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Self-induced Interval Exchanges and Piecewise Linear Maps of the Interval

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Transformações de intercâmbio de intervalos autoinduzidas e aplicações , lineares por partes, do intervalo

Carlos Gutierrez e Milton Edwin Cobo Cortez

Resumo

Dada uma transformação de intercâmbio de intervalos autoinduzida T (ou seja, um ponto periódico do operador de Rauzy-Zorich) consideramos o conjunto $C(T)$ das aplicações bijetivas, lineares por partes, do intervalo, que são conjugadas a T. Provamos que a classe de diferenciabilidade da conjugação depende do vetor de derivadas de $F \in C(T)$. Veremos que, para quase todo F, a conjugação não é absolutamente contínua, o que implica que a (única) medida invariante de F é singular com respeito à medida de Lebesgue.

Self-induced interval exchanges and piecewise linear maps of the interval

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Abstract

Given a self-induced interval exchange T (i.e, a periodic points of the Rauzy-Zorich operator) we consider the set $C(T)$ of bijective piecewise-linear maps of the interval that are conjugate with T. We proved that the class of differenciability of the conjugation depend on the vector of derivaties of $F \in C(T)$. We will also see that for most $F \in C(T)$ the conjugation is never absolutely continuous, which implies that the (unique) invariant measure of F is singular with respect to the Lebesgue one.

1 Affine interval exchanges

We will say that a map F is an affine interval exchange, if it is a bijective piecewise linear map of an interval I. Along this work we will always assume $I = [0, b), b > 0$ and consider only the case in which the derivatives of F are positive. In this way, associated to each affine interval exchange there are vectors $w \in \mathbb{R}_+^m, d \in \mathbb{R}^m$ and a sequence of points $a_0 = 0 < a_1 < \cdots < a_m = b$ such that

$$Fx = w_i x + d_i, \quad x \in [a_{i-1}, a_i).$$

We will usually denote $X_i(F) := [a_{i-1}, a_i)$. Clearly if $\mu_i = a_i - a_{i-1}$, a necessary condition for bijectivity of F is that $(\mu, w) = b$ (where $(\cdot, \cdot)$ is the usual inner product of $\mathbb{R}^m$). An

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interval exchange transformation is an affine one for which $w = (1, 1, \dots, 1)$.
We say that an affine interval exchange (or an interval exchange) is normalized if it is defined in the interval $[0, 1)$.

Let us introduce standard notation for interval exchanges. Let $\mathbb{R}^m_+$ be the positive cone of $\mathbb{R}^m$. $\mathfrak{S}_m$ will denote the group of irreducible permutations of $\{1, 2, \dots, m\}$, i.e, those without invariant subsets of the form $\{1, 2, \dots, k\}, k < m$. Let $|\cdot|$ denote, from now on, the L^1-norm of $\mathbb{R}^m$. Given $\lambda \in \mathbb{R}^m_+$ and $\pi \in \mathfrak{S}_m$ let λ^π be the vector of entries $\lambda^\pi_i = \lambda_{\pi^{-1}i}$, $1 \leq i \leq m$. Associated to the pair (λ, π) is the interval exchange $T = T(\lambda, \pi) : [0, |\lambda|) \to [0, |\lambda|)$,

$$Tx := x + \sum_{k=1}^{\pi i-1} \lambda^\pi_k - \sum_{k=1}^{i-1} \lambda_k, \ x \in [a_{i-1}, a_i) \tag{1.1}$$

where $a_i = \sum_{k=1}^{i} \lambda_k, 1 \leq i \leq m$.
In general, to define an affine interval exchange we need a permutation $\pi \in \mathfrak{S}_m$ and two vectors $\mu, w \in \mathbb{R}^m_+$ such that $(\mu, w) = |\mu|$. Then we define $F = F(\mu, \pi, w) : [0, |\mu|) \to [0, |\mu|)$ by

$$Fx = w_i x + \sum_{k=1}^{\pi i-1} w^\pi_k \, \mu^\pi_k - w_i \sum_{k=1}^{i} \mu_k, \ x \in [a_{i-1}, a_i)$$

where $a_i = \sum_{k=1}^{i} \mu_k, 1 \leq i \leq m$. The vector w will be called *the vector of derivaties of F.*

Let T and F be piecewise continuous maps of the interval $[a, b)$. We say that F is *semi-conjugate* to T if there is a continuous, surjective and non-decreasing map $h : [a, b) \to [a, b)$ such that $T \circ h(x) = h \circ F(x)$, $x \in [a, b)$.
If $\gamma \in \mathbb{R}^m$ we denote by $\exp(\gamma)$ the vector $(\exp(\gamma_1), \dots, \exp(\gamma_m))$. We denote by $C_\gamma(T)$ (resp. $S_\gamma(T)$) the set of affine maps F which are conjugate (resp. semi-conjugate) to T and have vector of derivaties $w = \exp(\gamma)$.

We will show in this work how the differenciabity of the conjugation between $F \in C_\gamma(T)$ and T depend on the vector γ.

This work has been inspired in the work of W. Veech (see [Vee, Vee1, Vee2, Vee3]). Also, some of our results complement those ones of [Cob].

2 Rauzy's induction

Rauzy's induction process is defined as follows. Given $\pi \in \mathfrak{S}_m$ and $\lambda \in \mathbb{R}^m_+$ a vector such that $\lambda_m \neq \lambda_{\pi^{-1}(m)}$, define $\nu(\lambda, \pi) := \min(\lambda_m, \lambda_{\pi^{-1}(m)})$. Let $T_{(\lambda,\pi)}$ and denote by $\widetilde{\mathcal{T}}(T)$ the Poincaré induced map of T on the interval $I(T) := [0, 1 - \nu(\lambda, \pi))$. According to [Rau], $\widetilde{\mathcal{T}}(T)$ is again an interval exchange transformation of m subintervals corresponding to a pair (λ', π'). There is a matrix $A(T)$ whose entries are non-negative integers and a bijective transformation $c(T) : \mathfrak{S}_m \to \mathfrak{S}_m$ (both $A(T)$ and $c(T)$ depending only on the pair (λ, π)) such that

$$\lambda' = A^{-1}(T)\lambda \quad \text{and} \quad \pi' = c(T)\pi. \tag{2.2}$$

Let Δ^{m-1} denote the standard simplex of $\mathbb{R}^m$, i,e, the set $\{\lambda \in \mathbb{R}^m_+ : |\lambda| = 1\}$. The Rauzy's induction map is the transformation

$$\mathcal{T} : \Delta^{m-1} \times \mathfrak{S}_m \mapsto \Delta^{m-1} \times \mathfrak{S}_m$$

such that $\mathcal{T}(T)$ is the normalization of $\widetilde{\mathcal{T}}(T)$. Notice that $\mathcal{T}$ is not defined on the set of null measure $\{\lambda \in \Delta^{m-1} : \lambda_m = \lambda_{\pi^{-1}(m)},\ \pi \in \mathfrak{S}_m\}$.

2.1 The self-induced situation and the main theorem

Suppose that $T = T(\lambda, \pi)$ is a periodic point for the Rauzy operator $\mathcal{T}$, this is, $\mathcal{T}^p(T) = T$, for some fixed $p \geq 1$. This implies that $\widetilde{\mathcal{T}}^p(T)$ is an interval exchange corresponding to a pair (λ', π') satisfying the relations $\pi = \pi'$ and $\lambda = \rho\,\lambda'$ for some $\rho > 1$. We will say that T is a self-induced interval exchange transformation. Let B denote the matrix $B = A(T)A(\mathcal{T}^1(T))\dots A(\mathcal{T}^p(T))$. Using recursibly (2.2) we get that λ is a positive eigenvector of the matrix B^{-1}, i.e, $B^{-1}\lambda = \lambda' = (1/\rho)\lambda, \rho > 1$.

In the following section we will introduce a matrix L^π with the property that the eigenspace of B associated to the eigenvalue 1 contains the kernel of L^π. We will suppose that outside the kernel of L^π, B has only real and different eigenvalues, and all this eigenvalues are different from 1. The kernel of L^π will be denote by $N(\pi)$ and its orthogonal complement will be denote by $H(\pi)$.

Given $\gamma \in \mathbb{R}^m$ we denote $\theta(\gamma) := \lim_{k\to\infty} \frac{1}{k} \log \| \, {}^tB^k\gamma \|$. Clearly $\theta(\gamma)$ is the logarithm of one of the eigenvalues of B.

It will be prove in Lemma 3.1 that if $T(\lambda, \pi)$ is a uniquely ergodic interval exchange then $S_\gamma(T) \neq \emptyset$ implies that γ is orthogonal to λ. In view of this, we only consider vectors of inclination γ in the orthogonal complement of λ, which will be denoted by H.

Let us denote by $C_\gamma^N(T)$ (resp. $S_\gamma^N(T)$) the set of normalized affine interval exchanges in $C_\gamma(T)$ (resp. $S_\gamma^N(T)$).

Theorem 1. *Let $\gamma \in H$. Then $S_\gamma(T) \neq \emptyset$ and*

(a) if $\theta(\gamma) < 0$ then there is a unique map $F \in S_\gamma^N(T)$. If $\gamma = L^\pi\lambda$, F is analitically conjugate *to T. In the other cases F is at least C^1-*conjugate *to T. In particular, F has an invariant measure which is equivalent to Lebesgue.*

(b) if $\theta(\gamma) = 0$ then there is a unique $F \in S_\gamma^N(T)$. If F is conjugate *to T then this conjugation is never absolutely continuous. In particular, F has an invariant measure which is singular with respect to Lebesgue.*

(c) let $\theta(\gamma) > 0$ and suppose that $F \in C_\gamma(T)$. Then the conjugation is never absolutely continuous. In particular, F has an invariant measure which is singular with respect to Lebesgue.

Observe that if F is semi-conjugate to T but not conjugate, this is, if h is not injective, then there is an interval $W \subset [0,1)$ such that $h(W)$ is reduced to a point x_0 and also by the semi-conjugacy $h(F^j(W)) = T^j(x_0), j \in \mathbb{Z}$. This implies that the iterates of W never reach the discontinuity points of F, i.e, $F^j(W)$ is an interval for each $j \in \mathbb{Z}$. If we suppose that T is minimal then F has no periodic point and we will say that W is a *wandering interval* for F. In [Cob] there have been constructed examples where $C_\gamma(T) = \emptyset$ for every γ such that $\theta(\gamma) > 0$. Then we ask whether this situation is typical or not,

Question 1. *In the self-induced situation if $\gamma \in H(\pi)$ and $\theta(\gamma) > 0$ then $C_\gamma(T) = \emptyset$?*

3 Some preliminary results

In [cam-gut] it is proved that, modulus redefining T at the discontinuities (in case $\theta(\gamma) > 0$), $S_\gamma(T)$ is always non-empty for $\gamma \in H$.

Observe that if F is semi-conjugate to T then we have

$$h(y) = x \quad \text{implies} \quad F^k(y) \in X_i(F) \iff T^k(x) \in X_i(T).$$

We recall the following easy result:

Lemma 3.1. *Let $T_{(\lambda,\pi)}$ be a uniquely ergodic interval exchange transformation and $F \in S_\gamma(T)$. Then the vector γ is orthogonal to λ. In particular,*

$$\lim_{n\to\infty} (DF^n(y))^{\frac{1}{n}} = 1, \forall y \in [0,1).$$

Proof. Clearly T and F have null topological entropy and thus zero Lyapunov exponent. If $h(y) = x$ then

$$\begin{aligned} DF^n(y) &= \exp(\sum_{i=1}^m \chi_i(y,n)\,\gamma_i) \\ &= \exp(\sum_{i=1}^m \chi_i(x,n)\,\gamma_i) \end{aligned}$$

where $\chi_i(y,n) := \text{card}\{0 \le j \le m : F^j(y) \in X_i(F)\} = \text{card}\{0 \le j \le m : T^j(x) \in X_i(T)\}$. Observe finally that the Lyapunov exponent of F,

$$\lim_{n\to\infty} \frac{1}{n} \log DF^n(y) = \lim_{n\to\infty} \sum_{i=1}^m \frac{\chi_i(x,n)}{n} \gamma_i = 0$$

is also equal to $\sum_{i=1}^m \lambda_i \gamma_i$ because the unique ergodicity of T implies that $\frac{\chi_i(x,n)}{n}$ converges to λ_i when $n \to \infty$. □

Let us denote $\widetilde{\mathcal{T}}^n(T) = T(\lambda^{(n)}, \pi^{(n)}), n \ge 1$. Then there is a sequence $I(\widetilde{\mathcal{T}}^n(T)) = [0, |\lambda^{(n)}|)$, $n \ge 1$ of nested intervals shrinking to zero such that $\widetilde{\mathcal{T}}^n(T)$ is the induced map of T on the interval $I(\widetilde{\mathcal{T}}^n(T))$. Observe also that (2.2) implies that

(3.3) $$\lambda = A(T)A(\mathcal{T}^1(T)) \ldots A(\mathcal{T}^{n-1}(T))\lambda^{(n)}$$

Let $F \in S^N_\gamma(T)$. Observe that necessarily F has the same associated permutation π as T. Let $h : [0,1) \to [0,1)$ be a semi-conjugation between F and T and let $I = I(T)$. It is not difficult to see that the induced map of F on the interval $K = h^{-1}(I)$ is again an affine interval exchange F' associated to the triple (μ', π', γ') where $\pi' = c(T)\pi$ and we can easily compute that

(3.4)
$$\begin{aligned} &(a) \quad \gamma' = {}^tA(T)\cdot\gamma, \ ({}^tA(T) \text{ is the traspose of } A(T)) \\ &(b) \quad \mu' = A_\gamma^{-1}(T)\mu \end{aligned}$$

and the matrix $A_\gamma(T)$ is equal to A_1 if $\lambda_m < \lambda_{\pi^{-1}(m)}$ and equal to A_2 if $\lambda_m > \lambda_{\pi^{-1}(m)}$; the matrices A_1 and A_2 are given in (3.6).

Define the vector $\gamma^{(n)}$ as the traspose of the matrix $A(T)A(\mathcal{T}^1(T))\dots A(\mathcal{T}^{n-1}(T))$ apply to the vector $\gamma, n \geq 1$. Then using recursively (3.4(b)) we can easily see that

$$\mu \in \bigcap_{n\geq 0} A_\gamma(T)A_{\gamma^{(2)}}(\mathcal{T}(T))\dots A_{\gamma^{(n)}}(\mathcal{T}^{n-1}(T))\,\mathbb{R}^m_+ . \tag{3.5}$$

$$A_1 := \left(\begin{array}{c|cccccc}
 & 0 & 0 & \dots & \dots & \dots & 0\\
I_{\pi^{-1}(m)} & \vdots & & \dots & \dots & \dots & \vdots\\
 & 1 & 0 & \dots & \dots & \dots & 0\\ \hline
 & 0 & 1 & 0 & \dots & \dots & 0\\
 & 0 & 0 & 1 & 0 & \dots & 0\\
 & \vdots & \vdots & \ddots & \ddots & \dots & \vdots\\
0 & & & & \ddots & & \\
 & 0 & 0 & \dots & \ddots & 1 & 0\\
 & 0 & 0 & \dots & \dots & 0 & 1\\
 & e(\gamma,\pi) & 0 & \dots & \dots & 0 & 0
\end{array}\right) \tag{3.6}$$

$$A_2 := \left(\begin{array}{ccccc|c}
 & & & & & 0\\
 & & I_{m-1} & & & \vdots\\
 & & & & & 0\\ \hline
0 & \dots & e(\gamma,\pi) & \dots & 0 & 1
\end{array}\right)$$

Here $e(\gamma, \pi)$ is the number $\exp(\gamma_{\pi^{-1}(m)})$ and appears at the $(\pi^{-1}(m))$ position.

3.1 The matrix L^π

When studing the suspension of interval exchanges to flows on boundaryless compact surfaces, Veech ([Vee]) introduced the skew-symmetric matrix $L^\pi, \pi \in \mathfrak{S}_m$,

$$L^\pi_{ij} := \begin{cases} +1 & \text{if } \; i < j \;\text{ and }\; \pi(i) > \pi(j) \\ -1 & \text{if } \; i > j \;\text{ and }\; \pi(i) < \pi(j) \\ \;\;0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Then proceeding as in [Vee3] we may easily see that $(L^\pi\lambda)_i = \sum_{k=1}^{\pi i-1} \lambda^\pi_k - \sum_{k=1}^{i-1} \lambda_k$ and then the (λ, π)-interval exchange T is given by

$$Tx = x + (L^\pi\lambda)_i, \quad x \in X_i(T). \tag{3.8}$$

The matrix L^π plays an important role in the ergodic properties of interval exchanges (see [Vee3]). It is proved that L^π satisfies the equation

$${}^tA(T)L^\pi = L^\pi A(T)^{-1}, \;\; (\; {}^tA(T) \text{ is the traspose of } A(T)) \tag{3.9}$$

Hereafter we will suppose that $T = T(\lambda, \pi)$ is a periodic point of the Rauzy's operator with period p, this is, $\mathcal{T}^p(T) = T$. This implies that $\widetilde{T}^p(T) = T(\lambda', \pi)$ with $\lambda' = \rho^{-1}\lambda$, for some $\rho > 1$. In particular T is self-induced on the interval $J = [0, \rho^{-1})$.

Let B denote the matrix $B = A(T)A(\mathcal{T}^1(T))\ldots A(\mathcal{T}^p(T))$. We may suppose (by increasing the period p) that B is a positive matrix. Using recursibly (2.2) we get that λ is a positive eigenvector of the matrix B^{-1}, i.e, $B^{-1}\lambda = \lambda' = (1/\rho)\lambda, \rho > 1$.

Observe that T is also self-induced on each subinterval $J_n := [0, \rho^{-n}), n \geq 1$ with induction matrix B^n, i.e , $B^{-n}\lambda = \rho^{-n}\lambda$. T_n will also denote the induced map of T in the interval J_n.

From equation (3.9) we obtain that ${}^tBL^\pi = L^\pi B^{-1}$ then if $\eta \neq 1$ is an eigenvalue of B associated to the eigenvector w then $1/\eta$ is an eigenvalue of ${}^tB$ associated to the eigenvector $L^\pi w$.

As B is a positive matrix we know that ρ is the largest eigenvalue of B and then $1/\rho$ is the smaller eigenvalue of B and it is associated to the eigenvector $L^\pi\lambda \neq 0$.

Remember that $N(\pi)$ denote the kernel of L^π. In Section 5 we will see that $N(\pi)$ has a basis of eigenvectors of B associated to the eigenvalue 1. We will suppose that $N(\pi)$ is generated by all the eigenvectors of B associated to the eigenvalue 1. Denote by $l(\pi)$ the dimension of $N(\pi)$ and let $g := \frac{1}{2}(m - l(\pi))$. Then the the eigenvalues of B are of the form

$$\frac{1}{\rho} = {\eta_m}^{-1} < \cdots < \eta_{m-g}^{-1} < 1 = \eta_{g+1} = \cdots = \eta_{g+l(\pi)+1} = 1 < \eta_{m-g} < \cdots < \eta_m = \rho.$$

We denote by

$$\mathcal{A}^s = \{\gamma \in H : \theta(\gamma) < 0\}$$
$$\mathcal{A}^c = \{\gamma \in H : \theta(\gamma) = 0\}$$
$$\mathcal{A}^u = \{\gamma \in H : \theta(\gamma) > 0\}.$$

4 The stable case $\mathcal{A}^s$

From the equation ${}^tBL^\pi B = L^\pi$ and the fact that $B\lambda = \rho\lambda$, we get

$${}^tB(L^\pi\lambda) = \frac{1}{\rho}(L^\pi\lambda),$$

this is, $L^\pi\lambda$ is the eigenvector of ${}^tB$ associated to the minimum eigenvalue $1/\rho$. From the relation $Tx = x + (L^\pi\lambda)_i, x \in X_i(T)$, we can see that $\varphi(x) = \mathrm{e}^x$ satisfies the relation

$$\varphi(T(x)) = \mathrm{e}^{(L^\pi\lambda)_i}\varphi(x), \quad x \in X_i(T). \tag{4.10}$$

Consider now the (absolutely continuous) measure $\mu = \int \varphi(x)\,dx$, and let

$$h(x) = \frac{\mu[0,x]}{\mu[0,1]} = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}, \quad x \in [0,1).$$

Then we can easily see that $F = h \circ T \circ h^{-1}$ is the following affine interval exchange

$$F(x) = \mathrm{e}^{(L^\pi\lambda)_i}x + \frac{\mathrm{e}^{(L^\pi\lambda)_i} - 1}{\mathrm{e} - 1}, \quad x \in X_i(F) := h(X_i).$$

Let $v \in \mathcal{A}^s$ and let $\chi_i(x,n)$ be the number $\mathrm{card}\{0 \le j \le n : T^j(x) \in X_i(T)\}$. We define

$$\Gamma(x,n) := \exp(\sum_{i=1}^{m} \chi_i(x,n)\, v_i). \tag{4.11}$$

In particular denote $\Gamma(x) := \Gamma(x,0) = \exp(v_i)$, $x \in X_i(T)$.

W. Veech proved in [Vee1] that the equation

$$\varphi(Tx) = \Gamma(x) \cdot \varphi(x) \tag{4.12}$$

has a continuous non-negative solution for $v \in \mathbb{R}^m$, provided that

$$\sum_{k=0}^{\infty} \|vB^k\| < \infty$$

which is our case when $v \in \mathcal{A}^s$. Let us assume that $\varphi : [0,1] \to [0,1]$ is a solution for equation (4.12) and consider the probability $\mu := d^{-1} \int \varphi\, ds$, where $d = \int_0^1 \varphi(s)\, ds$. Then μ satisfies the relation $\int_{T\mathcal{A}} d\mu = \int_{\mathcal{A}} \Gamma\, ds$, where $\mathcal{A}$ is any Borel set of $[0,1)$. Observe that the function $h(x) = \mu[0,x)$ is a bijective C^1-map of $[0,1)$ such that $F = h \circ T \circ h^{-1} \in C_v^N(T)$. To see this, let $X_i(F) := h(X_i(T))$. Then if $B \subset X_i(F)$ is an interval and $h(A) = B$ we have

$$\begin{aligned} |F(B)| &= |h(T(h^{-1}(B))| = \mu(TA) \\ &= e^{v_i}\, \mu(A) = e^{v_i} |h(A)| = e^{v_i}\, |B| \end{aligned}$$

which implies that $DF(y) \equiv e^{v_i}$ for all $y \in X_i(F)$. Hence we have proved that for $\gamma \in \mathcal{A}^s$ there is $F \in C_\gamma(T)$ which is C^1-conjugate to T.

Finally observe that if $\gamma \in \mathcal{A}^s$ then $\gamma^{(n)}$ goes to zero when $n \to \infty$. Therefore $A_{\gamma^{(n)}}(\mathcal{T}^n(T))$ converges to $A(\mathcal{T}^n(T))$. Now if $F(\mu, \pi, \gamma), \gamma \in \mathcal{A}^s$ is an affine interval exchange semi-conjugate to T then by (3.5),

$$\mu \in \bigcap_{n \ge 0} A_\gamma(T) A_{\gamma^{(2)}}(\mathcal{T}(T)) \ldots A_{\gamma^{(n)}}(\mathcal{T}^{n-1}(T))\, \mathbb{R}_+^m .$$

Arguing as in [Vee3, Lemma 3.28] we obtain that this intersection is one-dimensional and so there is a unique $F \in C_\gamma^N(T)$.

We remark that R. Barreto proved in ([Barr]) that for every $x \in [0,1)$, $\sup_{n\in\mathbb{Z}} \Gamma(x,n) < \infty$. Then it is easy to see that the function $\varphi(x) := \sup_{n\in\mathbb{Z}} \frac{1}{\Gamma(x,n)}$ satisfies equation (4.12). In fact, R. Barreto proved that this φ is a Lipschitz function.

Proposition 4.1. *Associated to every vector $\gamma \in \mathcal{A}^s$ there is a unique affine interval exchange $F \in S_\gamma^N(T)$ which is C^1-conjugate to T. If $\gamma = L^\pi \lambda$ then F is even analitically conjugate to T.*

5 The central space $\mathcal{A}^c$

We will introduce now results from ([Vee]) where it is exhibited an explicit basis for $N(\pi)$, the kernel of L^π. In particular this basis is formed by eigenvectors of B associated to the eigenvalue 1.

Define $\pi(0) := 0$ and $\pi(m+1) := m+1$ and consider the new permutation $\sigma = \sigma(\pi)$ on the set $\{0, 1, 2, \ldots, m\}$ given by

$$\sigma(j) := \pi^{-1}(\pi(j)+1) - 1, \ 0 \leq j \leq m.$$

Let $\Sigma(\pi)$ denote the ciclic sets of σ, i.e, the periodic orbits of σ. For each $S \in \Sigma(\pi)$ assign the vector $b_S \in \mathbb{R}^m$ whose components are given by (see 2.9 in [Vee]),

$$b_S(j) = \chi_S(j-1) - \chi_S(j), \ 1 \leq j \leq m \tag{5.13}$$

where $\chi_S : \{0, 1, \ldots, m\} \to \{0, 1\}$ is the characteristic function of S. It has been shown in [Vee] that this vectors form a basis for the kernel of the matrix L^π defined in (3.7). It is proved that associated to transformation $c(T) : \mathfrak{G}_m \to \mathfrak{G}_m$ there is a map $\tilde{c}(T) : \Sigma(\pi) \to \Sigma(\pi)$ such that

$$b_S = A(T)\, b_{\tilde{c}(T)S}, \ S \in \Sigma(\pi).$$

Even more if $c(\mathcal{T}^k(T))c(\mathcal{T}^{k-1}(T))\ldots c(T)\pi = \pi$ then $\tilde{c}(\mathcal{T}^k(T))\tilde{c}(\mathcal{T}^{k-1}(T))\ldots\tilde{c}(T)$ is the identity map on $\Sigma(\pi)$. This implies that if $\mathcal{T}^k(T)$ has the same associated permutation π as T then

$$b_S = A(\mathcal{T}^k(T))A(\mathcal{T}^{k-1}(T))\ldots A(T)\, b_S, \ S \in \Sigma(\pi)$$

i.e, the b_S are eigenvectors of the matrix $A(\mathcal{T}^k(T))A(\mathcal{T}^{k-1}(T))\ldots A(T)$ associated to the eigenvalue 1. In particular, if T is a periodic point of $\mathcal{T}$, with associated matrix B, we obtain that $N(\pi)$ is contained in the eigenspace of B associated to the eigenvalue 1.

Lemma 5.1. *Let $v \in \mathbb{R}^m$ such that $vB = v$. Then there is a unique $F \in S_v^N(T)$. Let h be a semi-conjugation between F and T and define $K = h^{-1}(J)$. Then F is self-induced on K.*

Proof. Let $F \in S_v^N(T)$. From (3.5) we know that

$$\mu \in \bigcap_{n \geq 0} [B_v]^n\, \mathbb{R}_+^m.$$

As B_v is a positive matrix, this intersection is one-dimensional and so F is unique (remember that $\mu \in \Delta^{m-1}$).

Now by our hypotheses T is self-induced on $J = [0, \rho^{-1})$. Let $\widetilde{F} = F(\tilde{\mu}, \pi, v)$ be the induced map of F on K. Write $\theta = |\tilde{\mu}|^{-1} > 1$. Re-escaling we obtain that $F(\theta\tilde{\mu}, \pi, v)$ is clearly semi-conjugate to T, hence $\theta\tilde{\mu} = \mu$. This implies that $(B_v)^{-1}\mu = \tilde{\mu} = \theta^{-1}\mu$, and therefore F is also self-induced on the interval K. □

Lemma 5.2. *Let e_j, $1 \leq j \leq m$ be the canonical basis of $\mathbb{R}^m$. Then there are constants $C_1 > \rho$ and $C_2 > 0$ such that*

$$|B^k e_j| \geq C_1 \rho^k \text{ and } |B^k e_j| \leq C_2 \rho^k$$

for each $k \geq 1$ and $1 \leq j \leq m$.

Proof. Define $\vartheta(B^n) = \max_{i,j,k} \frac{B^n_{ij}}{B^n_{ik}}$. We may suppose that $\vartheta(B) \geq 1$. Then if $b^k_j = |B^k e_j| = \sum_{i=1}^m B^k_{ij}$ we get that

$$b^k_j \leq \vartheta(B^k) b^k_i \leq \vartheta(B) b^k_i, \quad 1 \leq j, i \leq m, \tag{5.14}$$

where in the last inequality we have used the fact that $\vartheta(B^k) \leq \vartheta(B)$ ([Vee1]). Finally from $\sum_{i=1}^m \lambda_j e_j = \lambda$ and the fact that $B^k \lambda = \rho^k \lambda$ we obtain that

$$|B^k \lambda| = \rho^k = \sum_{i=1}^m \lambda_i |B^k e_i|. \tag{5.15}$$

This implies that there exists j_0 and K_0 such that e_{j_0} satisfies

$$|B^k e_{j_0}| \geq 1/2\rho^k, \; k \geq K_0.$$

Using (5.14) we conclude that (for $k \geq K_0$)

$$|B^k e_j| \geq 1/2\vartheta(B)\rho^k, 1 \leq j \leq m.$$

Hence the existence of C_1. Similarly, using equation (5.15) we obtain the constant $C_2 > 0$ such that, for each $k \geq 1$ and $1 \leq j \leq m$, $|B^k e_j| \leq C_2 \rho^k$. □

We will show now that the affine interval exchange that appear in the central case don't have absolutely continuous invariant measures. We will use the following easy lemma.

Lemma 5.3. *Let T be a uniquely ergodic i.e.t and $F \in C_\gamma(T)$. Then there exist an absolutely continuous conjugation between F and T if and only if F has an invariant measure which is absolutely continuous with respect to Lebesgue.*

Lemma 5.4. *Let $v \in \mathcal{A}^c$ and $F \in S_v(T)$. Then the (unique) invariant measure of F is singular with respect to Lebesgue.*

Proof. Suppose that h is an absolutely continuous conjugation between F and T, i.e, $h \circ T \equiv F \circ h$. Let m be the Lebesgue measure. Then the measure $\tilde{m} = m \circ h$ defined by $\tilde{m}(\mathcal{A}) = m(h(\mathcal{A}))$ where $\mathcal{A}$ is a Borel set, is absolutely continuous with respect to Lebesgue, say $\tilde{m} = \int \varphi \, dm$. Observe that then φ satisfies the equation

$$\varphi(T(x)) = e^{v_i} \varphi(x), \quad x \in X_i(T). \tag{5.16}$$

a.e in $[0,1)$. In fact if $A \subset X_i(T)$ is such that $h(A) = B \subset X_i(F)$, then

$$\begin{aligned} \tilde{m}(TA) &= \int_{TA} \varphi \, dm = \int_A \varphi \circ T \, dm \\ &= m(h(TA)) = m(F(B)) = e^{v_i} \, m(B) \\ &= e^{v_i} \, m(h(A)) = \exp(v_i) \tilde{m}(A) = \int_A e^{v_i} \varphi \, dm. \end{aligned}$$

Veech had proved in [Vee1] that (5.16) has a measurable solution only if (v, b_S) is an integer for all $S \in \Sigma(\pi)$. On the other hand, for any positive real number α, φ^α is a solution for

$$f(T(x)) = e^{\alpha v_i} f(x), \quad x \in X_i(T),$$

which implies that $\alpha \cdot (v, b_S) \in \mathbb{Z}$ for all $\alpha \in \mathbb{R}$, i.e, $(v, b_S) = 0, \forall S \in \Sigma(\pi)$; in other words, equation (5.16) has a measurable solution only if the vector v belongs to $H(\pi)$, the orthogonal complement of $N(\pi)$. It is easy to see that no eigenvector of ${}^tA$ associated to the eigenvalues 1 belong to $H(\pi)$. In fact if $v \in \mathbb{R}^m$ satisfies $vB = \theta v$, $\theta \neq 1$, then for each $S \in \Sigma(\pi)$ we have

$$(vB^k, b_S) = \theta^k(v, b_S) = (v, B^{-k}b_S) = (v, b_S),\ k \geq 1$$

which implies that $(v, b_S) = 0$ for all $S \in \Sigma(\pi)$, i.e, $H(\pi)$ contain all the eigenvectors of ${}^tB$ associated to eigenvalues different from 1, as we are assuming that the number of this eigenvectors is exactly the dimension of $H(\pi)$, then there are no vectors of $\mathcal{A}^c$ in $H(\pi)$.

□

6 The unstable case $\mathcal{A}^u$

We will prove that if $\theta(\gamma) > 0$ and if $F \in C^N_\gamma(T)$ is an affine map *conjugate* to T then F has an invariant measure which is singular with respect to Lebesgue. By Lemma 5.3, this is equivalent to prove that no conjugation between F and T can be an absolutely continuous function. Suppose by contradiction that h is an absolutely continuous conjugation between F and T, i.e, we have that $h(F(y)) = T(h(y))$, $y \in [0, 1)$. Then the measure $\tilde{m} = m \circ h$ defined by $\tilde{m}(A) = m(h(A))$, for any Borel set A is absolutely continuous. Write $\tilde{m} = \int \varphi dm$. We can easily see that φ satisfies the relations (remember (4.11) and (5.16)),

$$\varphi(T^k x) = \Gamma(x, k)\varphi(x),\ x \in [0, 1), k \in \mathbb{Z}.$$

Using the ergodic theorem of Birkhoff we get that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k(x)) &= \varphi(x) \cdot \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Gamma(x, k) \\ &= \int_0^1 \varphi dm = 1\end{aligned}$$

and therefore,

$$\varphi(x)^{-1} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Gamma(x, k). \tag{6.17}$$

This implies in particular that for almost every point x the quotients $\Gamma(x, n)/n$ remain bounded when n goes to infinity. We will prove now that this is impossible when $\theta(\gamma) > 0$ which implies that F cannot have an absolutely continuous invariant measure.

Lemma 6.1. *The invariant set*

$$G = \{x \in [0, 1) : \limsup_{n\to\infty} \frac{\Gamma(x, n)}{n} = \infty\}$$

has Lebesgue measure one.

Proof. Let us suppose first that γ is an eigenvector of ${}^tB$ associated to an eigenvalue $\eta > 1$. As G is T-invariant the Lebesgue measure of G is zero or one. Suppose by contradiction that $m(G) = 0$ (m represent the Lebesgue measure). Egoroff's Theorem implies that for every $\delta > 0$ there exist $M_\delta > 0$ and a set B_δ with $m(B_\delta) > 1 - \delta$ such that

$$\frac{1}{n}\Gamma(x, n) \leq M_\delta, \ \forall x \in B_\delta \text{ and } n \geq n_0. \tag{6.18}$$

Remember that T is also self-induced on each subinterval $J_n = [0, \rho^{-n}), n \geq 1$. Let k_0 be an integer such that J_{k_0} is contained in the first interval $X_1(T)$ of continuity of T. This implies that J_{n+k_0} is contained in the first interval $X_1(T_n)$ of continuity of T_n. It is not difficult to see that $|B^n e_1|$ is the number of iterates that the first sub-interval $X_1(T_n)$ takes to return to J_n. In this way, the interval J_{n+k_0} can be iterated $|B^n e_1|$ times without meeting the discontinuity points of T, this is $T^j(J_{n+k_0})$ is an interval for each $0 \leq j \leq |B^n e_1|$. Then T is also self-induced on each of subintervals $T^j(J_{n+k_0})$ with the same associated matrix B^{n+k_0}. By Lemma (5.2), $|B^n e_1| > C_1 \rho^n$.

Let $d > 0$ be a constant such that $m(X_i(T)) > d$ for $1 \leq i \leq m$, which implies that $m(X_i(T_n)) > d m(J_n) = d\rho^n$, $n \geq 1$. Then the Lebesgue measure of the union

$$D_i^{n+k_0} := \bigcup_{j=0,\ldots,|B^n e_1|} T^j(X_i(T_{n+k_0}))$$

is greater than $dC_1 \rho^{-n-k_0} \rho^n = dC_1 \rho^{-k_0}$.

Consider $0 < \delta < dC_1 \rho^{-k_0}$ and B_δ, M_δ as in (6.18). If x belong to J_n let $r_n(x)$ denote the time that x spend to return to J_n. Observe that if $x \in X_i(T_n)$ then $r_n(x) = |B^n e_i|$ and then by Lemma 5.2, x will spend at most $C_2 \rho^n$ iterates to return.

As $\theta(\gamma) > 0$, there exist $C > 0$ and $\eta > 1$ such that

$$|\, {}^tB^n \gamma|_1 > C\eta^n, n \geq 1.$$

From this and the fact that $\gamma^n := {}^tB^n \gamma$ is orthogonal to λ we obtain that for $n \geq 1$ there exist a constant $K > 0$ and $i_n \in \{1, 2, \ldots, m\}$ such that $\gamma_{i_n}^n > K|\gamma^n|_1 > 0$. By the shoe-box principle i_n is constant in an infinite set E of integers.

If $x \in X_i(T_{n+k_0})$, $1 \leq i \leq m$, we can easily compute that $\Gamma(x, n + k_0) = \exp(\gamma_i^{n+k_0})$ and then

$$\frac{\Gamma(x, r_{n+k_0}(x))}{r_{n+k_0}(x)} \geq \frac{\exp(\gamma_i^{n+k_0})}{C_2 \rho^{n+k_0}}. \tag{6.19}$$

In particular if $n \in E$ is big enough the quotient in (6.19) is greater than M_δ.
Finally observe that all the estimates above work also for any x in $D_{i_n}^{n+k_0}$. But the Lebesgue measure of $D_{i_n}^{n+k_0}$ is greater than δ which implies that this set intersects B_δ. This is a contradiction with (6.18) and so the lemma is proved. □

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