

NON-STANDARD VERMA TYPE MODULES FOR $\mathfrak{q}(n)^{(2)}$

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Abstract. We study non-standard Verma type modules over the Kac–Moody queer Lie superalgebra $\mathfrak{q}(n)^{(2)}$. We give a sufficient condition under which such modules are irreducible. We also give a classification of all irreducible diagonal $\mathbb{Z}$-graded modules over certain Heisenberg Lie superalgebras contained in $\mathfrak{q}(n)^{(2)}$.

Introduction

Kac–Moody algebras and their representations play a very important role in many areas of mathematics and physics. The "super" version of these algebras was introduced in [Kac77]. Affine Kac–Moody superalgebras are those of finite growth. Affine symmetrizable superalgebras were described in [Ser11] and [vdL89]. Theory of Verma type modules for affine Lie superalgebras was developed in [ERF09] and [CF18]. In particular, given a Borel subsuperalgebra $\widehat{\mathfrak{b}}$ of the affine Lie superalgebra $\widehat{\mathfrak{g}}$ and a 1-dimensional representation $\mathbb{C}_\lambda$ of $\widehat{\mathfrak{b}}$ for some weight λ of the Cartan subalgebra of $\widehat{\mathfrak{g}}$, one can construct the *Verma type module*

$$M_{\widehat{\mathfrak{b}}}(\lambda) := \operatorname{Ind}_{\widehat{\mathfrak{b}}}^{\widehat{\mathfrak{g}}} \mathbb{C}_\lambda.$$

This module admits a unique maximal proper submodule, and thus, a unique simple quotient. The Verma type module is *non-standard* if $\widehat{\mathfrak{b}}$ does not contain all positive root subspaces for some basis of the root system of $\widehat{\mathfrak{g}}$. In the case in which the finite-dimensional Lie superalgebra associated to $\widehat{\mathfrak{g}}$ is a contragredient Lie superalgebra, all Borel subsuperalgebras of $\widehat{\mathfrak{g}}$ were described in [CF18], see also [DFG09]. The paper [CF18] also gives a criterion for the irreducibility for non-standard Verma type module.

Non-symmetrizable affine Lie superalgebras were classified in [HS07]. In particular, this classification includes a degenerate family of affine Lie superalgebras, series $\mathfrak{q}(n)^{(2)}$. These superalgebras are twisted affinizations of queer Lie superalgebras $\mathfrak{q}(n)$. Structure of Verma modules (= standard Verma type modules) over

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the twisted affine superalgebra $\mathfrak{q}(n)^{(2)}$ with $n \geq 3$ was studied in [GS08]. The current paper advances the theory of Verma type modules for the affine queer Lie superalgebra. We establish sufficient conditions for the irreducibility of all non-standard Verma type modules (Theorem 9 and Theorem 15). We also consider modules induced from analogs of Heisenberg subsuperalgebra and give a criterion of their irreducibility (Theorem 7, Corollary 16).

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Notation. The ground field is $\mathbb{C}$. All vector spaces, algebras, and tensor products are considered to be over $\mathbb{C}$, unless otherwise stated. For a vector space V we denote by $\Lambda(V)$ its Grassmann algebra (i.e., its exterior algebra). For any Lie superalgebra $\mathfrak{a}$ we let $\mathbf{U}(\mathfrak{a})$ denote its universal enveloping algebra.

1. Preliminaries

Let $\mathfrak{q} = \mathfrak{q}(n)$ for $n \geq 3$, be the queer Lie superalgebra, that is,

$$\mathfrak{q} := \left\{ \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right) \;\middle|\; A \in \mathfrak{gl}(n+1), B \in \mathfrak{sl}(n+1) \right\}.$$

Let $\mathfrak{q}_0$ and $\mathfrak{q}_1$ be the even and odd parts of $\mathfrak{q}$, respectively. Choose a Cartan subalgebra $\mathfrak{h}_{\mathfrak{q}} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ of $\mathfrak{q}$ (i.e., $\mathfrak{h}_0$ a Cartan subalgebra of $\mathfrak{q}_0$) and let $\mathfrak{q} = \mathfrak{h}_{\mathfrak{q}} \oplus \left(\bigoplus_{\alpha \in \dot{\Delta}} \mathfrak{q}^\alpha \right)$ be the root space decomposition of $\mathfrak{q}$, where $\mathfrak{q}^\alpha$ denotes the root space associated to the root $\alpha \in \dot{\Delta} \subseteq \mathfrak{h}_0^*$. Recall that every root of $\dot{\Delta}$ is both even and odd, meaning that, for any $\alpha \in \dot{\Delta}$, $\mathfrak{q}^\alpha \cap \mathfrak{q}_i \neq 0$, for $i = 0, 1$. Recall also that $\dot{\Delta} = \dot{\Delta}_0 = \dot{\Delta}_1 = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$.

Although neither $\mathfrak{q}$ nor its affinization $\mathfrak{q}^{(1)}$ are Kac–Moody Lie superalgebras, i.e., admit a set of simple generators, after a twist of $\mathfrak{q}^{(1)}$ by an involution we obtain a regular quasisimple Kac–Moody superalgebra $\widehat{\mathfrak{g}} := \mathfrak{q}^{(2)}$ (see [Ser11]). As a super vector space we have that

$$\widehat{\mathfrak{g}} = L(\mathfrak{sl}(n)) \oplus \mathbb{C}K \oplus \mathbb{C}D, \quad \widehat{\mathfrak{g}}_0 = \mathfrak{sl}(n) \otimes \mathbb{C}[t^{\pm 2}] \oplus \mathbb{C}K \oplus \mathbb{C}D, \quad \text{and} \quad \widehat{\mathfrak{g}}_1 = \mathfrak{sl}(n) \otimes t\mathbb{C}[t^{\pm 2}],$$

where for any Lie superalgebra $\mathfrak{k}$, $L(\mathfrak{k}) := \mathfrak{k} \otimes \mathbb{C}[t^1, t^{-1}]$ is its associated loop superalgebra, K is a central element, and, for all $x(k) := x \otimes t^k \in L(\mathfrak{k})$ with $x \in \mathfrak{k}$ and $k \in \mathbb{Z}$, we have $[D, x(k)] = kx(k)$. Let $\mathfrak{g} = \mathfrak{sl}(n)$. Then for any $x, y \in \mathfrak{g}$, the bracket of $\widehat{\mathfrak{g}}$ is given as follows:

$$[x(k), y(m)] = (xy - yx)(k+m),$$

if km is even; and if we define

$$\iota : \mathfrak{gl}(n) \to \mathfrak{sl}(n), \quad x \mapsto x - \frac{\operatorname{tr}(x)}{n} I_n$$

where I_n is the $n \times n$ identity matrix, then

$$[x(k), y(m)] = \iota(xy + yx)(k+m) + 2\delta_{-k,m} \operatorname{tr}(xy) K,$$

if km is odd. Notice that K does not lie in $[\widehat{\mathfrak{g}}_0, \widehat{\mathfrak{g}}]$, but it lies in $[\widehat{\mathfrak{g}}_1, \widehat{\mathfrak{g}}_1]$. For convenience we set

$$(xy - yx) := [x, y]_0, \text{ and } \iota(xy + yx) := [x, y]_1.$$

Hence, in this notation we have that

$$[x(k), y(m)] = [x, y]_0(k + m), \text{ and } [x(k), y(m)] = [x, y]_1(k + m) + 2\delta_{-k,m} \operatorname{tr}(xy)K,$$

if km is even/odd, respectively.

Remark 1. Notice that if we assume $m \in 2\mathbb{Z}$, then the bracket between any two elements $x(m), y(k) \in L(\mathfrak{g})$ reduces to the bracket in the loop Lie algebra $L(\mathfrak{g})$.

Fix a Cartan subalgebra of $\widehat{\mathfrak{g}}$

$$\widehat{\mathfrak{h}} := \mathfrak{h} \otimes 1 \oplus \mathbb{C}K \oplus \mathbb{C}D$$

where $\mathfrak{h}$ is the Cartan subalgebra of diagonal matrices in $\mathfrak{g}$, and for each $\alpha \in \dot{\Delta}$, choose $f_\alpha \in \mathfrak{g}^{-\alpha}$, $e_\alpha \in \mathfrak{g}^\alpha$ and $h_\alpha \in \mathfrak{h}$ such that $[f_\alpha, e_\alpha]_0 = h_\alpha$.

Notice that, for $g_{\varepsilon_i - \varepsilon_j} \in \mathfrak{g}^{\varepsilon_i - \varepsilon_j}$, we have

$$[h, g_{\varepsilon_i - \varepsilon_j}]_1 = (\varepsilon_i + \varepsilon_j)(h) g_{\varepsilon_i - \varepsilon_j}, \text{ for all } h.$$

For simplicity, if $\alpha = \varepsilon_i - \varepsilon_j$, then we set $\bar{\alpha} := \varepsilon_i + \varepsilon_j$. Thus, in this notation, we have that

$$[h, g_\alpha]_1 = \bar{\alpha}(h) g_\alpha, \text{ for all } h \in \mathfrak{h}.$$

Moreover, if $\alpha_i \neq -\alpha_j$, then

$$[e_{\alpha_i}, f_{\alpha_j}]_1 = g_{\alpha_i + \alpha_j},$$

where $g_{\alpha_i+\alpha_j} = 0$ if $\alpha_i + \alpha_j \notin \dot{\Delta}$, $g_{\alpha_i+\alpha_j} = f_{\alpha_i+\alpha_j}$ if $\alpha_i + \alpha_j \in \dot{\Delta}^-$ and $g_{\alpha_i+\alpha_j} = e_{\alpha_i+\alpha_j}$ if $\alpha_i + \alpha_j \in \dot{\Delta}^+$. Finally, for $\alpha = \varepsilon_i - \varepsilon_j$ we have

$$[e_\alpha, f_\alpha]_1 = \iota(h'_\alpha),$$

where $h'_\alpha = E_{i,i} + E_{j,j}$.

If we identify K with $(1/n)I_n$, then $\mathfrak{h} \otimes 1 \oplus \mathbb{C}K$ can be identified with the Cartan subalgebra of diagonal matrices of $\mathfrak{gl}(n)$. Let $H_1, \ldots, H_n$ denote the standard basis of it (i.e., $H_i = E_{ii}$). The root system of $\widehat{\mathfrak{g}}$ with respect to $\widehat{\mathfrak{h}}$ is given by $\Delta = \{\alpha + k\delta,\ m\delta \mid \alpha \in \dot{\Delta},\ k \in \mathbb{Z},\ m \in \mathbb{Z} \setminus \{0\}\}$. Moreover, $p(\alpha + k\delta) = p(k)$ and $p(m\delta) = p(m)$, where $p(k)$ denotes the parity of k, and by abuse of notation we are denoting the parity of a root β also by $p(\beta)$. Finally, for a subalgebra $\mathfrak{a} \subseteq \widehat{\mathfrak{g}}$ we set

$$\Delta(\mathfrak{a}) := \{\alpha \in \Delta \mid \widehat{\mathfrak{g}}_\alpha \subseteq \mathfrak{a}\}.$$

Consider the subalgebra $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1$ generated by the imaginary root spaces of $\widehat{\mathfrak{g}}$. Then

$$\widehat{\mathcal{H}}_0 = \sum_{r \in \mathbb{Z}} \mathfrak{h} \otimes t^{2r} \oplus \mathbb{C}K, \quad \widehat{\mathcal{H}}_1 = \sum_{r \in \mathbb{Z}} \mathfrak{h} \otimes t^{2r+1}.$$

Notice that the center of $\widehat{\mathcal{H}}$ equals $\widehat{\mathcal{H}}_0$, the odd part $\widehat{\mathcal{H}}_1$ is spanned by $\{(H_i - H_{i+1})(2r + 1) \mid r \in \mathbb{Z}\}$ and the relations in $\widehat{\mathcal{H}}_1$ are given by

$$[x(2r+1), y(-2r-1)] = 2xy, \quad [x(2r+1), y(2s+1)] = \iota(2xy)(2(r+s+1))$$

for $r+s+1 \neq 0$. In particular, differently from the case of basic classical Lie superalgebras, the subalgebra $\widehat{\mathcal{H}}$ is not isomorphic to a Heisenberg algebra.

2. Generalized Verma type modules

Since the root system Δ of $\widehat{\mathfrak{g}}$ is the same as that of $\widehat{\mathfrak{sl}}(n)$, the sets of positive roots of Δ are obtained in the same way: fix $\Pi \subseteq \dot{\Delta}$ a set of simple roots, pick a subset $X \subseteq \Pi$, and let $\mathcal{W}$ denote the Weyl group of $\widehat{\mathfrak{sl}}(n)$. Let $\dot{\Delta}^+ = \langle \Pi \rangle_{\mathbb{Z}_{>0}} \cap \dot{\Delta}$, $\dot{\Delta}(X)^+ = \langle X \rangle_{\mathbb{Z}_{>0}} \cap \dot{\Delta}$, and $\dot{\Delta}(X) = \langle X \rangle_{\mathbb{Z}} \cap \dot{\Delta}$. Associated to X we define

$$\begin{aligned}\Delta(X)^+ := \{\alpha + k\delta \mid \alpha \in \dot{\Delta}^+ \setminus \dot{\Delta}(X)^+,\ k \in \mathbb{Z}\} \\ \cup \{\alpha + k\delta \mid \alpha \in \dot{\Delta}(X) \cup \{0\},\ k \in \mathbb{Z}_{>0}\} \cup \dot{\Delta}(X)^+.\end{aligned}$$

Then $\Delta(X)^+$ is a set of positive roots of Δ, and up to $\mathcal{W} \times \{\pm 1\}$-conjugation, every set of positive roots is of this form for some set of simple roots Π and some subset $X \subseteq \Pi$.

Consider the following subalgebras associated to X:

$$\mathfrak{m}(X) := \mathfrak{m}(X)^- \oplus \mathfrak{h} \oplus \mathfrak{m}(X)^+, \quad \mathfrak{m}(X)^{\pm} := \bigoplus_{\alpha \in \dot{\Delta}(X)^{\pm}} \mathfrak{g}_\alpha,$$
$$\mathfrak{u}(X)^{\pm} := \bigoplus_{\alpha \in \dot{\Delta}^{\pm} \setminus \dot{\Delta}(X)^{\pm}} \mathfrak{g}_\alpha.$$

Thus

$$\mathfrak{g} = \mathfrak{u}(X)^- \oplus \mathfrak{m}(X) \oplus \mathfrak{u}(X)^+ \quad \text{and} \quad \widehat{\mathfrak{g}} = L(\mathfrak{u}(X)^-) \oplus \widehat{\mathfrak{m}}(X) \oplus L(\mathfrak{u}(X)^+),$$

where $\widehat{\mathfrak{m}}(X) = L(\mathfrak{m}(X)) \oplus \mathbb{C}K \oplus \mathbb{C}D$.

Consider now the subalgebra

$$\mathfrak{k}(X) := \mathfrak{m}(X)^- \oplus \mathfrak{h}_X \oplus \mathfrak{m}(X)^+, \text{ where } \mathfrak{h}_X := \bigoplus_{\alpha \in \dot{\Delta}(X)^+} [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha].$$

Then $\mathfrak{m}(X) = \mathfrak{k}(X) \oplus \mathfrak{h}^X$, where $\mathfrak{h}^X := \{h \in \mathfrak{h} \mid \alpha(h) = 0,\ \forall \alpha \in \dot{\Delta}(X)\}$ is the center of $\mathfrak{m}(X)$. Set

$$\widehat{\mathfrak{k}}(X) := L(\mathfrak{k}(X)) \oplus \mathbb{C}K \oplus \mathbb{C}D \oplus \mathfrak{h}^X$$

with standard triangular decomposition

$$\widehat{\mathfrak{k}}(X) = \widehat{\mathfrak{k}}(X)^- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{k}}(X)^+, \quad \widehat{\mathfrak{k}}(X)^{\pm} = (\mathfrak{k}(X) \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}]) \oplus \mathfrak{m}(X)^{\pm}.$$

In particular, we have that

$$\widehat{\mathfrak{m}}(X) = \mathfrak{h}^X \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus (\widehat{\mathfrak{k}}(X)^- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{k}}(X)^+) \oplus \mathfrak{h}^X \otimes t\mathbb{C}[t]$$

and

$$\widehat{\mathfrak{g}} = (L(\mathfrak{u}(X)^-) \oplus \mathfrak{h}^X \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus \widehat{\mathfrak{k}}(X)^- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{k}}(X)^+ \oplus (\mathfrak{h}^X \otimes t\mathbb{C}[t] \oplus L(\mathfrak{u}(X)^+)).$$

Remark 2.

(1) Differently from the case of basic classical Lie superalgebras (this includes all simple Lie algebras), the imaginary subalgebra

$$\mathcal{H}(X) := (\mathfrak{h}^X \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus (\mathfrak{h}^X \oplus \mathbb{C}K) \oplus (\mathfrak{h}^X \otimes t\mathbb{C}[t])$$

is not a Heisenberg algebra. Another difference (from the Lie algebra case) is that we do not have that $[\mathcal{H}(X), \widehat{\mathfrak{k}}(X)] = 0$. In fact,

$$[\mathcal{H}_0(X), \widehat{\mathfrak{k}}(X)] = [\mathcal{H}(X), \widehat{\mathfrak{k}}(X)_0] = 0, \text{ but } [\mathcal{H}_1(X), \widehat{\mathfrak{k}}(X)_1] \neq 0.$$

Compare also with the isotropic case of [CF18].

(2) Heisenberg algebras admit a family of triangular decompositions parametrized by maps $\varphi : \mathbb{N} \to \{\pm\}^d$, where d is a certain dimension. It is worth noting that the algebra $\mathcal{H}(X)$ does not admit such decompositions, except the trivial ones (i.e., when $\varphi(i) = (+, \dots, +)$ for all $i \in \mathbb{N}$, or $\varphi(i) = (-, \dots, -)$ for all $i \in \mathbb{N}$).

Consider the triangular decomposition of $\mathcal{H}(X)$

$$\mathcal{H}(X) = \mathcal{H}(X)^- \oplus (\mathfrak{h}^X \oplus \mathbb{C}K) \oplus \mathcal{H}(X)^+,$$

where

$$\mathcal{H}(X)^\pm := \mathfrak{h}^X \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}],$$

and define $\mathcal{H}(X)_i^\pm := \mathcal{H}(X)_i \cap \mathcal{H}(X)^\pm$, for $i \in \mathbb{Z}_2$. Then we have a commutative algebra

$$\mathcal{S}(X) := \mathbf{U}(\mathcal{H}(X)_{\bar{0}}^-),$$

and we let $\mathcal{S}(X)^+$ denote the augmentation ideal of $\mathcal{S}(X)$.

Consider the triangular decompositions

$$\widehat{\mathfrak{m}}(X) = \widehat{\mathfrak{m}}(X)^- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{m}}(X)^+, \quad \text{where} \quad \widehat{\mathfrak{m}}(X)^\pm = \mathcal{H}(X)^\pm \oplus \widehat{\mathfrak{k}}(X)^\pm$$

and

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}(X)^- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{g}}(X)^+, \quad \text{where} \quad \widehat{\mathfrak{g}}(X)^\pm = L(\mathfrak{u}(X)^\pm) \oplus \widehat{\mathfrak{m}}(X)^\pm.$$

Fix the subalgebra $\widehat{\mathfrak{b}}(X) := \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{g}}(X)^+$ of $\widehat{\mathfrak{g}}$. Notice that $\widehat{\mathfrak{g}}(X)^+ \cap \widehat{\mathfrak{m}}(X) = \widehat{\mathfrak{m}}(X)^+$, $\widehat{\mathfrak{g}}(X)^+ \cap \widehat{\mathfrak{k}}(X) = \widehat{\mathfrak{k}}(X)^+$, and $\widehat{\mathfrak{g}}(X)^+ \cap \mathcal{H}(X) = \mathcal{H}(X)^+$. In what follows, we fix a set $X \subseteq \Pi$ and we drop the X from the notation above (for instance, we write $\widehat{\mathfrak{m}}^+$ instead of writing $\widehat{\mathfrak{m}}(X)^+$).

Let $\lambda \in \widehat{\mathfrak{h}}^*$, $\widehat{\mathfrak{s}} \in \{\widehat{\mathfrak{g}}, \widehat{\mathfrak{m}}, \widehat{\mathfrak{k}}, \mathcal{H}\}$, and $\widehat{\mathfrak{r}} = \widehat{\mathfrak{s}} \cap \widehat{\mathfrak{b}}$. Then we define the Verma $\widehat{\mathfrak{s}}$-module

$$M(\widehat{\mathfrak{s}}, \lambda) := \mathbf{U}(\widehat{\mathfrak{s}}) \otimes_{\mathbf{U}(\widehat{\mathfrak{r}})} \mathbb{C}v_\lambda,$$

where $\mathbb{C}v_\lambda$ is the $\widehat{\mathfrak{r}}$-module whose action of $\widehat{\mathfrak{h}}$ is determined by λ and the action of the nilpotent radical of $\widehat{\mathfrak{r}}$ is trivial. The unique irreducible quotient of $M(\widehat{\mathfrak{s}}, \lambda)$ will be denoted $L(\widehat{\mathfrak{s}}, \lambda)$. Also, for $\widehat{\mathfrak{s}}, \widehat{\mathfrak{r}}$ such that either $\widehat{\mathfrak{r}} = \widehat{\mathfrak{k}}$ and $\widehat{\mathfrak{s}} = \widehat{\mathfrak{m}}$, or $\widehat{\mathfrak{r}} = \widehat{\mathfrak{m}}$ and $\widehat{\mathfrak{s}} = \widehat{\mathfrak{g}}$, and an $\widehat{\mathfrak{r}}$-module N we define the module

$$M(\widehat{\mathfrak{s}}, \widehat{\mathfrak{r}}; N) := \mathbf{U}(\widehat{\mathfrak{s}}) \otimes_{\mathbf{U}(\widehat{\mathfrak{r}})} N,$$

where $\mathcal{H}(X)^+$ is assumed to act trivially on N if $\widehat{\mathfrak{r}} = \widehat{\mathfrak{k}}$ and $\widehat{\mathfrak{s}} = \widehat{\mathfrak{m}}$, and $L(\mathfrak{u}_+)$ is assumed to act trivially on N if $\widehat{\mathfrak{r}} = \widehat{\mathfrak{m}}$ and $\widehat{\mathfrak{s}} = \widehat{\mathfrak{g}}$. Notice that

$$M(\widehat{\mathfrak{g}}, \lambda) \cong M(\widehat{\mathfrak{g}}, \widehat{\mathfrak{m}}; M(\widehat{\mathfrak{m}}, \lambda)) \text{ and } M(\widehat{\mathfrak{m}}, \lambda) \cong M(\widehat{\mathfrak{m}}, \widehat{\mathfrak{k}}; M(\widehat{\mathfrak{k}}, \lambda)).$$

Using the terminology of [Fut97], the module $M(\widehat{\mathfrak{g}}, \widehat{\mathfrak{m}}; N)$ is called a *generalized Verma type module*, or a *generalized Imaginary Verma module*. When N is an irreducible weight $\widehat{\mathfrak{r}}$-module, $M(\widehat{\mathfrak{s}}, \widehat{\mathfrak{r}}; N)$ admits a unique irreducible quotient which will be denoted by $L(\widehat{\mathfrak{s}}, \widehat{\mathfrak{r}}; N)$.

3. Irreducible $\mathcal{H}$-modules

Consider the triangular decomposition

$$\mathcal{H} = \mathcal{H}^- \oplus \mathbb{C}K \oplus \mathcal{H}^+.$$

Then we have the following character formula

$$\text{ch } M(\mathcal{H}, \lambda) = e^{\lambda} \prod_{\alpha \in \Delta(\mathcal{H}^-)_0} (1 - e^{-\alpha})^{-1} \prod_{\alpha \in \Delta(\mathcal{H}^-)_1} (1 + e^{-\alpha}).$$

Notice that the subalgebra $\mathcal{S}$ lies in the center of $\mathbf{U}(\mathcal{H})$ and acts freely on $M(\mathcal{H}, \lambda)$. Then any ideal J of $\mathcal{S}$ defines the $\mathcal{H}$-submodule $JM(\mathcal{H}, \lambda)$ of $M(\mathcal{H}, \lambda)$. On the other direction, for any $\mathcal{H}$-submodule $N \subseteq M(\mathcal{H}, \lambda)$ we define an ideal J_N of $\mathcal{S}$ by requiring the equality:

$$N \cap \mathcal{S} v_\lambda = J_N v_\lambda.$$

In other words, $J_N = \{a \in \mathcal{S} \mid a v_\lambda \in N\}$.

Let D_δ^X be the matrix determined by the pairing

$$(\mathfrak{h}^X \otimes \mathbb{C}t) \times (\mathfrak{h}^X \otimes \mathbb{C}t^{-1}) \to \widehat{\mathfrak{h}}, \quad (x, y) \mapsto [x, y],$$

and consider $\det D_\delta^X$ as an element of the symmetric algebra $S(\widehat{\mathfrak{h}})$.

Example 1. If $n = 3$ and $X = \{\varepsilon_1 - \varepsilon_2\}$, then $\mathfrak{h}^X = \mathbb{C}h^1$, where $h^1 = H_1 + H_2 - 2H_3$. In particular, $\det D_\delta^X = 2(H_1 + H_2 - 2H_3)^2$. If $X = \varnothing$ and $n \geq 3$, then $\mathfrak{h}^X = \mathfrak{h}$, and $\det D_\delta^X = 2^{n-1} H_1 \cdots H_n (1/H_1 + \cdots + 1/H_n)$ (see [GS08]).

Proposition 1. *The $\mathcal{H}$-module $M(\mathcal{H}, \lambda)$ is reducible. If $\det D_\delta^X(\lambda) \neq 0$, then there is a bijection between submodules of $M(\mathcal{H}, \lambda)$ and ideals of $\mathcal{S}$. In particular, $L(\mathcal{H}, \lambda) \cong \Lambda(\mathcal{H}_1^-)$ as vector spaces.*

Proof. The fact that $\mathcal{H}_0^-$ is in the center of $\mathcal{H}$ implies that any ideal J of $\mathcal{S}$ defines a submodule of $M(\mathcal{H},\lambda)$, namely, $JM(\mathcal{H},\lambda)$. Thus the first statement follows.

Now, if we assume $\det D_\delta^X(\lambda) \neq 0$, then we can use similar arguments to those of [GS08, Proposition 3] to prove that there is a bijection between ideals of $\mathcal{S}$ and submodules of $M(\mathcal{H},\lambda)$. Namely, let $M = M(\mathcal{H},\lambda)$, and let N be a submodule of M. We claim that $N = J_N M$. Indeed, let $jm \in J_N M$. Then, writing $m = uv_\lambda$ with $u \in \mathbf{U}(\mathcal{H}^-)$, we get that

$$jm = juv_\lambda = ujv_\lambda \in uN \subseteq N.$$

Thus $J_N M \subseteq N$. In order to prove the other inclusion, we consider the canonical projection $\pi : M \to V := M/J_N M$, $W = \pi(N)$, and $R = \mathcal{S}/J_N$. Notice that V is free as an R-module, and that $W \cap Rv_\lambda = \pi(N \cap \mathcal{S}v_\lambda) = \pi(J_N v_\lambda) = 0$. Now we suppose that $W \neq 0$ to get a contradiction.

Let $h^1, \ldots, h^t$ be any fixed basis of $\mathfrak{h}^X$, and set $X_{i,m} := h^i(m)$, and $Y_{i,m} := h^i(-m)$. Recall from the commutation relations of $\mathcal{H}$ that $[X_{i,m}, Y_{k,m}] = [X_{i,0}, Y_{k,0}]$, and since we are assuming that $\det D_\delta^X(\lambda) \neq 0$, we may consider that the basis elements $h^1, \ldots, h^t$ were chosen so that $\lambda([X_{i,j}, Y_{k,j}]) = \delta_{i,k}$. Notice that the elements $X_{i,j}$ for $i = 1, \ldots, r$ and $m \geq 0$ form a basis for $\mathcal{H}_1^-$. In particular, if we let $X_{i,m} \geq X_{k,n}$ if $m \geq n$ or $m = n$ and $i \geq k$, then the monomials $X_{i_1,m_1} \cdots X_{i_s,m_s}$ with $X_{i_1,m_1} > \cdots > X_{i_s,m_s}$ form a basis B of V over R.

Since we are assuming $W \neq 0$, and since $W \cap Rv_\lambda = 0$, we can choose a nonzero $v \in W$ such that the maximal $X_{i,m}$ that occurs in the expression of v as a linear combination of elements of B is minimal among all nonzero vectors of W. Now we write $v = X_{i,m}w + u$ for nonzero w and u such that all factors occurring in w and u are less than $X_{i,m}$. Thus

$$Y_{i,m}u = Y_{i,m}w = 0 \text{ and } Y_{i,m}v = w,$$

as $[X_{i,m}, Y_{i,m}]$ is in the center of $\mathcal{H}$ and it acts as $\lambda([X_{i,m}, Y_{k,m}]) = 1$ on v_λ. But this implies $0 \neq w \in W$, and all factors occurring in w are less than $X_{i,m}$, which is a contradiction. □

Corollary 2. *Suppose that* $\det D_\delta^X(\lambda) \neq 0$. *Then we have the character formula*

$$\mathrm{ch}\ L(\mathcal{H},\lambda) = \mathrm{ch}\ \Lambda(\mathcal{H}_1^-) = e^\lambda \prod_{\alpha \in \Delta(\mathcal{H}^-)_1} (1 + e^{-\alpha}).$$

Proof. This follows from the isomorphism of vector spaces $L(\mathcal{H},\lambda) \cong \Lambda(\mathcal{H}_1^-)$. □

1. Modules for Heisenberg Lie superalgebra

In this section we consider the special case where $X = \varnothing$, and, in particular, $\mathfrak{h}^X = \mathfrak{h}$ and $\mathcal{H} = \mathcal{H}(X) = L(\mathfrak{h}) \oplus \mathbb{C}K$.

Define

$$\mathcal{H}_0' := \bigoplus_{r \in \mathbb{Z}} \mathfrak{h} \otimes t^{2r}.$$

It is clear that $\mathcal{H}_0'$ is an ideal of $\mathcal{H}$, and $K \notin \mathcal{H}_0'$. Define

$$\widetilde{\mathcal{H}} := \mathcal{H}/\mathcal{H}_0'.$$

Lemma 3. *Let $\pi : \mathcal{H} \to \widetilde{\mathcal{H}}$ be the canonical projection. Then there exists a basis $\{h^1, \ldots, h^{n-1}\}$ of $\mathfrak{h}$ such that $\pi(h^i h^j) = \delta_{ij} K$.*

Proof. The set $\{H_1 + \cdots + H_i - iH_{i+1} \mid 1 \le i \le n-1\}$ is a basis of $\mathfrak{h}$ such that $H^i H^j \in \mathfrak{h}$ if and only if $i \neq j$. Then a suitable normalization of this basis gives the required one. $\square$

Now we have the following result:

Proposition 4. *$\widetilde{\mathcal{H}}$ is an infinite-dimensional Heisenberg Lie superalgebra such that*

$$\widetilde{\mathcal{H}} \cong \mathbb{C}K \oplus \bigoplus_{r \in \mathbb{Z}} \mathfrak{h} \otimes t^{2r+1}$$

as vector spaces, where $[h \otimes t^{2r+1}, h' \otimes t^{-2r-1}]$ is a multiple of K and $[h \otimes t^{2r+1}, h' \otimes t^{2s+1}] = 0$ for all $h, h' \in \mathfrak{h}$ and all integer r, s with $r + s + 1 \neq 0$.

Fix a basis $h^1, \ldots, h^{n-1}$ of $\mathfrak{h}$ as in Lemma 3, and let $\varphi : \mathbb{N} \to \{\pm\}^{n-1}$ be a map of sets. Then φ induces a triangular decomposition on $\widetilde{\mathcal{H}}$:

$$\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_{\varphi}^{-} \oplus \mathbb{C}K \oplus \widetilde{\mathcal{H}}_{\varphi}^{+},$$

where

$$\widetilde{\mathcal{H}}_{\varphi}^{\pm} = \left(\bigoplus_{\substack{n \in \mathbb{N},\ 1 \le i \le t, \\ \varphi(n)_i = \pm}} \mathbb{C}h^i \otimes t^{2n+1} \right) \oplus \left(\bigoplus_{\substack{m \in \mathbb{N},\ 1 \le i \le t, \\ \varphi(m)_i = \mp}} \mathbb{C}h^i \otimes t^{-(2m+1)} \right),$$

and $\varphi(n) = (\varphi(n)_1, \ldots, \varphi(n)_{n-1})$. The Verma module associated to such a decomposition is called the φ-*Verma module* and it is denoted by $M_{\varphi}(\widetilde{\mathcal{H}}, a)$, where $a \in \mathbb{C}$ is the value of K on $M_{\varphi}(\widetilde{\mathcal{H}}, a)$. The module $M_{\varphi}(\widetilde{\mathcal{H}}, a)$ is isomorphic (as a vector space) to $\mathbf{U}(\widetilde{\mathcal{H}}_{\varphi}^{-})$ which is nothing but the Grassmann algebra $\Lambda(\widetilde{\mathcal{H}}_{\varphi}^{-})$. Finally let $L_{\varphi}(\widetilde{\mathcal{H}}, a)$ denote the unique irreducible quotient of $M_{\varphi}(\widetilde{\mathcal{H}}, a)$.

Remark 3. Notice that every $\widetilde{\mathcal{H}}$-module can (and will) be regarded as an $\mathcal{H}$-module via the canonical projection $\mathcal{H} \twoheadrightarrow \widetilde{\mathcal{H}}$.

Corollary 5. *If $\lambda(\mathfrak{h}) = 0$, then the action of $\mathcal{H}$ on $L(\mathcal{H}, \lambda)$ factors through the epimorphism*

$$\mathcal{H} \twoheadrightarrow \widetilde{\mathcal{H}}.$$

In particular, if $\lambda(K) := a \neq 0$, then $L(\mathcal{H}, \lambda) \cong M_{\varphi}(\widetilde{\mathcal{H}}, a)$ as $\widetilde{\mathcal{H}}$-modules, where $\varphi(i) = (+, \ldots, +)$ for all $i \in \mathbb{N}$ (i.e., $M_{\varphi}(\widetilde{\mathcal{H}}, a)$ is nothing but the standard Verma module of $\widetilde{\mathcal{H}}$).

Proof. We have $(\mathfrak{h} \otimes t^2\mathbb{C}[t])L(\mathcal{H}, \lambda) = 0$, since $\mathfrak{h} \otimes t^2\mathbb{C}[t]$ is in the center of $\mathcal{H}$ and it acts trivially on v_{λ}. Next, $\mathfrak{h} \otimes t^{-2}\mathbb{C}[t^{-1}]$ is contained in the maximal ideal $\mathcal{S}^+$ of $\mathcal{S}$, and then, by Proposition 1, we must have $(\mathfrak{h} \otimes t^{-2}\mathbb{C}[t^{-1}])L(\mathcal{H}, \lambda) = 0$. Finally, since $\lambda(\mathfrak{h}) = 0$, we conclude that $\mathcal{H}_0' L(\mathcal{H}, \lambda) = 0$ and the first statement follows.

Using similar arguments as those of [BBFK13, Proposition 3.3] one easily shows that $M_{\varphi}(\widetilde{\mathcal{H}}, a)$ is an irreducible $\widetilde{\mathcal{H}}$-module if and only if $a \neq 0$. Thus the result follows. $\square$

Let N be an irreducible $\mathcal{H}$-module such that $\mathfrak{h}N = 0$. We are interested in the case when N is $\mathbb{Z}$-graded. Then we can define the action of D on N by $D|_{N_i} = i\,\mathrm{Id}$. Notice that under such conditions $\mathcal{H}_0'$ must act trivially on N (indeed, N is irreducible and $\mathbb{Z}$-graded, $\mathcal{H}_0'$ is central in $\mathcal{H}$, $\mathfrak{h}N = 0$ and any element of $\mathfrak{h} \otimes t^{2r}$ with $r \in \mathbb{Z}^\times$ has degree different from 0).

Set $x_k^j = h^j \otimes t^k$, $k \in \mathbb{Z}$, $j = 1, \ldots, n-1$, so that

$$\widetilde{\mathcal{H}} \cong \mathbb{C}K \oplus \bigoplus_{\substack{r\in\mathbb{Z},\\ j=1,\ldots,t}} \mathbb{C}x_{2r+1}^j$$

and $[x_{2r+1}^j, x_{2s-1}^i] = \delta_{ij}\delta_{r,-s}K$, after suitable rescaling (see Lemma 3 and Proposition 4). Also set

$$d_{2r+1}^j := x_{-2r-1}^j x_{2r+1}^j, \quad r \in \mathbb{Z}_{\geq 0},\ j = 1, \ldots, n-1.$$

Since K is central and N is irreducible, we have that K acts on $N = \sum_{i\in\mathbb{Z}} N_i$ via multiplication by some $a \in \mathbb{C}$. Assume that $a \neq 0$, and fix a nonzero $v \in N_i$ for some i. Then

$$\begin{aligned}(d_{2r+1}^j)^2 v &= (x_{-2r-1}^j x_{2r+1}^j)(x_{-2r-1}^j x_{2r+1}^j)\\ &= x_{-2r-1}^j (K - x_{-2r-1}^j x_{2r+1}^j) x_{2r+1}^j = a d_{2r+1}^j v,\end{aligned}$$

that is, d_{2r+1}^j is diagonalizable on N_i and has eigenvalues a or 0. Now we have:

Lemma 6. *If $d_{2r+1}^j v = av$, then $x_{-2r-1}^j v = 0$. On the other hand, if $d_{2r+1}^j v = 0$, then $x_{2r+1}^j v = 0$.*

Proof. The fact that $x_{-2r-1}^j d_{2r+1}^j = 0$ implies the first statement. For the second statement observe that $d_{2r+1}^j v = 0$ implies $x_{2r+1}^j x_{-2r-1}^j v = av$. Hence the result follows. □

A non-zero $\mathbb{Z}$-graded $\mathcal{H}$-module N is *diagonal* if all d_{2r+1}^j are simultaneously diagonalizable for $r \in \mathbb{Z}_{\geq 0}$, $j = 1, \ldots, n-1$. Let N_i be a graded component of a diagonal $\mathbb{Z}$-graded $\mathcal{H}$-module N. We associate to N_i a t-tuple $(\mu^1, \ldots, \mu^{n-1})$ of infinite sequences $\mu^j = (\mu_{2r+1}^j)$ consisting of the eigenvalues μ_{2r+1}^j of d_{2r+1}^j, $r \in \mathbb{Z}_{\geq 0}$, $j = 1, \ldots, n-1$. In what follows we classify all diagonal irreducible modules with trivial action of $\mathfrak{h}$, and we describe their structure.

Theorem 7. *Let N be an irreducible diagonal $\mathbb{Z}$-graded $\mathcal{H}$-module, such that $\mathfrak{h}N = 0$ and $Kv = av$ for some $a \in \mathbb{C}$ and all $v \in N$. Then the following hold:*

1. *$\mathcal{H}_0'$ acts trivially on N, which is an irreducible $\widetilde{\mathcal{H}}$-module.*
2. *If $v \in N$ is a nonzero homogeneous element, then v is φ_μ-highest vector, where φ_μ is determined by the eigenvalues of d_{2r+1}^j on v, and $N \simeq L_{\varphi_\mu}(\widetilde{\mathcal{H}}, a)$ up to a shift of gradation. In particular, if $a \neq 0$, then $N \simeq M_{\varphi_\mu}(\widetilde{\mathcal{H}}, a)$ up to a shift of gradation.*
3. *If $a = 0$, then N is the trivial 1-dimensional module.*
4. *If $a \neq 0$, then $M_{\varphi_\mu}(\widetilde{\mathcal{H}}, a)$ has finite-dimensional graded components if and only if φ_μ differs from φ_ν only in finitely many places, where $\nu_{2k+1}^j = 0$ for all $k \in \mathbb{Z}_{\geq 0}$, $j = 1, \ldots, n-1$, or $\nu_{2k+1}^j \neq 0$ for all $k \in \mathbb{Z}_{\geq 0}$, $j = 1, \ldots, n-1$.*

Proof. Part(a): this follows from the fact that N is irreducible and $\mathbb{Z}$-graded, $\mathcal{H}'_0$ is central and its elements have degree different from 0. Part(b): let $N_i \neq 0$ such that all d^j_{2r+1} are simultaneously diagonalizable with eigenvalues μ^j_{2r+1}. Set $\mu^j = (\mu^j_{2r+1})$, $r \in \mathbb{Z}_{\geq 0}$, $j = 1, \dots, n-1$. By Lemma 6, each $(\mu^1, \dots, \mu^{n-1})$ defines a function $\varphi_\mu : \mathbb{N} \to \{\pm\}^{n-1}$, where $\varphi_\mu(k)_j = +$ if $\mu^j_{2k+1} = 0$ and $\varphi_\mu(k)_j = -$ if $\mu^j_{2k+1} = a$. Then v is a φ_μ-highest vector and $N \simeq L_{\varphi_\mu}(\widetilde{\mathcal{H}}, \lambda)$ up to a shift of gradation. Part(c) is clear. Part(d): without loss of generality we assume that $\nu^j_{2k+1} = 0$ for all $k \in \mathbb{Z}_{\geq 0}$, $j = 1, \dots, n-1$. Clearly, $M_{\varphi_\nu}(\widetilde{\mathcal{H}}, \lambda)$ has finite-dimensional graded components. Suppose that φ_μ differs from φ_ν only in s places. Consider a nonzero φ_μ-highest vector v. If $w = x^j_{2k+1} v \neq 0$ for some $k \geq 0$ and $j = 1, \dots, n-1$, then $x^j_{2k+1} w = 0$ and thus w is a $\varphi_{\mu'}$-highest vector where $\varphi_{\mu'}$ differs from φ_ν in $s-1$ places. Continuing we find a φ_ν-highest vector in $M_{\varphi_\mu}(\widetilde{\mathcal{H}}, \lambda)$. Since $M_{\varphi_\mu}(\widetilde{\mathcal{H}}, \lambda)$ is irreducible when $a \neq 0$ we conclude that $M_{\varphi_\mu}(\widetilde{\mathcal{H}}, \lambda) \simeq M_{\varphi_\nu}(\widetilde{\mathcal{H}}, \lambda)$ and hence it has finite-dimensional graded components. Conversely, assume that $M_{\varphi_\mu}(\widetilde{\mathcal{H}}, \lambda)$ has finite-dimensional graded components and let v be a nonzero φ_μ-highest vector. Denote by Ω_μ the subset of odd integers defined as follows: $k \in \Omega_\mu$ if $x^j_k v \neq 0$ for at least one $j = 1, \dots, n-1$. A sequence $(k_1, \dots, k_r)$ of Ω_μ is called *cycle* if $\sum_{i=1}^r k_i = 0$. Suppose Ω contains infinitely many positive as well as negative odd integers. Then one can form infinitely many cycles. Each such cycle $(k_1, \dots, k_r)$ yields a basis element $\Pi_{i=1}^r x^{j_i}_{k_i} v$ of $M_{\varphi_\mu}(\widetilde{\mathcal{H}}, \lambda)$ which is a contradiction. Hence, Ω contains only finitely many positive or only finitely negative odd integers. This means that φ_μ differs from φ_ν only in finitely many places, where $\nu^j_{2k+1} = 0$ for all $k \in \mathbb{Z}_{\geq 0}$, $j = 1, \dots, n-1$, or $\nu^j_{2k+1} \neq 0$ for all $k \in \mathbb{Z}_{\geq 0}$, $j = 1, \dots, n-1$. □

Remark 4. We conjecture that any irreducible $\mathbb{Z}$-graded $\widetilde{\mathcal{H}}$-module is diagonal.

We also have the following isomorphism criterion.

Proposition 8. *We have that $M_{\varphi_\mu}(\widetilde{\mathcal{H}}, a) \simeq M_{\varphi_{\mu'}}(\widetilde{\mathcal{H}}, a')$ (up to a shift of gradation) if and only if $a = a'$ and φ_μ and $\varphi_{\mu'}$ differ only in finitely many places.*

Proof. The condition $a = a'$ is clear. Assume that for some r and j, d^j_{2r+1} has an eigenvector $v \in M_{\varphi_\mu}(\widetilde{\mathcal{H}}, a)$ with eigenvalue $\mu^j_{2r+1} = a$. Set $w = x^j_{2r+1} v \neq 0$. Then $x^j_{2r+1} w = 0$ and hence w is a φ_ν-highest vector where $\nu^i_{2k+1} = \mu^i_{2k+1}$ if $k \neq r$ or $i \neq j$, while $\nu^j_{2r+1} = 0$. We have $M_{\varphi_\nu}(\widetilde{\mathcal{H}}, a) \simeq M_{\varphi_\mu}(\widetilde{\mathcal{H}}, a)$. Similarly, we can change finitely many nonzeros μ's to zeros.

Conversely, if we have the isomorphism, then one can obtain a $\varphi_{\mu'}$-highest weight vector by finitely many actions of elements $x^j_{\pm(2r+1)}$ on a φ_μ-highest weight vector. This implies the statement. □

4. Irreducibility of generalized Verma type modules

In this section we prove our main result which is the following theorem.

Theorem 9.

(1) *$M(\widehat{\mathfrak{m}},\widehat{\mathfrak{k}};L(\widehat{\mathfrak{k}},\lambda))$ and $M(\widehat{\mathfrak{s}},\lambda)$ are reducible for any $\widehat{\mathfrak{s}}\in\{\widehat{\mathfrak{g}},\widehat{\mathfrak{m}},\widehat{\mathfrak{k}},\mathcal{H}\}$.*
(2) *If $\det D^X_\delta(\lambda)\neq 0$, then there is a bijection between submodules of the module $M(\widehat{\mathfrak{m}},\widehat{\mathfrak{k}};L(\widehat{\mathfrak{k}},\lambda))$ and ideals of the algebra $\mathcal{S}$.*
(3) *If $\det D^X_\delta(\lambda)\neq 0$, then $M(\widehat{\mathfrak{g}},\widehat{\mathfrak{m}};L(\widehat{\mathfrak{m}},\lambda))$ is irreducible.*

The next two results imply Theorem 9 items (1) and (2).

Corollary 10. *$M(\widehat{\mathfrak{s}},\lambda)$ is reducible for any $\widehat{\mathfrak{s}}\in\{\widehat{\mathfrak{g}},\widehat{\mathfrak{m}},\widehat{\mathfrak{k}},\mathcal{H}\}$.*

Proof. It follows from Proposition 1. □

Proposition 11. *Let $M=M(\widehat{\mathfrak{m}},\widehat{\mathfrak{k}};L(\widehat{\mathfrak{k}},\lambda))$, $L=L(\widehat{\mathfrak{k}},\lambda)$, and assume that we have $\det D^X_\delta(\lambda)\neq 0$. Then there is a bijection between submodules of M and ideals of $\mathcal{S}$; $\mathcal{S}^+M$ is a maximal proper submodule of M; and $L(\widehat{\mathfrak{m}},\widehat{\mathfrak{k}};L)\cong\Lambda(\mathcal{H}_1^-)\otimes_{\mathbb{C}}L$ as vector spaces.*

Proof. Let J be an ideal of $\mathcal{S}$. Since $[\mathcal{H}_0,\widehat{\mathfrak{m}}]=0$, it is clear that JM defines a submodule of M. In the other direction, for a submodule $N\subseteq M$, we consider the ideal $J_N\subseteq\mathcal{S}$ such that $N\cap\mathcal{S}L=J_NL$. We claim that $J_N=\{a\in\mathcal{S}\mid av_\lambda\in N\}$. Indeed, let $a\in J_N$, and write an arbitrary $v\in L$ as uv_λ for some $u\in\mathbf{U}(\widehat{\mathfrak{k}}^-)$. Then we have $av=auv_\lambda=uav_\lambda\in N$, and hence $J_NL\subseteq N\cap\mathcal{S}L$. For the other inclusion, write a general element $v=\sum_{i=1}^m a_iv_i\in N\cap\mathcal{S}L$ with $a_i\in\mathcal{S}$ and assume that $v_1,\dots,v_m\in L$ are linearly independent. The fact that $[\mathcal{H}_0,\widehat{\mathfrak{k}}]=0$ along with the fact that L is a simple $\widehat{\mathfrak{k}}$-module with countable dimension allows us to apply the Jacobson density theorem to find, for each $i=1,\dots,m$, an element $u_i\in\mathbf{U}(\widehat{\mathfrak{k}}^+)$ for which $u_iv=a_iv_\lambda\in N\cap\mathcal{S}L$. In particular, $a_i\in J_N$ for every i, and the claim is proved.

Now we claim that $N=J_NM$. Indeed, let $jm\in J_NM$. Then, writing $m=ul$ with $u\in\mathbf{U}(\mathcal{H}^-)$ and $l\in L$, we get that

$$jm=jul=ujl\in uN\subseteq N.$$

Thus $J_NM\subseteq N$. For the other inclusion, consider the canonical projection $\pi: M\to V:=M/J_NM$, $W=\pi(N)$, and $R=\mathcal{S}/J_N$. Notice that V is free as an R-module, and that $W\cap RL=\pi(N\cap\mathcal{S}L)=\pi(J_NL)=0$. Now if we suppose that $W\neq 0$, then we can use the fact that $\det D^X_\delta(\lambda)\neq 0$, and that $[\mathcal{H}_0,\widehat{\mathfrak{m}}]=0$, to get a contradiction just as in the proof of Proposition 1. Thus $W=0$ and the proof is complete. □

Corollary 12. *If $\det D^X_\delta(\lambda)\neq 0$, then $L(\widehat{\mathfrak{m}},\lambda)\cong\Lambda(\mathcal{H}_1^-)\otimes_{\mathbb{C}}L(\widehat{\mathfrak{k}},\lambda)$ as vector spaces.*

Proof. This follows from $L(\widehat{\mathfrak{m}},\lambda)\cong L(\widehat{\mathfrak{m}},\widehat{\mathfrak{k}};L(\widehat{\mathfrak{k}},\lambda))$ and Proposition 11. □

Corollary 13. *Suppose that $\det D^X_\delta(\lambda)\neq 0$. Then we have the character formula*

$$\mathrm{ch}\ L(\widehat{\mathfrak{m}},\lambda)=e^\lambda\prod_{\alpha\in\Delta(X)^+_{\mathrm{re},0}}(1-e^{-\alpha})^{-1}\prod_{\alpha\in\Delta(X)^+_1}(1+e^{-\alpha})\prod_{\alpha\in\Delta(\mathcal{H}^-)_1}(1+e^{-\alpha}),$$

where $\Delta(X)^+_{\mathrm{re},0}$ denotes the set of real positive even roots of $\widehat{\mathfrak{k}}$.

Proof. This follows from the results of [GS08] along with the fact that $L(\widehat{\mathfrak{m}},\lambda)\cong\Lambda(\mathcal{H}_1^-)\otimes_{\mathbb{C}}L(\widehat{\mathfrak{k}},\lambda)$. □

From now on we assume that

$$\det D_\delta^X(\lambda)\neq 0, \text{ and hence, by Corollary 12,}$$
$$L(\widehat{\mathfrak{m}},\lambda)\cong\Lambda(\mathcal{H}_1^-)\otimes_{\mathbb{C}}L(\widehat{\mathfrak{k}},\lambda) \text{ as vector spaces.}$$

Before proving the irreducibility of $M(\widehat{\mathfrak{g}},\widehat{\mathfrak{m}};L(\widehat{\mathfrak{m}},\lambda))$, we introduce an ordered basis of $M(\widehat{\mathfrak{g}},\widehat{\mathfrak{m}};L(\widehat{\mathfrak{m}},\lambda))$. Recall that for a subalgebra $\mathfrak{a}\subseteq\widehat{\mathfrak{g}}$ we defined $\Delta(\mathfrak{a})=\{\alpha\in\Delta\mid\widehat{\mathfrak{g}}_\alpha\subseteq\mathfrak{a}\}$. Let $B(\mathfrak{u}^-)=\{f_i\in\mathfrak{g}_{\alpha_i}\mid\alpha_i\in\dot{\Delta}(\mathfrak{u}^-)\}$ be a basis of $\mathfrak{u}^-$ such that

$$f_i<f_j \text{ if } \alpha_i<\alpha_j.$$

Now we order the basis $B(L(\mathfrak{u}^-))=\{f_i(m)\mid m\in\mathbb{Z}\}$ of $L(\mathfrak{u}^-)$ so that

(1) if m is odd and n is even, then $f_i(m)<f_j(n)$,
(2) if m,n are both even or both odd, then $f_i(m)<f_j(n)$ if $m<n$, or $m=n$ and $f_i<f_j$.

For $r\geq 1$ and $(\mathbf{i},2\mathbf{m},\mathbf{p})=(i_1,\dots,i_r,m_1,\dots,m_r,p_1,\dots,p_r)\in\mathbb{Z}_t^r\times 2\mathbb{Z}^r\times\mathbb{Z}_{\geq 0}^r$, we set $f_{\mathbf{i},2\mathbf{m},\mathbf{p}}:=f_{i_1}(m_1)^{p_1}\cdots f_{i_r}(m_r)^{p_r}\in\mathbf{U}(L(\mathfrak{u}^-)_0)$ and we define $\deg f_{\mathbf{i},2\mathbf{m},\mathbf{p}}:=\sum p_i$. For monomials of the different degree we let $f_{\mathbf{i},2\mathbf{m},\mathbf{p}}<f_{\mathbf{i}',2\mathbf{m}',\mathbf{p}'}$ if $\deg f_{\mathbf{i},2\mathbf{m},\mathbf{p}}<\deg f_{\mathbf{i}',2\mathbf{m}',\mathbf{p}'}$; for monomials of same degree we define $f_{\mathbf{i},2\mathbf{m},\mathbf{p}}<f_{\mathbf{i}',2\mathbf{m}',\mathbf{p}'}$ if $(\mathbf{i},2\mathbf{m},\mathbf{p})<(\mathbf{i}',2\mathbf{m}',\mathbf{p}')$, where the latter order is the reverse lexicographical order. This provides us a totally ordered basis $B(\mathbf{U}(L(\mathfrak{u}^-)_0))=\{f_{\mathbf{i},2\mathbf{m},\mathbf{p}}:=f_{i_1}(m_1)^{p_1}\cdots f_{i_r}(m_r)^{p_r}\}$ of $\mathbf{U}(L(\mathfrak{u}^-)_0)$. For $r\geq 1$ and $(\mathbf{i},\mathbf{m})=(i_1,\dots,i_r,m_1,\dots,m_r)\in\mathbb{Z}_t^r\times(2\mathbb{Z}^r+1)$, we set $f_{\mathbf{i},\mathbf{m}}:=f_{i_1}(m_1)\cdots f_{i_r}(m_r)$ and we define $\deg f_{\mathbf{i},\mathbf{m}}:=r$. For monomials of the different degree we let $f_{\mathbf{i},\mathbf{m}}<f_{\mathbf{i}',\mathbf{m}'}$ if $\deg f_{\mathbf{i},\mathbf{m}}<\deg f_{\mathbf{i}',\mathbf{m}'}$; for monomials of same degree we define $f_{\mathbf{i},\mathbf{m}}<f_{\mathbf{i}',\mathbf{m}'}$ if $(\mathbf{i},\mathbf{m})<(\mathbf{i}',\mathbf{m}')$, where the latter order is the reverse lexicographical order. Finally, we let $f_{\mathbf{i}',\mathbf{m}'}<f_{\mathbf{i},2\mathbf{m},\mathbf{p}}$ for all such monomials. By PBW Theorem, we have that $B(\mathbf{U}(L(\mathfrak{u}^-)))=\{f_{\mathbf{i},2\mathbf{m},\mathbf{p}}f_{\mathbf{i}',\mathbf{m}'}\}$ is a totally ordered basis of $\mathbf{U}(L(\mathfrak{u}^-))$.

Let $h_1,\dots,h_t$ be a basis of $\mathfrak{h}^X$. Then $H_{i,m}:=h_i(-m)$ for $i=1,\dots,t$ and $m\in\{2\mathbb{Z}_{\geq 0}+1\}$ form a basis for $\mathcal{H}_1^-$. In particular, if we let $H_{i,m}\geq H_{k,n}$ if $m\geq n$ or $m=n$ and $i\geq k$, then the monomials $H_{i_1,m_1}\cdots H_{i_s,m_s}$ with $H_{i_1,m_1}>\cdots>H_{i_s,m_s}$ form a basis $B(\mathcal{H}_1^-)$ of $\Lambda(\mathcal{H}_1^-)$.

Since we are assuming $\det D_\delta^X(\lambda)\neq 0$, we have by Corollary 12 that $L(\widehat{\mathfrak{m}},\lambda)\cong\Lambda(\mathcal{H}_1^-)\otimes_{\mathbb{C}}L(\widehat{\mathfrak{k}},\lambda)$ as vector spaces. Let $\{v_i\mid i\in I\}$ be an ordered basis of $L(\widehat{\mathfrak{m}},\lambda)$, where the order is induced by the order of $\Lambda(\mathcal{H}_1^-)$. We say $f_{\mathbf{i},\mathbf{m},\mathbf{p}}f_{\mathbf{i}',\mathbf{m}'}v_i<f_{\mathbf{i}_1,\mathbf{m}_1,\mathbf{p}_1}f_{\mathbf{i}_1',\mathbf{m}_1'}v_j$ if $f_{\mathbf{i},\mathbf{m},\mathbf{p}}f_{\mathbf{i}',\mathbf{m}'}<f_{\mathbf{i}_1,\mathbf{m}_1,\mathbf{p}_1}f_{\mathbf{i}_1',\mathbf{m}_1'}$ or if $f_{\mathbf{i},\mathbf{m},\mathbf{p}}f_{\mathbf{i}',\mathbf{m}'}=f_{\mathbf{i}_1,\mathbf{m}_1,\mathbf{p}_1}f_{\mathbf{i}_1',\mathbf{m}_1'}$ and $i<j$. Finally, for an element

$$u=\sum u^j_{\mathbf{i},\mathbf{m},\mathbf{p}}f_{\mathbf{i},\mathbf{m},\mathbf{p}}f_{\mathbf{i}',\mathbf{m}'}v_j, \text{ with } u^j_{\mathbf{i},\mathbf{m},\mathbf{p}}\in\mathbb{C},$$

we define

$$\operatorname{LinSpan}(u) := \operatorname{Span}\{f_{\mathbf{i},\mathbf{m},\mathbf{p}} f_{\mathbf{i}',\mathbf{m}'} \mid u^j_{\mathbf{i},\mathbf{m},\mathbf{p}} \neq 0\}.$$

For the next result recall that $L(\widehat{\mathfrak{m}},\lambda) \cong \Lambda(\mathcal{H}_1^-) \otimes_{\mathbb{C}} L(\widehat{\mathfrak{k}},\lambda)$ as vector spaces when $\det(D_\delta^X(\lambda)) \neq 0$. Also recall that for $\alpha_i \in \Delta$ we have a triple $f_i \in \mathfrak{g}^{-\alpha_i}$, $e_i \in \mathfrak{g}^{\alpha_i}$, $h_i \in \mathfrak{h}$ such that $[f_i, e_i]_0 = h_i$.

Lemma 14. *Let* $\bar{f}=\bar{f}_0\bar{f}_1=f_{i_1}(m_1)^{p_1}\cdots f_{i_r}(m_r)^{p_r} f_{i'_1}(m'_1)\cdots f_{i'_{r'}}(m'_{r'})\in B(\mathbf{U}(L(\mathfrak{u}^-)))$, $v \in L(\widehat{\mathfrak{m}},\lambda)$ *be a nonzero vector, and assume that all factors occurring in* $\bar{f}$ *are simple. For any such factor* f_{i_l}*, we consider* $e_{i_l} \in \mathfrak{n}^+ = \mathfrak{m}^+ \oplus \mathfrak{u}^+$*. If* $\det D_\delta^X(\lambda) \neq 0$*, then the following hold:*

(1) *If* $\deg \bar{f}_1 = 0$*, then there is* $0 \gg m_l \in \{2\mathbb{Z}+1\}$ *or* $0 \ll m \in \{2\mathbb{Z}+1\}$ *such that*

$$\begin{aligned} e_{i_l}(m)\bar{f}v \equiv \sum_{\substack{1\le j\le r\\ i_j=i_l}}^{r} -p_j(p_j-1) f_{i_j}(m+2m_j)\widehat{\widehat{\bar{f}^{jj}}} v \\ + \sum_{\substack{1\le j\le r\\ i_j=i_l}}^{r} \sum_{\xi=j+1}^{r} p_j p_\xi \alpha_{i_\xi}(h_{i_l}) f_{i_\xi}(m_j+m_\xi+m)\widehat{\widehat{\bar{f}^{j\xi}}} v \\ + \sum_{\substack{1\le j\le r\\ i_j=i_l}}^{r} p_j \widehat{\bar{f}^{j}} h_{i_l}(m+m_j)v \mod \mathbf{U}(L(\mathfrak{u}^-))_{(p-2)} \otimes L(\widehat{\mathfrak{m}},\lambda). \end{aligned}$$

(2) *If* $\deg \bar{f}_1 \geq 1$*, then there is* $0 \gg m \in 2\mathbb{Z}$ *or* $0 \ll m \in 2\mathbb{Z}$ *such that*

$$\begin{aligned} e_{i_l}(m)\bar{f}v \equiv \Bigg(\sum_{\substack{1\le j\le r\\ i_j=i_l}}^{r} -p_j(p_j-1) f_{i_j}(m+2m_j)\widehat{\widehat{\bar{f}^{jj}}} v \\ + \sum_{\substack{1\le j\le r\\ i_j=i_l}}^{r} \sum_{\xi=j+1}^{r} p_j p_\xi \alpha_{i_\xi}(h_{i_l}) f_{i_\xi}(m_j+m_\xi+m)\widehat{\widehat{\bar{f}^{j\xi}}} v \Bigg) \\ + \Bigg(\sum_{\substack{1\le j\le r'\\ i_j=i_l}}^{r'} \sum_{\xi=j+1}^{r'} (-1)^{\xi-(j+1)} \bar{\alpha}_{i_\xi}(h_{i_l}) \bar{f}_0 f_{i_\xi}(m'_j+m'_\xi+m)\widehat{\widehat{\bar{f}_1^{j\xi}}} v \\ + \sum_{\substack{1\le j\le r'\\ i_j=i_l}}^{r'} (-1)^{(r'-j)} \widehat{\bar{f}^{j}} h_{i_l}(m+m'_j)v \Bigg) \\ \mod \mathbf{U}(L(\mathfrak{u}^-))_{(p+r'-2)} \otimes L(\widehat{\mathfrak{m}},\lambda). \end{aligned}$$

Proof. We prove part (b) first, as part (a) follows from it. Choose $0 \gg m \in 2\mathbb{Z}$ or $0 \ll m \in 2\mathbb{Z}$ such that $h_{i_l}(m+m'_j)$ is in $B(\mathcal{H}_1^-)$. Since $\operatorname{ad}(e_{i_l}(m))$ is a derivation of even degree, we have that

$$
\begin{aligned}
&e_{a_l}(m)\bar{f}v\\
&=\sum_{j=1}^{r}\sum_{\gamma=0}^{p_j-1} f_{i_1}(m_1)^{p_1}\cdots f_{i_j}(m_j)^{\gamma}[e_{i_l},f_{i_j}]_0(m+m_j)f_{i_j}(m_j)^{p_j-\gamma-1}\cdots f_{i_r}(m_r)^{p_r}\bar{f}_1 v\\
&\quad+\sum_{j=1}^{r'}\bar{f}_0 f_{i'_1}(m'_1)\cdots f_{i'_{j-1}}(m'_{j-1})[e_{i_l},f_{i'_j}]_0(m+m'_j)f_{i'_{j+1}}(m'_{j+1})\cdots f_{i'_{r'}}(m'_{r'})v\\
&=\sum_{\substack{1\le j\le r\\ i_j=i_l}}\sum_{\gamma=0}^{p_j-1} f_{i_1}(m_1)^{p_1}\cdots f_{i_j}(m_j)^{\gamma}h_{i_j}(m+m_j)f_{i_j}(m_j)^{p_j-\gamma-1}\cdots f_{i_r}(m_r)^{p_r}\bar{f}_1 v\\
&\quad+\sum_{\substack{1\le j\le r'\\ i_j=i_l}}^{r'}\bar{f}_0 f_{i'_1}(m'_1)\cdots f_{i'_{j-1}}(m'_{j-1})h_{i_j}(m+m'_j)f_{i'_{j+1}}(m'_{j+1})\cdots f_{i'_{r'}}(m'_{r'})v\\
&\equiv\Bigg(\sum_{\substack{1\le j\le r\\ i_j=i_l}}^{r}-p_j(p_j-1)f_{i_j}(m+2m_j)\bar{f}^{\widehat{j}\widehat{j}}v\\
&\qquad+\sum_{\substack{1\le j\le r\\ i_j=i_l}}^{r}\sum_{\xi=j+1}^{r}p_jp_\xi\alpha_{i_\xi}(h_{i_l})f_{i_\xi}(m_j+m_\xi+m)\bar{f}^{\widehat{j}\widehat{\xi}}v+\sum_{\substack{1\le j\le r\\ i_j=i_l}}^{r}p_j\bar{f}^{\widehat{j}}h_{i_l}(m+m_j)v\Bigg)\\
&\quad+\Bigg(\sum_{\substack{1\le j\le r'\\ i_j=i_l}}^{r'}\sum_{\xi=j+1}^{r'}(-1)^{\xi-(j+1)}\bar{\alpha}_{i_\xi}(h_{i_l})\bar{f}_0 f_{i_\xi}(m'_j+m'_\xi+m)\bar{f}_1^{\widehat{j}\widehat{\xi}}v\\
&\qquad+\sum_{\substack{1\le j\le r'\\ i_j=i_l}}^{r'}(-1)^{(r'-j)}\bar{f}^{\widehat{j}}h_{i_l}(m+m'_j)v\Bigg)\mod \mathbf{U}(L(\mathfrak{u}^-))_{(p+r'-2)}\otimes L(\widehat{\mathfrak{m}},\lambda)\\
&\equiv\Bigg(\sum_{\substack{1\le j\le r\\ i_j=i_l}}^{r}-p_j(p_j-1)f_{i_j}(m+2m_j)\bar{f}^{\widehat{j}\widehat{j}}v\\
&\qquad+\sum_{\substack{1\le j\le r\\ i_j=i_l}}^{r}\sum_{\xi=j+1}^{r}p_jp_\xi\alpha_{i_\xi}(h_{i_l})f_{i_\xi}(m_j+m_\xi+m)\bar{f}^{\widehat{j}\widehat{\xi}}v\Bigg)\\
&\quad+\Bigg(\sum_{\substack{1\le j\le r'\\ i_j=i_l}}^{r'}\sum_{\xi=j+1}^{r'}(-1)^{\xi-(j+1)}\bar{\alpha}_{i_\xi}(h_{i_l})\bar{f}_0 f_{i_\xi}(m'_j+m'_\xi+m)\bar{f}_1^{\widehat{j}\widehat{\xi}}v\\
&\qquad+\sum_{\substack{1\le j\le r'\\ i_j=i_l}}^{r'}(-1)^{(r'-j)}\bar{f}^{\widehat{j}}h_{i_l}(m+m'_j)v\Bigg)\mod \mathbf{U}(L(\mathfrak{u}^-))_{(p+r'-2)}\otimes L(\widehat{\mathfrak{m}},\lambda),
\end{aligned}
$$

where the first equivalence follows from the fact that $\mathrm{ad}(h_{i_l}(m+m_j))$ is an even derivation, $\mathrm{ad}(h_{i_j}(m+m'_j))$ is an odd derivation, and $f_{i'_\xi}(m'_\xi)$ is an odd element for any m'_ξ; and the second equivalence follows from the fact that $h_{i_l}(m+m_j)v=0$ for all $1\le j\le r$, since either $h_{i_l}(m+m_j)\in\mathcal{H}_0^+$ that implies $h_{i_l}(m+m_j)v=0$, or $h_{i_l}(m+m_j)\in\mathcal{S}^+$, and hence $h_{i_l}(m+m_j)v$ lies in the maximal proper submodule of $M(\widehat{\mathfrak{k}},\lambda)$.

For part (a), we notice that the second parentheses above does not appear in the expression of $e_{a_l}(m)\bar{f}v$. Moreover, despite the fact that $\mathrm{ad}(e_{i_l}(m))$ and $\mathrm{ad}(h_{i_l}(m+m_k))$ are odd derivations (as $m\in\{2\mathbb{Z}+1\}$ and $m_j\in 2\mathbb{Z}$ for all $1\le j\le r$), they behave as regular derivations when applied on factors of $\bar{f}_0$, since $m_j\in 2\mathbb{Z}$ for all $1\le j\le r$. Thus the proof follows from the above equation. □

We now state our key result.

Theorem 15. *If* $\det D^X_\delta(\lambda)\neq 0$*, then* $M(\widehat{\mathfrak{g}},\widehat{\mathfrak{m}};L(\widehat{\mathfrak{m}},\lambda))$ *is irreducible.*

Proof. We claim that any non-trivial submodule N of $M(\widehat{\mathfrak{g}},\widehat{\mathfrak{m}};L)$ intersects $L(\widehat{\mathfrak{m}},\lambda)$ non-trivially. Assuming that the claim holds, the result follows from the simplicity of $L(\widehat{\mathfrak{m}},\lambda)$.

To prove the claim, let $0\neq v\in N_\mu$, and let $\bar{f}_{\max}x_{\max}=f_{\mathbf{a},2\mathbf{b},\mathbf{c}}f_{\mathbf{a}',\mathbf{b}'}x_d$ be the maximal monomial occurring in v. We now reduce the proof to the case where all factors f_{i_j} of maximal degree monomials occurring in v are simple root vectors. Indeed, consider all factors f_{i_j} that occur in the monomials of maximal degree of v, and let f_{i_k} be the minimal among them (i.e., its associated root α_{i_k} is such that $|\alpha_{i_k}|$ is maximal among them). Let $\bar{f}_{\min}x_{\min}=f_{\mathbf{d},2\mathbf{g},\mathbf{k}}f_{\mathbf{d}',\mathbf{g}'}h_{\mathbf{d}'',\mathbf{g}''}x_{\min}=\bar{f}_{0,\min}\bar{f}_{1,\min}x_{\min}$ be an element (occurring in v) of maximal degree having f_{i_k} as a factor, and let $z\in\mathfrak{n}^+=\mathfrak{m}^+\oplus\mathfrak{u}^+$ be such that $0\neq[z,f_{i_k}]\in\mathfrak{u}^-$ (such z exists by [Cox94, Lem. 4.2]). Let $J_{\min}$ the set of indexes j for which f_{i_j} is a factor of $\bar{f}_{\min}$ and $[z,f_{i_j}]\in\mathfrak{u}^-$. Let $0\gg m\in 2\mathbb{Z}$ (if $z\in\mathfrak{u}^+$) or $0\ll m\in 2\mathbb{Z}$ (if $z\in\mathfrak{m}^+$) (here $m\ll 0$ (resp. $m\gg 0$) means m so that for every fixed j, $m+m_j\notin\{g_k,\ g'_l\mid 1\le k\le r,\ 1\le l\le r'\}$). Then, using that $\mathrm{ad}(z(m))$ is an even derivation, we obtain

$$
\begin{aligned}
&z(m)\bar{f}_{\min}x_{\min}\\
&=\sum_{j=1}^{r}\sum_{\gamma=0}^{k_j-1} f_{d_1}(g_1)^{k_1}\cdots f_{d_j}(g_j)^{\gamma}[z,f_{d_j}]_0(m+g_j)f_{d_j}(g_j)^{k_j-\gamma-1}\cdots f_{d_r}(g_r)^{k_r}\bar{f}_{1,\min}x_{\min}\\
&\quad+\sum_{j=1}^{r'}\bar{f}_{0,\min}f_{d'_1}(g'_1)\cdots f_{d'_{j-1}}(g'_{j-1})[z,f_{d'_j}]_0(m+g'_j)f_{d'_{j+1}}(g'_{j+1})\cdots f_{d'_{r'}}(g'_{r'})x_{\min}\\
&\equiv\sum_{j\in J_-}k_j[z,f_{i_j}]_0(m+g_j)\bar{f}^{\widehat{j}}_{\min}x_{\min}\\
&\quad+\sum_{j\in J_-}(-1)^{j-1}\bar{f}_{0,\min}[z,f_{i_j}]_0(m+g'_j)\bar{f}^{\widehat{j}}_{1,\min}x_{\min}\mod \mathbf{U}(L(\mathfrak{u}^-))_{(k+d'-1)}\otimes L(\widehat{\mathfrak{m}},\lambda),
\end{aligned}
$$

where $k+d'=\deg\bar{f}_{\min}$. Now if S_1 denotes this summation, then it is nonzero since $[z,f_{i_k}]_0\neq 0$ and $m+m_j\notin\{g_k,\ g'_l\mid 1\le k\le r,\ 1\le l\le r'\}$. Moreover, if

$\bar{f}x = \bar{f}_0\bar{f}_1x$ is a different monomial occurring in v, then, similarly we have that

$$z(m)\bar{f}x \equiv \sum_{j\in J_-} p_j[z, f_{i_j}]_0(m+m_j)\bar{f}^{\widehat{j}}x + \sum_{j\in J_-}(-1)^{j-1}\bar{f}_0[z, f_{i_j}]_0(m+m_j')\bar{f}_1^{\widehat{j}}x \mod \mathbf{U}(L(\mathfrak{u}^-))_{(p+r-1)}\otimes L(\widehat{\mathfrak{m}},\lambda),$$

where $p+r = \deg \bar{f}$. Since $\bar{f}_{\min}$ has maximal degree among monomials in v, we have that $p+r \leq k+d'$. Hence, if T_1 is the summation above, then $S_1 \notin \mathrm{LinSpan}(T_1) + \mathbf{U}(L(\mathfrak{u}^-))_{(p-1)} \otimes L(\widehat{\mathfrak{m}},\lambda)$, since this could happen only if $p+r = k+d'$; $\mathbb{C}[z, f_{i_j}]_\ell = \mathbb{C}[z, f_{i_l}]_\ell$ for $\ell = 0, 1$; $m_j = g_l$; $\bar{f}_{\min}^{\widehat{j}} = \bar{f}^{\widehat{l}}$; and $x_{\min} = x$. But this would imply $\bar{f}_{\min} = \bar{f}$, and $x_{\min} = x$, which contradicts the fact that $\bar{f}x \neq \bar{f}_{\min}x_{\min}$.

We may now assume that factors of all maximal degree monomials occurring in v are simple. In particular, this is the case for

$$\bar{f}_{\max}x_{\max} = \bar{f}_{0,\max}\bar{f}_{1,\max}x_{\max} = f_{a_1}(b_1)^{c_1}\cdots f_{a_s}(b_s)^{c_s}f_{a_1'}(b_1')\cdots f_{a_{s'}}(b_{s'})x_{\max}.$$

Moreover, we may also assume that $\deg f_{1,\max} \geq 1$ (as otherwise the proof is the same as that of [Cox94, Proposition 4.5], using Lemma 14 and a suitable $e \in \{2\mathbb{Z}+1\}$ in his notation). By Lemma 14, for each simple root factor f_{a_l} of $\bar{f}_{\max}$, there is $0 \gg m \in 2\mathbb{Z}$ or $0 \ll m \in 2\mathbb{Z}$ for which

$$\begin{aligned} e_{a_l}(m)\bar{f}_{\max}x_{\max} &\equiv \Bigg(\sum_{\substack{1\leq j\leq s\\ i_j=i_l}}^{s} -c_j(c_j-1)f_{a_j}(m+2b_j)\bar{f}_{\max}^{\widehat{j}\widehat{j}}x_{\max} \\ &\quad + \sum_{\substack{1\leq j\leq s\\ i_j=i_l}}^{r}\sum_{\xi=j+1}^{r} c_jc_\xi\alpha_{a_\xi}(h_{a_l})f_{a_\xi}(b_j+b_\xi+m)\bar{f}_{\max}^{\widehat{j}\widehat{\xi}}x_{\max}\Bigg) \\ &\quad + \Bigg(\sum_{\substack{1\leq j\leq s'\\ i_j=i_l}}^{s'}\sum_{\xi=j+1}^{s'}(-1)^{\xi-(j+1)}\bar{\alpha}_{a_\xi}(h_{a_l})\bar{f}_{0,\max}f_{a_\xi}(b_j'+b_\xi'+m)\bar{f}_{1,\max}^{\widehat{j}\widehat{\xi}}x_{\max} \\ &\quad + \sum_{\substack{1\leq j\leq s'\\ i_j=i_l}}^{s'}(-1)^{(s'-j)}\bar{f}_{\max}^{\widehat{j}}h_{i_l}(m+b_j')x_{\max}\Bigg) \mod \mathbf{U}(L(\mathfrak{u}^-))_{(c+s'-2)}\otimes L(\widehat{\mathfrak{m}},\lambda) \end{aligned}$$

Finally, it is not hard to prove that for any fixed index l, the summand

$$w_l = \bar{f}_{\max}^{\widehat{l}}h_l(m+b_l')x_{\max}$$

is not in the LinSpan of the remaining monomials occurring in $e_l(m)v$. Therefore, $e_l(m)v \neq 0$, the maximal monomial occurring in $e_l(m)v$ has degree less than that of the maximal monomial occurring in v, and thus the result follows by induction. □

Applying Theorem 15 in the case $X = \varnothing$ gives:

Corollary 16. *If* $\det D_\delta^{\varnothing}(\lambda) \neq 0$*, then* $M(\widehat{\mathfrak{g}}, \mathcal{H}; L(\mathcal{H}, \lambda))$ *is irreducible.*

Remark 5. Notice that differently from the other cases studied in the literature, we do not need the central charge to be nonzero in order to have $M(\widehat{\mathfrak{g}}, \widehat{\mathfrak{m}}; L(\widehat{\mathfrak{m}}, \lambda))$ be irreducible (compare with [Cox94, Fut94, CF18]). This is due to the fact that the central element K does not play a role in the action of the imaginary subalgebra $\mathcal{H}$ on $L(\widehat{\mathfrak{m}}, \lambda)$. On the other hand, the condition $\det D_\delta^X(\lambda) \neq 0$ is essential in our context. Without this condition we do not necessarily have that $L(\widehat{\mathfrak{m}}, \lambda) \cong \Lambda(\mathcal{H}_1^-) \otimes_{\mathbb{C}} L(\widehat{\mathfrak{k}}, \lambda)$ as vector spaces (see Corollary 12).

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