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"THE INFLUENCE OF THE KINETIC ENERGY IN EQUILIBRIUM OF HAMILTONIAN SYSTEMS"

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The Influence of the Kinetic Energy in Equilibrium of Hamiltonian Systems

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Abstract

We provide a simple and explicit example of the influence of the kinetic energy in the stability of the equilibrium of classical hamiltonian systems of the type $H(q,p) =< B(q)p;p > + \pi(q)$. We construct a potential energy π of class C^k with a critical point at 0 and two different positive defined matrices B_1 and B_2, both independent of q, and show that the equilibrium $(0,0)$ is stable according to Lyapunov for the hamiltonian $H_1 =< B_1(q)p;p > + \pi(q)$, while for $H_2 =< B_2(q)p;p >$ $+ \pi(q)$ the equilibrium is unstable. Moreover, we give another example showing that even in the analytical situation the kinetic energy has influence in the stability, in the sense that there is an analytic potential energy π and two kinetic energies, also analytic, T_1 and T_2 such that the attractive basin of $(0,0)$ is a two dimensional manifold in the system of hamiltonian $\pi + T_1$ and a one dimensional manifold in the system of hamiltonian $\pi + T_2$.

1 Introduction

The study of the stability according to Lyapunov of equilibrium points of classical hamiltonian system of the type $H(q,p) = T(q,p) + \pi(q)$, where $T(q,p) =< B(q)p;p >$ is a C^2 positive defined quadratic form on the momenta, called the kinetic energy and $\pi(q)$ is the potential energy, also of class C^2, is one of the oldest problems of mechanics, and has been studied by many famous mathematicians. While many of the relevant results were obtained back in the 19th century, some very important facts were not proved or known until the recent years.

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This field's landmark is the well-known Dirichlet-Lagrange theorem, which states that if q_0 is a local strict minimum for the potential energy π, then the equilibrium point $(q_0;0)$ is stable according to Liapounof. The converse to this theorem doesn't necessarily hold, not even when the potential energy is of class C^∞ as was shown by Painlévè, but a result by Tagliaferro (see [T]) assures the instability of $(q_0;0)$ when q_0 is a local maximum for π.

The case in which π is analytic is somewhat better understood.

First Laloy and Pfeiffer (see [LP]) showed that if the system has two degrees of freedom, π is analytic and q_0 is a local minimum of π, but not a strict local minimum of π, then the equilibrium $(q_0;0)$ is unstable. On 1995, Palamodov (see[P]) showed that, if π is analytic and q_0 is a saddle point of π, then the associated equilibrium is unstable.

Later on, we provided (see [GT]), for systems with two degrees of freedom, a necessary and sufficient condition for the k-jet of π at q_0 to determine whether the equilibrium $(q;0)$ is unstable or not. The condition is that the k-jet of π at q_0 shows that q is not a local minimum of the potential, and it was also proved that whenever this condition holds there is an asymptotic trajectory to the equilibrium. Also, every analytical function with a critical point q that is not an local minimum satisfies this condition for some k (see [BGZ]).

A common trace to all this articles is the lack of further hypotheses on the kinetic energy. This fact led to the conjecture that the kinetic energy played no role in the stability of the equilibria, that is,

Conjecture 1 *If, for some potential energy $\pi \in C^2$ with a critical point at q_0 and a C^2 positive defined kinetic energy $T_1(q,p) =< B_1(q)p; p >$ the equilibrium $(q_0, 0)$ of the system $H_1 = T_1 + \pi$ is stable then, for every positive defined kinetic energy $T_2 =< B_2(q)p; p >$ the equilibrium $(q_0, 0)$ of the system $H_2 = T_2 = \pi$ is also stable.*

Strikingly, this conjecture was shown to be false by Bertotti and Bolotin (see [BB]) who showed that there are two C^∞ kinetic energies and a C^∞ potential energy with a critical point at the origin and they proved that $(0;0)$ was stable for one of the hamiltonian systems, while it was unstable for the other. In this work Bertotti and Bolotin did not exhibit an explicit example of this situation, rather they proved its existence by a limit process.

In section 2 of this work we provide a very simple and explicit counter-example for the stated conjecture. The example has a potential energy of class C^k, where k may be taken as large as desired, but not C^∞. On the other hand, the kinetic energies are both analytic and independent of q.

Another pertinent question is to determine up to which extend is the kinetic energy relevant when dealing with equilibria of analytic hamiltonians. While stability of the equilibrium is completly determined by the potential energy, a better understanding of the possible role of the kinetic energy in the behaviour of the system close to the equilibrium is lacking and highly desirable.

In section 3 of this note we present, to our knowledge, the first example of the importance of the kinetic energy in the study of analytical mechanical equilibrium. In our example, which has two degrees of freedom, we show a polynomial potential energy π with a saddle point at the origin and so, no matter what is the kinetic energy, there must be an asymptotic trajectory to (0;0). But we provide two different polynomial kinetic energies, T_1 and T_2, and we show that, for the system $H_1 = T_1 + \pi$, the attractive basin of (0;0) is a two dimensional manifold, while that there is only one asymptotic trajectory to the equilibrium which is a solution of the hamiltonian equations of the system H_2.

2 An explicit counter-example for the conjecture

In this section we consider the Hamiltonian $H_j = T_j + \pi,\ j = 1,2$, where the potential energy $\pi\colon \mathbf{R}^2 \longrightarrow \mathbf{R}$ is the continuous function determined by

$$\pi(x,y) = x^6 \sin(\ln(|x|)) - y^6(\sin(\ln(|y|)) + \frac{1}{2}),\ \text{if } xy \neq 0.$$

and the kinetics energies are given by

$$T_1(x,y,\dot{x},\dot{y}) = \frac{\dot{x}^2+\dot{y}^2}{2} \text{ and } T_2(x,y,\dot{x},\dot{y}) = \frac{(\dot{x}+\dot{y})^2 + c(\dot{x}-\dot{y})^2}{2},$$

where $c > 0$ is a constant that will be determined later. Observe that π is a function of class C^5, the origin is a saddle point of π and $\pi(x,x) = -\frac{x^6}{2}$.

We shall prove that origin is a stable point for H_1 and unstable for H_2.

The first statement is trivial since H_1 splits variables and it is clear that the origin is a stable equilibrium of the one dimensional Hamiltonian systems $x^6 \sin(\ln(|x|)) + \frac{\dot{x}^2}{2}$ and $y^6(\sin(\ln(|y|)) + \frac{1}{2}) + \frac{\dot{y}^2}{2}$.

In order to proof the instability of the origin in H_2 we start with the change of coordinates $u = x+y, w = x-y$. In this coordinates, we have $\pi(u,w) = (u+w)^6 \sin(\ln(|u+w|)) - (u-w)^6(\sin(\ln(|u-w|)) + \frac{1}{2})$ and T_2 becomes the trivial kinetic energy $\frac{\dot{u}^2 + c\dot{w}^2}{2}$. Then the Hamiltonian equations of H_2 are

$$\ddot{u} = -\pi_u,\ \ddot{w} = -\frac{\pi_w}{c}, \tag{1}$$

and a simple calculation shows that

$$\begin{aligned}\pi_u &= 6((u+w)^5\sin(\ln(|u+w|)) - (u-w)^5(\sin(\ln(|u-w|)) + \frac{1}{2})) + \\ & |u+w|^5\cos(\ln(|u+w|)) - |u-w|^5\cos(\ln(|u-w|)) \qquad (2)\end{aligned}$$

and

$$\begin{aligned}\pi_w &= 6((u+w)^5\sin(\ln(|u+w|)) + (u-w)^5(\sin(\ln(|u-w|)) + \frac{1}{2})) + \\ & |u+w|^5\cos(\ln(|u+w|)) + |u-w|^5\cos(\ln(|u-w|)). \qquad (3)\end{aligned}$$

The next technical result will be essential to us.

Lemma 1 *There is a real number $0 < \alpha < 1$ such that if $u > 0$ and $|w| < \alpha u$ then*

$$|(u+w)^5\sin(\ln|u+w|) - (u-w)^5\sin(\ln|u-w|)| \le \frac{u^5}{16}$$

and

$$||u+w|^5\cos(\ln(|u+w|)) - |u-w|^5\cos(\ln(|u-w|))| \le \frac{3u^5}{8}.$$

Proof: Since $0 < \alpha < 1$, $u > 0$ and $|w| < \alpha u$, we have $u - w > 0$ and $u + w > 0$.

The first inequality follows from the observation that there is $z = z(u, w)$ such that $\sin(\ln(u+w)) - \sin(\ln(u-w)) = \cos(z)\ln(\frac{u+w}{u-w})$, and so

$$\begin{aligned}&|(u+w)^5\sin(\ln(u+w)) - (u-w)^5\sin(\ln(u-w))| \le \\ &|(u+w)^5(\sin(\ln(u+w)) - \sin(\ln(u-w))| + \\ &|(u-w)^5 - (u+w)^5||\sin(\ln(u-w))| \le \\ &|(u+w)^5\cos(z)\ln(\frac{u+w}{u-w})| + |(u+w)^5 - (u-w)^5|. \qquad (4)\end{aligned}$$

Then, since $|w| < \alpha u$, we have $|u+w|^5 \le u^5 + |p(\alpha)|u^5$, where $p(\alpha)$ is a polinomial with $p(0) = 0$. Moreover, it's clear that there is another polinomial $q(\alpha)$ such that $q(0) = 0$ and $|(u+w)^5 - (u-w)^5| \le |q(\alpha)|u^5$. Finally, note that $|\frac{u+w}{u-w} - 1| < \frac{2\alpha}{1-\alpha}$, and this implies $|\ln\frac{u+w}{u-w}| \le r(\alpha)$, where $\lim_{\alpha\to 0} r(\alpha) = 0$.

From these observations and (4) it follows that there is $\alpha > 0$ small enough such that

$$\begin{aligned}&|(u+w)^5\sin(\ln(u+w)) - (u-w)^5\sin(\ln(u-w))| < \\ &|(u+w)^5(\cos(z)\ln\frac{u+w}{u-w})| + |(u+w)^5 - (u-w)^5| < \frac{u^5}{16}.\end{aligned}$$

The second inequality follows from analogous calculations and we omit the details. ■

By using this result we deduce the follow corollary.

Corollary 1 *There are real constants $c > 0$ and $0 < \alpha < 1$, with $(1-\alpha)^5 > \frac{1}{2}$ such that if $u > 0$ and $|w| < \alpha u$ then $\pi_u(u,w) < 0$ and $|\frac{\pi_w(u,w)}{c}| < \alpha|\pi_u(u,w)|$.*

Proof: If α obeys the hypothesis, $(u-w)^5 > (1-\alpha)^5 u^5 > \frac{1}{2}u^5$. So, by using (2) and the previous lemma one sees readly $\pi_u(u,w) < \frac{3u^5}{4} - 3(u-w)^5 < -\frac{3}{4}u^5 < 0$.

Note that the last inequality shows also that $|\pi_u(u,w)| > \frac{3}{4}u^5$.

Now we use (3) and proceed exactly as in the proof of lemma to show that we can find $K > 0$ such that, if $|w| < \alpha u$ and $u > 0$, we have $|\frac{\pi_w(u,w)}{c}| < \frac{Ku^5}{c}$.

Then if $c > 0$ obeys $\frac{K}{c} < \frac{3\alpha}{4}$ we have the thesis. ■

Now we can prove our main result. Suppose that we fix $\alpha > 0$ and $c > 0$ as above.

Theorem 1 *If we choose in T_2 this constant $c > 0$ then the origin is an unstable equilibrium of equations (1).*

Proof: Let be $\varepsilon > 0$ and consider a point $p_0 = (u_0, w_0, \dot{u}_0, \dot{w}_0)$ such that $0 < u_0 < \varepsilon$, $|w_0| < \alpha u_0$ and $\dot{u}_0 > 0$ and $|\dot{w}_0| < \alpha \dot{u}_0$.

We claim that the solution $\varphi(t) = (u(t), w(t))$ of (1) with initial condition p_0 at $t_0 = 0$ remains in the cone $C = \{(u,w) \colon u > 0,\ |w| < \alpha u\}$ for all $t > 0$.

In fact, since $0 < |\dot{w}(0)| < \alpha\dot{u}(0)$ it follows from corollary 1 that $|\dot{w}(t)| - \alpha|\dot{u}(t)| < 0$, for all $t > 0$. Then, as $|w(0)| - \alpha u(0) < 0$, we have that $\varphi(t) \in C$, for all $t > 0$.

Moreover, the corollary 1 shows that, in C, $\ddot{u} > 0$, then $\dot{u}(t) > \dot{u}_0 > 0$, for all $t > 0$. This implies immediately that there is a $T > 0$ such that $u(T) = \varepsilon$ and the proof is complete. ■

We conclude this section with the observation that if k is a positive integer, we can consider the potential energy of class C^k

$$\widetilde{\pi}(x,y) = x^{2k}\sin(\ln(|x|)) - y^{2k}(\sin(\ln(|y|)) + \frac{1}{2})$$

and is not difficult to prove an analogous lemma 1 and corollary 1 for this function, then considering the kinetic energies above and defining $\widetilde{H}_j = T_j + \widetilde{\pi}$, $j = 1, 2$, we can exhibit the same change of stability of the origin for Hamiltonian of an arbitrarily great class.

3 The analytical case

Our potential energy will be

$$\pi : \Omega \longrightarrow \mathbf{R} \qquad \pi(q_1;q_2) = \frac{q_1^3 - q_2^{10}}{2},$$

where $\Omega = \{(q_1, q_2) \in \mathbf{R}^2 : q_1 > -1\}$ and the kinetic energies

$$T_1, T_2 : \Omega \times \mathbf{R}^2 \longrightarrow \mathbf{R}^+ \qquad T_1 = \frac{p_1^2 + p_2^2}{2} \text{ and } T_2 = \frac{(1+q_1)(p_1^2 + p_2^2)}{2}$$

In the first example we get the set of equations

$$\dot{q}_1 = p_1 \quad \dot{p}_1 = \frac{-3(q_1)^2}{2}$$
$$\dot{q}_2 = p_2 \quad \dot{p}_2 = \frac{10(q_2)^9}{2}$$

And, since the system decouples into two autonomous one dimensional second order equations, it is easily seen that for every (q_1, q_2) with $q_1 \leq 0$, there exist a unique (p_1, p_2) such that the solution starting at (q_1, q_2, p_1, p_2) which tends to 0 as $t \to \infty$.

The second example will require a little more work. Its equations are

$$\dot{q}_1 = (1+q_1)p_1 \quad \dot{p}_1 = \frac{-3(q_1)^2}{2} - \frac{p_1^2+p_2^2}{2}$$
$$\dot{q}_2 = (1+q_1)p_2 \quad \dot{p}_2 = \frac{10(q_2)^9}{2}. \tag{5}$$

Clearly, there is a solution asymptotic to the origin for which $p_2(t) = q_2(t) = 0$. We show it is unique.

So we begin by taking a trajectory asymptotic to the origin $\phi(t) = (q_1, q_2, p_1, p_2)(t)$, and let $t > t_0$ be sufficiently great so that $|q_1(t)| < \frac{1}{2}$. We will assume $t_0 = 0$. Note that $\dot{p}_1(t) < 0$ and so we must have $p_1(t) > 0$ otherwise ϕ wouldn't be assymptotic to the origin.

The last statement implies that $\dot{q}_1(t) > 0$ for positive times so, by the same argument, we have $q_1(t) < 0$, for all $t \geq 0$. Also, we claim that, if for some positive t it happens that $q_2(t)p_2(t) > 0$, then the solution does not tend to 0, since it follows directly from (5) that $q_2\dot{p}_2$ and $p_2\dot{q}_2$ are positive.

Lemma 2 *Let $\phi(t) = (p_1(t), p_2(t), q_1(t), q_2(t))$ be a solution of (5) such that $\lim_{t\to\infty} \|\phi(t)\| = 0$. For all $t > 0$, we have $|q_1(t)| \geq |p_1(t)|^{\frac{2}{3}}$.*

Proof: We consider the auxiliary function $V(q_1,p_1)=\frac{q_1^3+p_1^2}{2}$, and we calculate its time derivative

$$\dot{V}=\frac{3q_1^2\dot{q}_1}{2}+p_1\dot{p}_1=\frac{p_1}{2}(3q_1^3-p_1^2-p_2^2)$$

so V decreases along the trajectories.

Also, since $\lim_{t\to\infty}V(q_1(t),p_1(t))=0$, we have that $\frac{q_1^3+p_1^2}{2}>0$, and the result follows ■

Lemma 3 *Let $\phi(t)=(p_1(t),p_2(t),q_1(t),q_2(t))$ be a solution of (5) such that $\lim_{t\to\infty}\|\phi(t)\|=0$. Then, for all $t>0$, we have $|p_2(t)|>|q_2(t)|^5$.*

Proof: We consider the auxiliary function $U(q_2,p_2)=\frac{-q_2^{10}+p_2^2}{2}$, and we calculate its time derivative

$$\dot{U}=\frac{-10q_2^9\dot{q}_2}{2}+p_2\dot{p}_2=\frac{-10p_1q_2^9p_2}{2}\leq 0,$$

the last inequality following from $q_1(t)<0, q_2(t)p_2(t)\leq 0$.

Again, since $\lim_{t\to\infty}U(q_2(t),p_2(t))=0$, we have that $\frac{-q_2^{10}+p_2^2}{2}>0$, as we stated. ■

Theorem 2 *Let $\phi(t)=(p_1(t),p_2(t),q_1(t),q_2(t))$ be a solution of (5) such that $\lim_{t\to\infty}\|\phi(t)\|=0$. Then $p_2(t)=q_2(t)=0$.*

Proof: Suppose that this is false. Thus, by lemma 3, for all $t>0$, $p_2(t)\neq 0$. Moreover, since $q_2(t)p_2(t)\leq 0$ for all $t>0$ and, if $q_2\equiv 0$, have from (5) $p_2\equiv 0$, we conclude that there is a time $t_0\geq 0$ such that $q_2(t_0)p_2(t_0)<0$. In order to simplify notation, we will assume that $t_0=0$ and $(q_1,p_1,q_2,p_2)(0)=(c_0,c_1,c_2,c_3)$, with $c_2<0<c_3$ (the case $c_3<0<c_2$ is analogous).

Since $p_2>|q_2|^5$, and $\dot{p}_2=10(q_2)^9$, we have $|\dot{p}_2|<10(p_2)^{\frac{9}{5}}$, and so

$$p_2(t)>(c_3^{-\frac{4}{5}}+8t)^{-\frac{5}{4}} \tag{6}$$

On the other hand we have that $\dot{p}_1<-3\frac{q_1^2}{2}$ and, from Lemma 2, $|q_1|\geq p_1^{\frac{2}{3}}$, so that

$$p_1(t)<(c_1^{-\frac{1}{3}}+\frac{1}{2}t)^{-3} \tag{7}$$

Now we define

$$f(t) = \int_t^{\infty} \frac{p_2^2(s)}{2} ds.$$

Then, by (6), there is a $k_1 > 0$ such that $f(t) > k_1 t^{-\frac{3}{2}}$, for $t \geq 1$.

If we note that $\dot{p}_1(t) < -\frac{p_2^2(t)}{2}$, then it follows that, if $t \geq 1$,

$$\lim_{s\to\infty} p_1(s) = p_1(t) + \int_t^{\infty} \dot{p}_1(w)dw < p_1(t) - f(t) < (c_1^{-\frac{1}{3}} + \frac{1}{2}t)^{-3} - k_1 t^{-\frac{3}{2}},$$

but the last inequality clearly implies that $\lim_{t\to\infty} p_1(t) < 0$, which is absurd in light of previous considerations ■

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