RT-MAT 96-24

Some remarks on Central Idempotents in Group Rings

Sônia P. Coelho
and
C. Polcino Milies

Novembro 1996

Some remarks on Central Idempotents in Group Rings

Sônia P. Coelho and C. Polcino Milies *
Universidade de São Paulo
Caixa Postal 66.281, 05389-970
São Paulo, Brazil
sonicoe@ime.usp.br
polcino@ime.usp.br

Abstract

Let G be a group, K a field of characteristic 0 and T the set of all elements of finite order in G. In this note we give necessary and sufficient conditions under which every idempotent of KT is central in KG.

Introduction

Let G be a group and K a field of characteristic 0. We denote by $U(KG)$ the group of units of the group ring of G over K. Also, if X is a group, we shall denote by $T(X)$ the *torsion* of X; i.e., the set of all elements of finite order in X. The study of group theoretical properties of $U(KG)$ has lead, on occasions, to the condition that $T = T(G)$ is a subgroup and that every idempotent of KT is central in KG. In what follows we study this condition and prove the following.

Theorem 1 *Let K be a field of characteristic 0 and let T be the set of elements of finite order of a group G. Then, every idempotent of KG with support in T is central in KG if and only if the following conditions hold:*

(i) For every element $t \in T$ and every $x \in G$ there exists a positive integer j such that $xtx^{-1} = t^j$.

*Both authors are partially supported by research grants from CNPq., (Proc. 300371/82-9(RN) and Proc. 300243/79-0(RN) respectively).

(ii) If $t \in T$ is an element of order n and ζ_n is a primitive nth root of unity over K, then, for each exponente j obtained as in (i), there exists a map $\sigma \in Gal(K(\zeta_n) : K)$ such that $\sigma(\zeta_n) = \zeta_n^j$.

(iii) T is either an abelian group or a Hamiltonian group such that for each element $t \in T$ of odd order k, the field $K(\zeta_k)$ contains no non-trivial solution of the equation $x^2 + y^2 + z^2 = 0$.

This problem was first studied in [1] where, due to an oversight, condition (ii) was stated in the following weaker form.

(ii') *For every non-central element t of order n, K contains no root of unity of order n.*

This condition is actually not sufficient, as is shown below. The authors are grateful to Prof. E. Jespers, who spotted the oversight and suggested this counterexample.

Example Let $G = \langle x, y \mid yxy^{-1} = x^5,\ x^8 = 1\rangle$; let ζ be a primitive root of unity of order 8 and set $K = \mathbf{Q}(\zeta + \zeta^{-1})$.

Write:

$$e_1 = \frac{1}{8}\sum_{i=0}^{7}(\zeta x)^i \quad \text{and} \quad e_2 = \frac{1}{8}\sum_{i=0}^{7}(\zeta^{-1}x)^i.$$

Then, e_1, e_2 are orthogonal idempotentes in $\mathbf{Q}(\zeta)\langle x\rangle$ and thus $e = e_1 + e_2 \in K\langle x\rangle$ is also an idempotent. Notice that the coefficient of x in e is equal to $\sqrt{2}$ and the coefficient of x^5 is equal to $-\sqrt{2}$. It follows that the coefficient of x of the element yey^{-1} is $-\sqrt{2}$ so $e \neq yey^{-1}$ and thus e is not central in KG. Since K is a real field, it contains no root of unity other than ± 1 and it is easily seen that conditions (i) and (iii) are also satisfied.

Proof of the Theorem

The fact that conditions (i) and (iii) are necessary follows as in [1]. To prove (ii), first notice that, given an element $t \in T$ we can write $K\langle t\rangle$ as a direct sum

$$K\langle t\rangle \cong K_1 \oplus \cdots \oplus K_s$$

with $K_i = (K\langle t\rangle)e_i \cong K(\zeta_i)$, where e_i, $1 \leq i \leq s$, is the set of principal idempotents of $K\langle t\rangle$, each ζ_i denotes a root of unity and at least one of them, ζ_1 say, is such that $o(\zeta_1) = o(t)$. Also notice that, in the isomorphism above, the element t corresponds with $(\zeta_1, \ldots, \zeta_s)$.

Since every idempotent of KT is central in KG, it follows that conjugation by an element $x \in G$ induces an automorphism $\theta : K\langle t\rangle \to K\langle t\rangle$ which, in turn, induces automorphisms θ_i on each simple component K_i, $1 \leq i \leq s$.

Each K_i contains $\overline{K}_i = Ke_i$,which is an isomorphic copy of K and, since $xtx^{-1} = x^j$ for some positive integer j, we see that, in particular, $\theta_1 : K(\zeta_1) \to K(\zeta_1)$ fixes K and is such that $\theta_1(\zeta_1) = \zeta_1^j$ so $\theta_1 \in Gal(K(\zeta_1) : K)$, as desired.

To prove sufficiency we begin again as in [1]. Assume first that T is an abelian group and let $e \in KT$ be an idempotent. By considering $supp(e)$ we may assume that T is finite. Furthermore, since every idempotent is a sum of primitive idempotents, we may restrict ourselves to the case where e is itself primitive. We wish to show that, for each fixed element $x \in G$, we have that $xex^{-1} = e$.

Write $T = \langle t_1\rangle \times \cdots \times \langle t_s\rangle$, a direct product of cyclic groups and set $t_0 = t_1.\cdots.t_s$. Then $xt_0x^{-1} = t_0^j$ for some positive integer j and thus also $xtx^{-1} = t^j$, for all $t \in T$. Notice that $o(t_0)$, the order of t_0, is equal to the exponent of T so, if ζ is a primitive root of unity whose order is equal to $o(t_0)$, then $K(\zeta)$ is a splitting field for T. Hence, $e \in KT \subset K(\zeta)T$ is a sum of primitive idempotents of $K(\zeta)T$. Let f be one of these idempotents.

Every K-automorphism of $K(\zeta)$ extends in a natural way to an automorphism of $K(\zeta)T$. We define:

$$H = \{\phi \in Gal(K(\zeta) : K) \mid \phi(f) = f\}$$

and take $\phi_1 = I, \phi_2, \ldots, \phi_r$ a transversal of H in $Gal(K(\zeta) : K)$.

Set $e* = \phi_1(f) + \cdots + \phi_r(f)$. Exactly as in [1], it can be shown that $e* = e$.

According to [4, Theorem 2.12], we can write f in the form:

$$f = \frac{1}{|T|}\sum_{t\in T} \chi(t^{-1})t$$

where χ is an irreducible character of T with values in $K(\zeta)$.

Since $e = \phi_1(f) + \cdots + \phi_r(f)$ with $\phi_1(f) = f$ it follows that the idempotent

$$xfx^{-1} = \frac{1}{|T|}\sum_{t\in T} \chi(t^{-1})xtx^{-1} = \frac{1}{|T|}\sum_{t\in T} \chi(t^{-1})t^j$$

is one of the summands of xex^{-1}. Now, notice that

$$xfx^{-1} = \frac{1}{|T|}\sum_{t\in T} \chi(t^{-j})t.$$

According to our hypothesis, there exists an automorphism $\sigma \in Gal(K(\zeta) : K)$ such that $\sigma(\zeta) = \zeta^j$. Since $\chi(t)$ is a power of ζ for all $t \in T$, it follows that $\chi(t^{-j}) = \sigma(\chi(t^{-1}))$, $\forall t \in T$ and thus $xfx^{-1} = \sigma(f)$. As it was shown that e is the sum of all distinct images of f under K-automorphisms of $K(\zeta)$, it follows that e and xex^{-1}, which are both primitive idempotents of KT, have f as a common summand when written as sums of primitive idempotents of $K(\zeta)T$. This readily implies that $e = xex^{-1}$, as desired.

The case where T is Hamiltonian now follows as in [1].

Final comments

We recall that the supercenter of a group G over a field K is defined as the set $S = S_K(G)$ of all elements in G having a finite number of conjugates in $U(KG)$, the group of units of KG. This subgroup was studied in [2] and its description, in the case where $char(K) = 0$ was obtained using the theorem on central idempotents given above. Though the statement of [2, Theorem C] is correct, it can now be stated in a more precise form.

Theorem 2 *Let K be a field of characteristic 0 and let G be a non torsion group. Then, one of the following holds:*
(i) $S = Z(G)$, the center of G.
(ii) $T(S)$ is an abelian group such that for all $t \in T(S)$ and all $x \in G$ we have that $xtx^{-1} = t^j$, for some positive integer j and, if ζ is a primitive root of unity of order $o(t)$, then there exists an element $\sigma \in Gal(K(\zeta) : K)$ such that $\sigma(\zeta) = \zeta^j$. Furthermore, if $T(S)$ is infinite, then $T(S) = \mathbb{Z}(q^\infty) \times B$ where q is a prime rational integer, B is finite central in G, $\mathbb{Z}(q^\infty)$ is central in S, $(G, S) \subset \mathbb{Z}(q^\infty)$ and there exists a positive integer k such that K does not contain roots of unity of order q^k.

Also, [3, Theorem 3.2] can now be stated as follows.

Theorem 3 *Let G be a nilpotent or FC group and let K be a field of characteristic 0. Then, $TU(KG)$ is a subgroup if and only if the following conditions hold:*
(i) T is abelian.
(ii) For each $t \in T$ and each $x \in G$, there exists a positive integer j such that $xtx^{-1} = t^j$ and, if ζ is a primitive root of unity of order $o(t)$, then there exists a map $\sigma \in Gal(K(\zeta) : K)$ such that $\sigma(\zeta) = \zeta^j$.

The proofs are essentially the same as the original ones, requiring only minor changes.

References

[1] S.P. Coelho and C. Polcino Milies, A note on central idempotents in group rings II, *Proc. Edinburgh Math. Soc.*, **31** (1988), 211-215.

[2] S.P. Coelho and C. Polcino Milies, Finite conjugacy in group rings, *Commun. Algebra*, 19, **3** (1991), 981-995.

[3] S.P. Coelho and C. Polcino Milies, Group rings whose torsion units form a subgroup, *Proc. Edinburgh Math. Soc.*, **37** (1994), 201-205.

[4] M. Isaacs, *Character theory of finite groups*, Academic Press, New York, 1976.

TRABALHOS DO DEPARTAMENTO DE MATEMÁTICA

TÍTULOS PUBLICADOS

95-01 BARROS, L.G.X. de and Juriaans, S.O. Loops whose Loop Algebras are Flexible. 25p.

95-02 GUIDORIZZI, H.L. - Jordan canonical form: an elementary proof, 12 p.

95-03 CATALAN A., and COSTA R. E-Ideals in Baric Algebras. 14p.

95-04 MARTIN, P.A. On the generating function of the decimal expansion of an irracional real numbers. 7p.

95-05 COELHO F.U., MARCOS E.N., MERKLEN H.A. and PLATZECK.M.I. Modules of Infinite Projective Dimension over Algebras whose Idempotent Ideals are Projective. 13p.

95-06 GUIDORIZZI, H. L. The family of functions $S_{,k}$ and the Liénard Equation. 22p.

95-07 GUIDORIZZI, H. L. On the Existence of Periodic Solution for the Equation $\ddot{x}+\alpha x^{2n+1}\dot{x}+x^{4n+3}=0$. 5p.

95-08 CORTIZO, S.F. Extensões Virtuais. 27p.

95-09 CORTIZO, S.F. Cálculo Virtual. 31p.

95-10 GUIDORIZZI, H. L. On Periodic Solutions of Systems of the Type $\ddot{x}=H(y),\ \dot{y}=-\sum_{i=1}^{n} f_i(x)H_i(y)-g(x)$. 16p.

95-11 OLIVA, S. M., PEREIRA, A. L. Attractors for Parabolic Problems with Non linear Boundary Conditions in Fractional Power Spaces. 28p.

95-12 CORDARO, P. D. Global hypoellipticity for $\overline{\partial}_b$ on certain compact three dimensional CR manifolds. 11p.

95-13 COELHO, F.U. and SKOWRONSKI, A. On Auslander-Reiten Components for Quasitilted Algebras. 16p.

95-14 COELHO, F.U. and HAPPEL, D. Quasitilted algebras admit a preprojective component. 12p.

95-15 GOODAIRE, E.G. and POLCINO MILIES, C. The torsion product property in alternative algebras. 10p.

95-16 GOODAIRE, E. G. and POLCINO MILIES, C. Central idempotents in alternative loop algebras. 7p.

95-17 GOODAIRE, E. G. and POLCINO MILIES, C. Finite conjugacy in alternative loop algebras. 7 p.

95-18 EXEL, R. Unconditional integrability for dual actions. 22p.

95-19 OLIVA, W.M. and SALLUM, E.M. The dynamic of malaria at a rice irrigation system. 11p.

95-20 FIGUEIREDO, L.M.V., GONÇALVES, J.Z. and SHIRVANI, M. Free Group Algebras in Certain Division Rings. 28p.

95-21 SHIRVANI, M. and GONÇALVES, J. Z. Algebraically Independent Orbits and Free Algebras.17p.

95-22 ARAGONA, J. Generalized functions on quasi-regular sets. 17p.

95-23 GUZZO JR., H. On normal and composition series for baric algebras. 18p.
95-24 DRUCK, I. de F. Um pouco da história de potências, exponenciais e logaritmos. 25p.
95-25 BARROS, L.G.X. de and JURIAANS, S.O. Integral Loop Rings of Code Loops. 7p.
95-26 GARCIA D., LOURENÇO M.L., MORAES L.A., and PAQUES O.W. The spectra of some algebras of analytic mappings. 15p.
95-27 BRASIL, A. JR Complete hipersurfaces of S^{n+1} with constant mean curvature and constant scalar curvature. 11p.
95-28 GUZZO JR., H. The bar-radical of baric algebras. 19p.
95-29 MARTINS, M.I.R. Composition factors of indecomposable modules. 26p.
95-30 CORTIZO, S.F. Cálculo Virtual - Parte II. 19p.
95-31 CORTIZO, S.F. Sobre o Cálculo Delta de Dirac. 18p.
95-32 COSTA, R. and SUAZO, A. The Multiplication Algebra of a Bernstein Algebra: Basic Results. 13p.
95-33 FABEL, E., GORODSKI, C. and RUMIN, M. Holonomy of Sub-Riemannian Manifolds. 34p.
95-34 MELO, S.T. Characterizations of Pseudodifferential Operators on the Circle. 9p.
96-01 GUZZO JR., H. On commutative train algebras of rank 3. 15p.
96-02 GOODAIRE, E. G. and POLCINO MILIES, C. Nilpotent Moufang Unit Loops. 9p.
96-03 COSTA., R. and SUAZO, A. The multiplication algebra of a train algebra of rank 3. 12p.
96-04 COELHO, S.P., JESPERS, E. and POLCINO MILIES, C. Automorphisms of Groups Algebras of Some Metacyclic Groups. 12p.
96-05 GIANNONI, F., MASIELLO, A. and PICCIONE, P. Sur une Théorie Variationelle pour Rayons de Lumière sur Variétés Lorentziennes Stablement Causales. 7p.
96-06 MASIELLO, A. and PICCIONE, P. Shortening Null Geodesics in Lorentzian Manifolds. Applications to Closed Light Rays, 17p.
96-07 JURIAANS, S.O. Trace Properties of Torsion Units in Group Rings II. 22p.
96-08 JURIAANS, S.O. and SEHGAL, S.K. On a conjecture of Zassenhauss for Metacyclic Groups. 13p.
96-09 GIANNONI, F. and MASIELLO, A. and PICCIONE, P. A Timelike Extension of Fermat's Principle in General Relativity and Applications. 21p.
96-10 GUZZO JR, H. and VICENTE, P. Train algebras of rank n which are Bernstein or Power-Associative algebras. 11p.
96-11 GUZZO JR, H. and VICENTE, P. Some properties of commutative train algebras of rank 3. 13p.
96-12 COSTA, R. and GUZZO JR., H. A class of exceptional Bernstein algebras associated to graphs. 13p.
96-13 ABREU, N.G.V. Aproximação de operadores não lineares no espaço das funções regradas. 17p.

96-14 HENTZEL, I.R. and PERESI. L.A. Identities of Cayley-Dickson Algebras. 22p.

96-15 CORDES, H.O. and MELO, S.T. Smooth Operators for the Action of SO(3) on $L^2(S^2)$. 12p.

96-16 DOKUCHAEV, M.A., JURIAANS, S.O. and POLCINO MILIES, C. Integral Group Rings of Frobenius Groups and the Conjectures of H.J. Zassenhaus. 22p.

96-17 COELHO, F., DE LA PEÑA, J.A. and TOMÉ, B. Algebras whose Tits Form weakly controls the module category. 19p.

96-18 ANGELERI-HÜGEL, L. and COELHO, F.U. Postprojective components for algebras in H_1, 11p.

96-19 GÓES, C.C., GALVÃO, M.E.E.L. A Weierstrass type representation for surfaces in hyperbolic space with mean curvature one. 17p.

96-20 BARROS, L.G.X. and JURIAANS, S.O. Units in Alternative Integral Loop Rings. 20p.

96-21 GONÇALVES, D.L. and RAPHAEL, D. Characterization of some co-Moore spaces. 17p.

96-22 COSTA, R., GUZZO JR., H. and VICENTE, P. Shape identities in train algebras of rank 3. 14p.

96-23 BARDZELL, M.J. and MARCOS, E.N. Induced Boundary Maps of the Cohomology of Monomial and Auslander Algebras. 11p.

96-24 COELHO, S.P. and POLCINO MILIES, C. Some remarks on Central Idempotents in Group Rings. 5p.

NOTA: Os títulos publicados nos Relatórios Técnicos dos anos de 1980 a 1994 estão à disposição no Departamento de Matemática do IME-USP.

Cidade Universitária "Armando de Salles Oliveira"
Rua do Matão, 1010 - Butantã
Caixa Postal - 66281 (Ag. Cidade de São Paulo)
CEP: 05389-970 - São Paulo - Brasil