

We state for convenience the following result of [CP].

Lemma 2 . ([CP], Lemma 2.2. *Let us assume that Λ is weakly triangular and that all $\tau_{\mathbf{Q}_i}(\Lambda)\,(i = 1, 2, \cdots, n)$ are in $\mathcal{P}^{<\infty}(\Lambda)$. Then, $M \in \mathcal{P}^{<\infty}(\Lambda)$ implies $\tau_{\mathbf{Q}_i}(M) \in \mathcal{P}^{<\infty}(\Lambda)$ for all $i \leq n$.*

Theorem 1 . *Let Λ be a weakly triangular artin algebra. Then, the following conditions are equivalent.*

$$(i)\ \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n \in \mathcal{P}^{<\infty}(\Lambda)$$

$$(ii)\ \tau_{\mathbf{Q}_i}(\Lambda) \in \mathcal{P}^{<\infty}(\Lambda)\ \forall i = 1, 2, \cdots, n$$

PROOF. Obviously, (ii) implies (i). To prove the converse we assume that (i) holds, and observe that proving (ii) amounts to proving that $\tau_{\mathbf{Q}_i}(\mathbf{P}_j)$ is in $\mathcal{P}^{<\infty}(\Lambda)$ for all $i, j = 1, 2, \cdots, n$. We prove this by induction on j. We observe that the statement holds for $j = 1$ and assume that it is true for $j \leq k - 1$. We will prove that $\tau_{\mathbf{Q}_i}(\mathbf{P}_k) \in \mathcal{P}^{<\infty}(\Lambda)\,\forall i = 1, 2, \cdots, n$ by induction on n.

Since we are assuming that the projective dimension of $\mathbf{A}_k$ is finite it follows that $\tau_{\mathbf{Q}_{k-1}}(\mathbf{P}_k)$ has finite projective dimension. Thus, by Corollary 1, we get that

$$\mathrm{pd}_{\Gamma_{k-1}}\Lambda(\mathbf{Q}_{k-1}, \tau_{\mathbf{Q}_{k-1}}(\mathbf{P}_k)) < \infty.$$

It follows from Lemma 1 that the induction hypothesis applies to the algebra Γ_{k-1}. Then we can apply Lemma 2 to the Γ_{k-1}-module $\Lambda(\mathbf{Q}_{k-1}, \tau_{\mathbf{Q}_{k-1}}(\mathbf{P}_k))$ and conclude that

$$\Lambda(\mathbf{Q}_{k-1}, \tau_{\mathbf{Q}_i}(\tau_{\mathbf{Q}_{k-1}}(\mathbf{P}_k))) = \tau_{\Lambda(\mathbf{Q}_{k-1},\mathbf{Q}_i)}\Lambda(\mathbf{Q}_{k-1}, \tau_{\mathbf{Q}_{k-1}}(\mathbf{P}_k))$$

has finite projective dimension over Γ_{k-1}, for all $i \leq k-1$. But this finishes the proof, because $\tau_{\mathbf{Q}_i}(\tau_{\mathbf{Q}_{k-1}}(\mathbf{P}_k)) = \tau_{\mathbf{Q}_i}(\mathbf{P}_k)$ and, applying Corollary 1 once more, we obtain $\tau_{\mathbf{Q}_i}(\mathbf{P}_k) \in \mathcal{P}^{<\infty}(\Lambda)$, as desired. □

The results proven in the following theorem were proven in **[CP]** for weakly triangular algebras such that $\mathrm{pd}_\Lambda \tau_{\mathbf{Q}_i}(\Lambda)$ is finite for all $i = 1, 2, \cdots, n$. In view of Theorem 1, we replace the last condition by the assumption that $\mathbf{A}_1, \cdots, \mathbf{A}_n$ have finite projective dimension.

Theorem 2 . [CP] *Let* Λ *be a weakly triangular artin ring such that all the* $\mathbf{A}_i$ $(i = 1, 2, ..., n)$ *have finite projective dimension and let* $M \in \mathrm{mod}(\Lambda)$. *Then*

1. *If* M *is in* $\mathcal{P}^{<\infty}(\Lambda)$ *then* $\tau_{\mathbf{Q}_j}(M)$ *is in* $\mathcal{P}^{<\infty}(\Lambda)$, *for all* $j = 1, 2, ..., n$, *and the factor* $\tau_{\mathbf{Q}_j}(M)/\tau_{\mathbf{Q}_{j-1}}(M)$ *is a free* $\mathbf{A}_j$*-module.*
2. M *is in* $\mathcal{P}^{<\infty}(\Lambda)$ *if and only if* M *admits a filtration with factors in* $\{\mathbf{A}_i / i = 1, 2, ..., n\}$.
3. *The finitistic projective dimension of* Λ *is the maximum of the projective dimensions of* $\mathbf{A}_1, \cdots, \mathbf{A}_n$.
4. $\mathcal{P}^{<\infty}(\Lambda)$ *is functorially finite, closed under extensions and hence has Auslander-Reiten sequences.*

It follows from the last example in **[CP]** that the hypothesis of the theorem do not imply that all idempotent ideals of Λ have finite projective dimension.

2 The $\mathcal{P}^{<\infty}(\Lambda)$-injective modules.

Throughout this section we assume that Λ is weakly triangular and the modules $\mathbf{A}_i = \mathbf{P}_i/\tau_{\mathbf{P}_i}(\mathbf{P}_i)$ have finite projective dimension, for $i = 1, \cdots, n$. These hypothesis are met by weakly triangular algebras having all idempotent ideals of finite projective dimension.

We will study the relative injective modules in $\mathcal{P}^{<\infty}(\Lambda)$. We know that the modules in $\mathcal{P}^{<\infty}(\Lambda)$ are those admitting a filtration in the set $\{\mathbf{A}_1, \cdots, \mathbf{A}_n\}$, and $\mathcal{P}^{<\infty}(\Lambda)$ is a functorially finite subcategory of mod Λ. The results in this section strengthen and generalize similar results proven in [CMMMP] for algebras with all idempotent ideals projective.

We will refer to notions relative to $\mathcal{P}^{<\infty}(\Lambda)$, such as $\mathcal{P}^{<\infty}(\Lambda)$ -injective, $\mathcal{P}^{<\infty}(\Lambda)$-projective, $\mathcal{P}^{<\infty}(\Lambda)$-simple, etc. We will show that one can define the $\mathcal{P}^{<\infty}(\Lambda)$-socle of a module in $\mathcal{P}^{<\infty}(\Lambda)$. It is clear that the $\mathcal{P}^{<\infty}(\Lambda)$-projective objects are the projective Λ-modules. On the other hand, using the fact that $\mathcal{P}^{<\infty}(\Lambda)$ is closed under cokernels of monomorphism, it follows that the $\mathcal{P}^{<\infty}(\Lambda)$-injective modules coincide with the Ext-injective modules in $\mathcal{P}^{<\infty}(\Lambda)$, that is, with the modules I such that $\operatorname{Ext}^1_\Lambda(X, I) = 0$ for all X in $\mathcal{P}^{<\infty}(\Lambda)$. Or, equivalently, with the modules I such that any exact sequence in $\mathcal{P}^{<\infty}(\Lambda)$ starting at I splits.

Since $\mathcal{P}^{<\infty}(\Lambda)$ is functorially finite we can give a different description of the $\mathcal{P}^{<\infty}(\Lambda)$-injective modules in terms of the $\mathcal{P}^{<\infty}(\Lambda)$-approximations of the indecomposable injective Λ-modules. Although these approximations may decompose, we will prove that they have only one indecomposable summand, up to isomorphism. The multiplicity of such summand will also be determined.

Definition 1 . *We say that the $\mathcal{P}^{<\infty}(\Lambda)$-socle of $M \in \mathcal{P}^{<\infty}(\Lambda)$ is defined and is equal to V when $M \in \mathcal{P}^{<\infty}(\Lambda)$ has a unique maximal $\mathcal{P}^{<\infty}(\Lambda)$-semisimple submodule, V.*

We start by proving that the $\mathcal{P}^{<\infty}(\Lambda)$-socle is defined for all $M \in \mathcal{P}^{<\infty}(\Lambda)$.

Proposition 4 *Assume Λ is weakly triangular and the modules $\mathbf{A}_1, \cdots, \mathbf{A}_n$ have finite projective dimension. Then, for $M \in \mathcal{P}^{<\infty}(\Lambda)$, the family of $\mathcal{P}^{<\infty}(\Lambda)$-semisimple submodules of M has a maximum element.*

PROOF. We write again $\mathbf{Q}_j = \oplus_{i \leq j} \mathbf{P}_i$. Let $M \in \mathcal{P}^{<\infty}(\Lambda)$. We know by Theorem 1 that $\tau_{Q_j}(M) \in \mathcal{P}^{<\infty}(\Lambda)$ and

$$\tau_{Q_j}(M)/\tau_{Q_{j-1}}(M) \simeq \mathbf{A}_j^{n_j},$$

for some $n_j \geq 0$ and for all $j = 1, 2, ..., n$.

We will prove the proposition by induction on the minimal number k such that $\tau_{Q_k}(M) = M$. If $k = 1$ then $M \simeq \mathbf{A}_1^{n_1}$ and the result is true. Let k be greater than 1. We assume that the proposition holds for modules N such that $\tau_{Q_{k-1}}(N) = N$ and let $M \in \mathcal{P}^{<\infty}(\Lambda)$ be such that $\tau_{Q_k}(M) = M$. We may assume that M is indecomposable. Since $\tau_{Q_{k-1}}(M) \in \mathcal{P}^{<\infty}(\Lambda)$ and $\tau_{Q_{k-1}}(\tau_{Q_{k-1}}(M)) = \tau_{Q_{k-1}}(M)$ we can apply the induction hypothesis and conclude that the socle of $\tau_{Q_{k-1}}(M)$ is defined.

We will prove that either $\mathbf{A}_k \simeq M$ or $\mathrm{soc}_{\mathcal{P}^{<\infty}(\Lambda)}(M)$ is defined and coincides with $\mathrm{soc}_{\mathcal{P}^{<\infty}(\Lambda)}(\tau_{Q_{k-1}}(M))$. To do so we consider a $\mathcal{P}^{<\infty}(\Lambda)$-simple submodule X of M and prove that either $M \simeq \mathbf{A}_k$ or $X \subseteq \tau_{Q_{k-1}}(M)$. If $X \simeq \mathbf{A}_i$ with $i < k$ then the second assertion holds. So we assume $X \simeq \mathbf{A}_k$, and let $j : X \to M$ be the inclusion and $\pi : M \to \tau_{Q_k}(M)/\tau_{Q_{k-1}}(M)$ the canonical map. Since the only composition factor of X is $\mathbf{S}_k$, which is not a composition factor of $\tau_{Q_{k-1}}(M)$ it follows that $X \cap \tau_{Q_{k-1}}(M) = 0$, so πj is a monomorphisms.

Since $\mathbf{A}_k$ is a local artinian ring then any monomorphism $f : \mathbf{A}_k \to \mathbf{A}_k^{n_k}$ splits. This follows from the fact that $0 \to \mathbf{A}_k \to \mathbf{A}_k^{n_k} \to \mathrm{Coker}(f) \to 0$ is a projective resolution of Coker(f) and is not minimal.

Since $X \simeq \mathbf{A}_k$ and $\tau_{Q_k}(M)/\tau_{Q_{k-1}}(M) \simeq \mathbf{A}_k^{n_k}$, for some $n_k \geq 0$, it follows that the monomorphism πj splits. Thus $j : X \to M$ splits. Since M is indecomposable this proves that $M \simeq X \simeq \mathbf{A}_k$, ending the proof of the proposition. □

We go on now to study the $\mathcal{P}^{<\infty}(\Lambda)$-injective modules. The following proposition proves in particular that the $\mathcal{P}^{<\infty}(\Lambda)$-injective indecomposable modules have $\mathcal{P}^{<\infty}(\Lambda)$-simple socle and are the $\mathcal{P}^{<\infty}(\Lambda)$-injective envelopes (in the sense that the corresponding inclusions are minimal morphisms) of $\mathbf{A}_1, \cdots, \mathbf{A}_n$.

Proposition 5 . *Assume Λ is weakly triangular and the modules $\mathbf{A}_1, \cdots, \mathbf{A}_n$ have finite projective dimension. Let*

$$\widetilde{\mathbf{I}}_i \xrightarrow{\phi_i} \mathbf{I}_i$$

be the minimal right $\mathcal{P}^{<\infty}(\Lambda)$-approximation of the indecomposable injective Λ-module $\mathbf{I}_i$, and let $\widetilde{\mathbf{I}}_i = \coprod_j \widetilde{\mathbf{I}}_{ij}$, with $\widetilde{\mathbf{I}}_{ij}$ indecomposable. Then

1. *$\{\widetilde{\mathbf{I}}_{ij}\}_{i,j}$ is the set of indecomposable $\mathcal{P}^{<\infty}(\Lambda)$-injective objects, up to isomorphism. Thus the category $\mathcal{P}^{<\infty}(\Lambda)$ has enough injectives.*
2. *$\mathbf{A}_i$ is not a $\mathcal{P}^{<\infty}(\Lambda)$-composition factor of $\widetilde{\mathbf{I}}_j$, for $i < j$.*
3. *$\widetilde{\mathbf{I}}_{nj} \cong \mathbf{A}_n$, for all j.*
4. *$\widetilde{\mathbf{I}}_{ij} \simeq \widetilde{\mathbf{I}}_{i1}$, for all i, j.*
5. *$\mathrm{soc}_{\mathcal{P}^{<\infty}(\Lambda)}(\widetilde{\mathbf{I}}_{i1}) = \mathbf{A}_i$.*

6. $\tilde{\mathbf{I}}_i = (\tilde{\mathbf{I}}_{i1})^{n_i}$, *where* n_i *is the smallest number of copies of* $\mathbf{A}_i$ *necessary to cover the* $\mathbf{A}_i$*-injective envelope* $\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i)$ *of* $\mathbf{S}_i$. *That is,* n_i *is the length of* $\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i)/rad(\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i))$.

1) Since $\mathcal{P}^{<\infty}(\Lambda)$ contains the projective Λ-modules, all the approximations $\tilde{\mathbf{I}}_i \xrightarrow{\phi_i} \mathbf{I}_i$ are epimorphism. We start by proving that all $\tilde{\mathbf{I}}_i$ are $\mathcal{P}^{<\infty}(\Lambda)$-injective. Let

$$0 \to \tilde{\mathbf{I}}_i \xrightarrow{f} X \xrightarrow{g} Y \to 0$$

be an exact sequence in $\mathcal{P}^{<\infty}(\Lambda)$. According to the remark at the beginning of this section we only need to prove that this sequence splits. Since f is a monomorphism and $\mathbf{I}_i$ is injective there is a morphism $t : X \to \mathbf{I}_i$ such that $\phi_i = tf$. Since X is in $\mathcal{P}^{<\infty}(\Lambda)$ then $t : X \to \mathbf{I}_i$ factors through the $\mathcal{P}^{<\infty}(\Lambda)$-approximation $\tilde{\mathbf{I}}_i \xrightarrow{\phi_i} \mathbf{I}_i$ That is, there is $h : X \to \tilde{\mathbf{I}}_i$ such that $t = \phi_i h$. So $\phi_i = \phi_i h f$. The minimality of ϕ_i implies that hf is the identity map, so f splits, as required.

To prove the converse, let J be an indecomposable $\mathcal{P}^{<\infty}(\Lambda)$-injective module in $\mathcal{P}^{<\infty}(\Lambda)$, and let $j : J \to I(J)$ be its injective envelope in mod Λ. Since J is in $\mathcal{P}^{<\infty}(\Lambda)$ the map j factors through the $\mathcal{P}^{<\infty}(\Lambda)$-approximation

$$\tilde{\mathbf{I}}(J) \xrightarrow{\phi} \mathbf{I}(J)$$

Thus there is $h : J \to \tilde{\mathbf{I}}(J)$ such that $j = \phi h$. Since j is a monomorphism h is a monomorphism too and therefore it splits, because J is $\mathcal{P}^{<\infty}(\Lambda)$-injective. This ends the proof of 1).

2) We observe first that $\Lambda(\mathbf{P}_i, \mathbf{I}_j) = 0$ for $i < j$, because in this case $\Lambda(\mathbf{P}_j, \mathbf{P}_i) = 0$. So the minimal left $\mathcal{P}^{<\infty}(\Lambda)$ -approximation

$$\tilde{\mathbf{I}}_j \xrightarrow{\phi_j} \mathbf{I}_j$$

vanishes on the trace of $\mathbf{Q}_i$ in $\tilde{\mathbf{I}}_j$, for $i < j$

Hence, ϕ_j factors through $\tilde{\mathbf{I}}_j/\tau_{\mathbf{Q}_i}(\tilde{\mathbf{I}}_j)$. Since this quotient is in $\mathcal{P}^{<\infty}(\Lambda)$ (see Theorem 1) we deduce from the minimality of ϕ_j that the trace of $\mathbf{Q}_i$ in $\tilde{\mathbf{I}}_j$ is 0, proving 2).

3) We know by 2) that the only $\mathcal{P}^{<\infty}(\Lambda)$-composition factor of $\tilde{\mathbf{I}}_n$ is $\mathbf{A}_n$ So for each summand $\tilde{\mathbf{I}}_{nj}$ of $\mathbf{I}_n$ there is an epimorphism $f : \tilde{\mathbf{I}}_{nj} \to \mathbf{A}_n$ in mod $\mathbf{A}_n$. Since $\tilde{\mathbf{I}}_{nj}$ is indecomposable it follows that f is an isomorphism, proving 3).

4) Since the approximation $\tilde{\mathbf{I}}_i \xrightarrow{\phi_i} \mathbf{I}_i$ is minimal it follows that $\mathbf{S}_i$ is a composition factor of all summands $\tilde{\mathbf{I}}_{ij}$ of $\tilde{\mathbf{I}}_i$. Thus $\tau_{P_i}(\tilde{\mathbf{I}}_{ij}) \neq 0$ for all j. Since we know by 2) that $\mathrm{Hom}_{\Lambda}(\mathbf{P}_k, \tilde{\mathbf{I}}_i) = 0$ for all $k < i$, it follows that $\tau_{\mathbf{P}_i}(\tilde{\mathbf{I}}_{ij}) =$

$\tau_{\mathbf{Q}_i}(\tilde{\mathbf{I}}_{ij})/\tau_{\mathbf{Q}_{i-1}}(\tilde{\mathbf{I}}_{ij})$. We know then by Theorem 1 that $\tau_{\mathbf{P}_i}(\tilde{\mathbf{I}}_{ij})$ is a free $\mathbf{A}_i$-module. That is, $\tau_{\mathbf{P}_i}(\tilde{\mathbf{I}}_{ij}) = \mathbf{A}_i^{k_i}$, for some $k_i > 0$.

We can therefore consider monomorphisms $\mathbf{A}_i \to \tilde{\mathbf{I}}_{ij}$, $\mathbf{A}_i \to \tilde{\mathbf{I}}_{i1}$. Since both $\tilde{\mathbf{I}}_{ij}$ and $\tilde{\mathbf{I}}_{i1}$ are $\mathcal{P}^{<\infty}(\Lambda)$-injective and indecomposable, we get that they are isomorphic. This proves 4).

5) Using that $\Lambda(\mathbf{A}_j, \mathbf{I}_i) = 0$ and the minimality of the approximation ϕ_i it follows that $\mathbf{A}_j$ is not contained in $\tilde{\mathbf{I}}_i$, for $j \neq i$. Thus, to prove 5) we only need to prove that the integer k_i above considered is 1. We start by observing the following consequence of 2). If $M \in \mathcal{P}^{<\infty}(\Lambda)$ has $\mathcal{P}^{<\infty}(\Lambda)$-composition factors in $\{\mathbf{A}_i, \cdots, \mathbf{A}_n\}$, then all the modules that occur in a minimal injective $\mathcal{P}^{<\infty}(\Lambda)$-coresolution of M have the same property. This fact follows by induction on the length of the coresolution, and using 2). We know that the length of the coresolution is finite because the projective dimension of the modules in $\mathcal{P}^{<\infty}(\Lambda)$ is bounded.

To prove 5) we proceed by decreasing induction on i, using that the statement is true for $i = n$, by 3). Let

$$0 \to \mathbf{A}_i \to E_1 \to E_2 \to \cdots \to E_t \to 0$$

be a minimal $\mathcal{P}^{<\infty}(\Lambda)$-injective coresolution of $\mathbf{A}_i$.

Let m_k denote the number of times that $\tilde{\mathbf{I}}_{i1}$ occurs as a summand of E_k, and let n_i be the multiplicity of $\mathbf{A}_i$ as $\mathcal{P}^{<\infty}(\Lambda)$ -composition factor of $\tilde{\mathbf{I}}_{i1}$. We want to prove that $n_i = 1$. From the above observation we know that the E_i's have $\mathcal{P}^{<\infty}(\Lambda)$-composition factors in $\{\mathbf{A}_i, \cdots, \mathbf{A}_n\}$. So the only possible summands of E_i are $\tilde{\mathbf{I}}_{i1}, \cdots, \tilde{\mathbf{I}}_{n1}$. From 2) we know that $\mathbf{A}_i$ is not a $\mathcal{P}^{<\infty}(\Lambda)$-composition factor of $\tilde{\mathbf{I}}_{j1}$ for $j > i$. Thus the multiplicity of $\mathbf{A}_i$ in E_k is $m_k n_i$.Then, from the above coresolution, we obtain

$$1 + \sum_{k=1}^{t} (-1)^k m_k n_i = 0$$

from which it follows that n_i divides 1. So $n_i = 1$, as required.

6) Since $\mathbf{A}_i = \Lambda/\tau_{\mathbf{P}_i}\Lambda$ then the $\mathbf{A}_i$-injective envelope of $\mathbf{S}_i$ is $\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i) = \{x \in \mathbf{I}_i : \tau_{\mathbf{P}_i}\Lambda.x = 0\}$.

Let $\pi : \mathbf{A}_i^{k_i} \to \mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i)$ be the $\mathbf{A}_i$-projective cover of $\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i)$, and let $j : \mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i) \to \mathbf{I}_i$ be the inclusion map. Then $j\pi$ factors through the minimal approximation $\phi_i : \tilde{\mathbf{I}}_i \to \mathbf{I}_i$. Since the domain of π is $\mathcal{P}^{<\infty}(\Lambda)$ -semisimple then it factors also through the $\mathcal{P}^{<\infty}(\Lambda)$-socle $\mathbf{A}_i^{n_i}$ of $\tilde{\mathbf{I}}_i$. But this says that $\mathbf{A}_i^{n_i}$ also covers $\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i) = j(\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i))$, implying that $\phi_i \mid_{\mathbf{A}_i^{n_i}}$ factors as $\pi\pi'$, where π' is a split epimorphism. Let $\mathbf{A}_i^{n_i} \cong \mathbf{A}_i^{k_i} \oplus \mathbf{A}_i^{n_i - k_i}$ be the induced factorization. Since $\tilde{\mathbf{I}}_{i1}$ is the $\mathcal{P}^{<\infty}(\Lambda)$-injective envelope of $\mathbf{A}_i$ we obtain a decomposition $\tilde{\mathbf{I}}_i \simeq \tilde{\mathbf{I}}_{i1}^{k_i} \oplus \tilde{\mathbf{I}}_{i1}^{n_i - k_i}$ so that ϕ_i vanishes on the second summand

$\tilde{\mathbf{I}}_{i1}^{n_i-k_i}$. Since the approximation ϕ_i is minimal, this implies that $n_i - k_i = 0$, as desired. □

Proposition 5 shows that though the approximation of an indecomposable injective module has only one indecomposable summand up to isomorphism, it is not indecomposable. In fact the multiplicity of such summand can be arbitrarily large, as shown in the following example.

Example 1 . *Let Λ be a basic radical square zero local artin algebra of length n. Let $\tilde{\mathbf{I}}$ be the minimal $\mathcal{P}^{<\infty}(\Lambda)$-approximation of the indecomposable injective Λ-module* **I**. *Then the multiplicity of the indecomposable $\mathcal{P}^{<\infty}(\Lambda)$-injective module as a summand of $\tilde{\mathbf{I}}$ is $n-1$.*

3 On the defining relations for the Grothendieck group.

As in the preceding section we assume throughout this one that the artin algebra Λ is weakly triangular and the modules $\mathbf{A}_i = \mathbf{P}_i/\tau_{\hat{\mathbf{P}}_i}(\mathbf{P}_i)$ have finite projective dimension, for $i = 1, \cdots, n$. We know that under these hypotheses $\mathcal{P}^{<\infty}(\Lambda)$ is functorially finite, and has therefore almost split sequences.

In [**A**] M. Auslander has shown for an artin algebra Λ that the relations given by the almost split sequences generate the group of all relations for the Grothendieck group of Λ if and only if A is of finite representation type. The sufficiency of this latter condition had been previously proved by Butler in [**B**]. In this section we will prove this statement for the category $\mathcal{P}^{<\infty}(\Lambda)$, using methods similar to those used by Auslander in [**A**].

We will denote by $\mathbf{K}_0(\mathcal{P}^{<\infty})$ the Grothendieck group of the category $\mathcal{P}^{<\infty}(\Lambda)$ and by $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ the free abelian group with basis a complete set of indecomposable objects in $\mathcal{P}^{<\infty}(\Lambda)$.

We observe that $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ is the quotient of the free abelian group generated by the isomorphism classes $[M]$ of elements M in $\mathcal{P}^{<\infty}(\Lambda)$ by the subgroup generated by all relations of the form $[A] + [C] - [B]$, where $0 \to A \to B \to C \to 0$ is a short split exact sequence in $\mathcal{P}^{<\infty}(\Lambda)$. We denote also by $[M]$ the element of $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$, and by (M) the element of $\mathbf{K}_0(\mathcal{P}^{<\infty})$, determined by the module M in $\mathcal{P}^{<\infty}(\Lambda)$.

Then $\mathbf{K}_0(\mathcal{P}^{<\infty})$ is the quotient of $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ by the subgroup generated by the relations of the form $[A]+[C]-[B]$ such that there is an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{P}^{<\infty}(\Lambda)$.

In all that follows **ES** will denote this latter subgroup and **AR** the subgroup of $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ generated by the elements of the form $[A] + [C] - [B]$ such that there is a (relative) almost split sequence $0 \to A \to B \to C \to 0$ in $\mathcal{P}^{<\infty}(\Lambda)$. We will say that $\mathcal{P}^{<\infty}(\Lambda)$ is of finite type if $\mathcal{P}^{<\infty}(\Lambda)$ contains only a finite number of nonisomorphic indecomposable objects. We will prove that $\mathbf{AR} = \mathbf{ES}$ if and only if $\mathcal{P}^{<\infty}(\Lambda)$ is of finite type.

We fix here some further notations which will be used in all that follows.

Following [**A**] we consider the bilinear form $< , >$ on $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ which satisfies that $< [M], [N] >$ is the length of the R-module $\Lambda(M, N)$, for all M, N in $\mathcal{P}^{<\infty}(\Lambda)$.

For each indecomposable and not projective module C in $\mathcal{P}^{<\infty}(\Lambda)$, r_C denotes the element $[A]+[C]-[B]$ of $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$, where $0 \to A \to B \to C \to 0$ is the almost split sequence in $\mathcal{P}^{<\infty}(\Lambda)$ ending at C. Dually, if $A \in \mathcal{P}^{<\infty}(\Lambda)$ is indecomposable and not injective we write $l_A = [A] + [C] - [B]$, where $0 \to A \to B \to C \to 0$ is the almost split sequence in $\mathcal{P}^{<\infty}(\Lambda)$ beginning at A.

For each $i \leq n$ we consider in $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ the element $a_i = [\mathbf{P}_i] - [\tau_{\hat{\mathbf{P}}_i}(\mathbf{P}_i)]$. Then the element in $\mathbf{K}_0(\mathcal{P}^{<\infty})$ corresponding to a_i is $(\mathbf{A}_i)$.

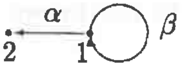
The following proposition follows directly from the fact that the modules in $\mathcal{P}^{<\infty}(\Lambda)$ are those having a filtration with factors amongst the $\mathbf{A}_i$'s.

Proposition 6 . *$\mathbf{K}_0(\mathcal{P}^{<\infty})$ is a free abelian group with basis the classes of the modules $\mathbf{A}_1, \cdots, \mathbf{A}_n$. Hence $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ is generated by $\mathbf{ES}$ and the elements $a_1, \cdots, a_n$.*

For nonprojective indecomposable modules in $\mathcal{P}^{<\infty}(\Lambda)$ the elements r_C above defined have the following important property: If $X \in \mathcal{P}^{<\infty}(\Lambda)$ then $< [X], r_C >= 0$ if and only if X is not isomorphic to C. A dual result holds for the elements l_A.

Many of the results and arguments here are a direct generalization of those in [**A**]. However, there are some essential differences, so that the proof in [**A**] cannot be directly carried over. One important difference is the following. For each projective indecomposable module $\mathbf{P}_i$ Auslander considers the element $r_{\mathbf{P}_i} = [\mathbf{P}_i] - [r\mathbf{P}_i]$ in the Grothendieck group of Λ. Then if X is an indecomposable Λ-module, $< [X], r_{\mathbf{P}_i} > \neq 0$ implies that X is isomorphic to $\mathbf{P}_i$. However, the element we consider in $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ instead of $r_{\mathbf{P}_i}$ is $a_i = [\mathbf{P}_i] - [\tau_{\hat{\mathbf{P}}_i}(\mathbf{P}_i)]$, and it doesn't have the corresponding property. To see this, we consider the algebra Λ and the module M of the example in [**P**] p. 2529.

Example 2 . *Let Λ be the k-algebra given by the following quiver Q with the relation $\beta^2 = 0$.*



Let M be given by the following representation

$$\begin{tikzcd} k & k^2 \arrow[l, "(0\,1)"'] \arrow[loop right, "\begin{pmatrix}0&1\\0&0\end{pmatrix}"] \end{tikzcd}$$

Then $a_1 = [\mathbf{P}_1] - [\tau_{\mathbf{P}_2}(\mathbf{P}_1)]$ and

$$\Lambda(M, \mathbf{P}_1) \neq 0, \ \Lambda(M, \tau_{\mathbf{P}_2}(\mathbf{P}_1)) = 0.$$

Thus $< [M], a_1 > \neq 0$, with M not isomorphic to $\mathbf{P}_1$.

We also observe that in this example the $\mathcal{P}^{<\infty}(\Lambda)$-approximation of $\mathbf{S}_1$ is not $\mathbf{A}_1$ but M, as observed in [P],p 2529.

The following result will be useful to decide if two elements of $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ are equal.

Proposition 7 . *Let* $x = \sum_M a_M[M]$, $y = \sum_M b_M[M]$ *in* $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$, *where* M *runs over the indecomposable modules in* $\mathcal{P}^{<\infty}(\Lambda)$. *Then the following conditions are equivalent.*

- *(i)* $x = y$
- *(ii)* $< [N], x > = < [N], y >$ *for all indecomposable module* N *in* $\mathcal{P}^{<\infty}(\Lambda)$.
- *(iii)* $< x, [N] > = < y, [N] >$ *for all indecomposable module* N *in* $\mathcal{P}^{<\infty}(\Lambda)$.

PROOF. It is clear that (i) implies (ii) and (iii). If (iii) is true, we have that $< x, r_C > = < y, r_C >$ for every non-projective indecomposable C in $\mathcal{P}^{<\infty}(\Lambda)$. Since $< -, r_C >$ annihilates all summands of x and y except the one with index C we obtain that $a_M = b_M$ if M is not projective. So in order to deduce (i) we may assume that $x = \sum_{i=1}^{n} \alpha_i[\mathbf{P}_i]$ and $y = \sum_{i=1}^{n} \beta_i[\mathbf{P}_i]$. The fact that Λ is weakly triangular implies that $< x, [\mathbf{P}_n] > = \alpha_n . < [\mathbf{P}_n], [\mathbf{P}_n] >$ and $< y, [\mathbf{P}_n] > = \beta_n < [\mathbf{P}_n], [\mathbf{P}_n] >$. Since we are assuming that (iii) holds we get $\alpha_n = \beta_n$. We can then apply the same argument to $x_1 = x - \alpha_n[\mathbf{P}_n]$, $y_1 = y - \beta_n[\mathbf{P}_n]$ and use the projective module $\mathbf{P}_{n-1}$ to conclude that $\alpha_{n-1} = \beta_{n-1}$. Iterating the procedure we prove that (i) holds.

If (ii) is true, using the l_A's instead of the r_C's, the question of proving (i) is reduced to the case when the modules which occur in both x and y are

$\mathcal{P}^{<\infty}(\Lambda)$-injective. But then (i) follows from Prop. 2 by successive application of $<\mathbf{P}_1, - >, \cdots, <\mathbf{P}_n, - >$. □

The next proposition describes a linearly independent family in $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$. We will prove later that this family is a basis of $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ if and only if $\mathcal{P}^{<\infty}(\Lambda)$ is of finite type.

Proposition 8 . *The subset* $\{a_i : i = 1, \cdots, n\} \bigcup \{r_C : C$ *is indecomposable nonprojective in* $\mathcal{P}^{<\infty}(\Lambda)\}$ *of* $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)$ *is linearly independent. Moreover, any linear combination* x *of the elements* r_C *is of the form*

$$x = \sum_C < C, x > / < C, r_C > r_C$$

PROOF. The proof uses arguments similar to those above. Consider a linear combination of elements in the given set. That is, let

$$x = \sum_i k_i a_i + \sum_C b_C r_C$$

with k_i, b_C integers. We observe that, since $< D, r_C >= 0$ for any indecomposable D in $\mathcal{P}^{<\infty}(\Lambda)$ not isomorphic to C, $< [\mathbf{P}_i], - >$ annihilates all summands except the one of index i in the first sum. Thus, since $< [\mathbf{P}_i], a_i > \neq 0$ we obtain that

$$k_i =< [\mathbf{P}_i], x > / < [\mathbf{P}_i], a_i > .$$

Hence, the first sum above is determined by x, and we will denote it with $\overline{x}$. That is, we write $\overline{x} = \sum_i k_i a_i$

We obtain next in a similar way that

$$b_C =< [C], x - \overline{x} > / < [C], r_C >,$$

and the proof is complete. □

For the proof of the main theorem in this section we introduce the following notation. Let us observe that if M is a module in $\mathcal{P}^{<\infty}(\Lambda)$, then the number of times m_i that the $\mathcal{P}^{<\infty}(\Lambda)$-composition factor $\mathbf{A}_i$ appears in M, is exactly

$$< [\mathbf{P}_i], [M] > / < [\mathbf{P}_i], a_i >=< [\mathbf{P}_i], [M] > / < a_i, a_i > .$$

This can be proven by induction, applying the functor $\Lambda(\mathbf{P}_i, -)$ to an exact sequence of the form $0 \to N \to M \to \mathbf{A}_i \to 0$.

In what follows, given M in $\mathcal{P}^{<\infty}(\Lambda)$, we denote by $\hat{M}$ the module $\oplus_i \mathbf{A}_i^{m_i}$, where m_i is the multiplicity of $\mathbf{A}_i$ in M.

Then M and $\hat{M}$ define the same element $\sum_i m_i[A_i]$ in $\mathbf{K}_0(\mathcal{P}^{<\infty})$. Moreover, we can prove the following result.

Lemma 3 . *The kernel* **ES** *of the natural map from* $\mathrm{K}_0(\mathcal{P}^{<\infty},0)$ *onto* $\mathrm{K}_0(\mathcal{P}^{<\infty})$ *is generated by the elements of the form* $[M]-[\hat{M}]$, *with* M *in* $\mathcal{P}^{<\infty}(\Lambda)$.

PROOF. Let M in $\mathcal{P}^{<\infty}(\Lambda)$. We know that $[M]-[\hat{M}]$ is in **ES**. Consider now an exact sequence $0\to A\to B\to C\to 0$ in $\mathcal{P}^{<\infty}(\Lambda)$. Since the $\mathcal{P}^{<\infty}(\Lambda)$-multiplicity of $\mathbf{A}_i$ in B is the sum of $\mathcal{P}^{<\infty}(\Lambda)$-multiplicities of $\mathbf{A}_i$ in A and C, then $[\hat{B}]=[\hat{A}]+[\hat{C}]$. Thus $[A]+[C]-[B]=[A]+[C]-[B]-([\hat{A}]+[\hat{C}]-[\hat{B}])=([A]-[\hat{A}])+([C]-[\hat{C}])-([B]-[\hat{B}])$, proving the lemma. □

We prove next the main theorem in this section.

Theorem 3 . *The relations given by the almost split sequences generate the defining relations of the Grothendieck group of* $\mathcal{P}^{<\infty}(\Lambda)$, *that is* **AR** = **ES**, *if and only if* $\mathcal{P}^{<\infty}(\Lambda)$ *is of finite type. Moreover, if* $\mathcal{P}^{<\infty}(\Lambda)$ *is of finite type then*

a) $\{a_i : i=1,\cdots,n\}\bigcup\{r_C : C$ *is indecomposable nonprojective in* $\mathcal{P}^{<\infty}(\Lambda)\}$ *is a free basis for* $\mathrm{K}_0(\mathcal{P}^{<\infty},0)$

b) $\{r_C : C$ *is indecomposable nonprojective in* $\mathcal{P}^{<\infty}(\Lambda)\}$ *is a free basis for the kernel of the natural map from* $\mathrm{K}_0(\mathcal{P}^{<\infty},0)$ *onto* $\mathrm{K}_0(\mathcal{P}^{<\infty})$.

PROOF. Let us suppose that $\mathcal{P}^{<\infty}(\Lambda)$ is of finite type and let M be in $\mathcal{P}^{<\infty}(\Lambda)$. According to the preceding lemma we only need to prove that $[M]-[\hat{M}]$ is in **AR**. So we need to write $[M]-[\hat{M}]$ as a linear combination of the elements r_C. By Prop. 8 this amounts to prove that

$$[M]-[\hat{M}]=\sum_C(<[C],[M]-[\hat{M}]>/<[C],r_C>)r_C.$$

So we write

$$x=[\hat{M}]+\sum_C(<[C],[M]-[\hat{M}]>/<[C],r_C>)r_C,$$

and prove that $x=[M]$.

¿¿From the definition of $\hat{M}$ and the properties of the r_C's it follows that $<\mathbf{P}_i,x>=<\mathbf{P}_i,[M]>$ for all $i=1,\cdots,n$ and $<[C],x>=<[C],[M]>$ for all indecomposable nonprojective C in $\mathcal{P}^{<\infty}(\Lambda)$. If follows then from Prop. 7 that $x=[M]$, as desired.

Let us assume now that **AR** = **ES**, so that each $[M]-[\hat{M}]$ is a linear combination of the r_C's. Then the coefficient of each r_C is a nonzero multiple of $<[C],[M]-[\hat{M}]>$, as we proved in Prop. 8. Hence for each M there is only a finite number of nonisomorphic indecomposable modules C in $\mathcal{P}^{<\infty}(\Lambda)$ such that $<[C],[M]-[\hat{M}]>\neq 0$. Let us denote by U the set of isomorphism classes of elements in ind $\mathcal{P}^{<\infty}(\Lambda)$ that satisfy this condition for at least one of the modules $\tilde{\mathbf{I}}_i$ or $\tilde{\mathbf{I}}_i/\mathbf{A}_i$. Then U is a finite set. According to the definition, $C \notin U$ if and only if $< [C], [\tilde{\mathbf{I}}_i] >=< [C], [\hat{\tilde{\mathbf{I}}}_i] >$ and $< [C], [\tilde{\mathbf{I}}_i/\mathbf{A}_i] >=< [C], [\tilde{\tilde{\mathbf{I}}}_i/\mathbf{A}_i] >$ for all $i = 1, \cdots, n$.

We claim now that if an indecomposable module C of $\mathcal{P}^{<\infty}(\Lambda)$ is not in U then C is projective. To prove this we apply $\Lambda(C, -)$ to the exact sequence $0 \to \mathbf{A}_i \to \tilde{\mathbf{I}}_i \to \tilde{\mathbf{I}}_i/\mathbf{A}_i \to 0$, so that $[\hat{\tilde{\mathbf{I}}}_i] = [\mathbf{A}_i] + [\hat{\tilde{\mathbf{I}}}_i/\mathbf{A}_i]$. Then, using that C is not in U, we obtain that the sequence $0 \to \Lambda(C, \mathbf{A}_i) \to \Lambda(C, \hat{\tilde{\mathbf{I}}}_i) \to \Lambda(C, \widehat{\tilde{\mathbf{I}}_i/\mathbf{A}_i}) \to 0$ is exact. This shows that $\mathrm{Ext}^1_\Lambda(C, \mathbf{A}_i) = 0$ for each i. With a standard induction argument on the $\mathcal{P}^{<\infty}(\Lambda)$-length of M it follows that that $\mathrm{Ext}^1_\Lambda(C, M) = 0$ for each M in $\mathcal{P}^{<\infty}(\Lambda)$. So C is projective, proving our claim.

Finally, we see that what we proved amounts to say that the set of isomorphism classes of indecomposable modules in $\mathcal{P}^{<\infty}(\Lambda)$ consists of the finite set U and the n indecomposable projective modules. Thus $\mathcal{P}^{<\infty}(\Lambda)$ is of finite type, and the first part of the theorem is proved. The remaining statements follow now from Prop. 8. □

Remark 1 . *We finish this section observing that in our situation the classes of indecomposable projective modules also comprise a free generating set for* $\mathbf{K}_0(\mathcal{P}^{<\infty})$. *Therefore if we write every indecomposable projective* $\mathbf{P}_i$ *in the basis of the* $\mathbf{A}'_j$*s, then we have an invertible matrix in the set of the integers.*

Obse. Let Λ be a finite dimensional algebra. Let $\{\Gamma_i\}_{i \in I}$ be the set of components of its Auslander Reiten quiver. Let us denote also by Γ_i the full subcategory of mod-Λ whose objects are the modules in Γ_i. Then the Γ_i's, clearly, have almost split sequences and the almost split sequences in Γ_i are the almost split sequences of mod-Λ whose terms are in Γ_i. It is not hard to see that $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)/\mathbf{AR}) \cong \coprod \mathbf{K}_o(\Gamma_i, 0)/\mathbf{AR}(\Gamma_i)$.

Moreover, for each component of the right hand side of the above sum we have a map to the integers induced by the length. Therefore we get that $rank(\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)/\mathbf{AR}) \geq \#I$.

We would like to observe that there are counterexamples to the property in Auslander-Butler theorem, in the sense that there exist subcategories with almost split sequences and of infinite type, such that the defining relations of their Grothendieck group are given by the group **AR**, cf([G]).

Let us take Λ to be a hereditary algebra and let $\mathcal{P}_0$ denote the subcategory defined by the preprojective component of the Auslander-Reiten quiver of Λ. Then one sees that the set of projective modules forms a generating set for $\mathbf{K}_0(\mathcal{P}_0, 0)/\mathbf{AR}$.

We prove this by induction in the following way. Let $C \in \mathcal{P}_0$, $C \notin \mathbf{AR}$ be such that the number n such that $DTr^n C$ is projective is minimal, and such that the projective in the orbit of C is of minimal length.

Let

$$0 \to DtrC \to \sum_i B_i \to C \to 0$$

be an almost split sequence. Then, $DTrC$ is in $\mathbf{AR}$, but (as we can assume), B_1 is not. Therefore, for $j = 1, \cdots, n-1$, $DTr^j C$ is not projective, implying that there is an arrow from $DTr^n B_1$ to $DTr^n C$, a contradiction to the minimality of the length of the projective $DTr^n C$.

From the natural maps

$$\mathbf{K}_o(\mathcal{P}_0, 0) \to \mathbf{K}_0(\mathcal{P}^{<\infty}, 0) \to \mathbf{K}_0(\mathcal{P}^{<\infty})$$

it follows that $\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)/\mathbf{AR}$ is isomorphic to $\mathbf{K}_0(\mathcal{P}^{<\infty})$.

Clearly, if Λ is of infinite representation type, so is $\mathcal{P}_0$ and we have the desired example.

Observe that we can get Auslander-Butler's theorem for hereditary algebras out of these simple remarks.

In a similar way we can show the following:

Proposition 9 . *If Λ is a finite dimensional hereditary algebra such that the classes of the projective modules form a generating set for*

$$(\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)/\mathbf{AR})$$

then Λ is of finite representation type.

Proof In this case it is clear that the set of classes of projectives forms a generating set for $\mathbf{K}_0(\mathcal{P}^{<\infty})$, so the natural map $(\mathbf{K}_0(\mathcal{P}^{<\infty}, 0)/\mathbf{AR}) \to \mathbf{K}_0(\mathcal{P}^{<\infty})$ is an isomorphism and we conclude that the only component on the Auslander Reiten quiver is the component which contains the projectives, therefore Λ is of finite representation type. □.

ACKNOWLEDGEMENTS. The first two authors had partial support from CNPq, Brazil, the first named author had also support from FAPESP, and the third, from ANPCyT, Argentina, and Fundación Antorchas, Argentina. During part of the work done, the second author was visiting Universidad del Sur, Bahía Blanca, Argentina, in the frame of project FOMEC, Argentina, and he is thankful for the warm hospitality received during this stay.

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