Categorial forms of the Axiom of Choice

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Abstract

In this work, we will be interested on the investigation of *categorial forms* of the Axiom of Choice (**AC**). We introduce a quite comprehensive list of categorial set-forms of **AC**, which are (by definition) all equivalent statements within the category **Set**—but which may behave differently in some other categories. We exemplify (either the presence or the absence of) all of these introduced categorial versions of **AC** in several categories; a number of applications of such forms, mostly related to the notion of initial objects, are also presented.

Keywords: Axiom of Choice, category theory.

1 Introduction

It is well-known that the Axiom of Choice is fundamental for classical mathematical reasoning and that it is questioned by constructivists because of its 'non-constructive nature'.[1] We will investigate in this article some categorial forms of such axiom. The problem of identifying statements from Category Theory which are equivalent to the Axiom of Choice — in the following abbreviated by **AC** — was previously addressed by a number of authors. We have, for instance, a kind of 'folklore' which is stated in [4] as follows:

A category $\mathcal{C}$ satisfies **AC** if every epic arrow has a right inverse.

This means, in other words, that every epimorphism has a *section*, or equivalently, every epic arrow *splits*. This previous statement is widely referred to the 'Axiom of Choice for Category Theory', and it is sometimes also referred to as an *external form of the Axiom of Choice*, in contrast to an *internal form of the Axiom of Choice*, which treats the version of the axiom in a category which has an *internal* logic, such as a topos, for example. In that case, the axiom is valid if the axiom is satisfied in this internal logic. In fact, it is known that, for a class of categories which properly

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[1] For a precise and detailed account on this subject, see [8].

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include toposes, the validity of the internal form is equivalent to the validity of the external form.[2] In this article, we will not treat this last mentioned internal form of the Axiom of Choice. We will see in the following that we have more than one *external* version of **AC**, if we consider a number of other statements. And, by introducing some new language and concepts, we will, in fact, introduce new *internal* and *external* versions in other related contexts and situations.

Another (external) version of **AC** is considered by Bell, [4], and Goldblatt, [10]: Let $\mathcal{C}$ be a category and a, b objects in this category. Consider the following statement:

If a is a non-initial object then for any object b and for any arrow $a \xrightarrow{f} b$, there exists $b \xrightarrow{g} a$ such that $f \circ g \circ f = f$.

This version is named as *Strong Axiom of Choice* in [10], and there it is attributed to Saunders Mac Lane.

Another statement related to **AC** was carefully considered by Freyd and Scredov, in [9], page 20, and treats *existence of skeletons*. It says that **AC** is equivalent to the statement: 'Every category has a skeleton'.

Recall that a *skeleton* of a category $\mathcal{C}$ is a full, isomorphism-dense subcategory $\mathcal{C}$ in which no two distinct objects are isomorphic; roughly speaking, it is a 'minimal' subcategory capturing the categorial properties of $\mathcal{C}$. Notice also that this last version is actually different from the other two mentioned above; in fact, while the two first versions are statements about *a given* category, this last one is a statement about *all* categories. Observe also that this last version of the Axiom of Choice can be also seen as some kind of *external* **AC**, perhaps we could speak of a *meta-external* version of **AC**.

In the literature, the first two versions of **AC** above mentioned are understood and presented as purely categorial versions of **AC**, while the third version is presented as an equivalence of **AC**. However, it is not hard to see that there are differences between the status of these statements. Let us try to summarize these differences as follows:

- A category $\mathcal{C}$ which satisfies the statement 'Every epic arrow has a section', or even the above mentioned Mac Lane's Strong Axiom of Choice, has a feature which, in **Set**, is equivalent to the validity of the Axiom of Choice *(for sets)*.
- On the other hand, we have: If *all* categories $\mathcal{C}$ satisfy the statement '$\mathcal{C}$ has a skeleton' then the Axiom of Choice *for classes*[3] does hold.

So, the first two statements (the *Axiom of Choice for Categories* and the *Strong Axiom of Choice)* describe an *external* feature of a given category $\mathcal{C}$ and witness some resemblance between $\mathcal{C}$ and the category of sets. However, from this moment on we will deal with a statement as 'Every epic arrow has a section' as a kind of *internal* form of the Axiom of Choice. We hope that this will not be confused with the *internal* forms mentioned in the beginning of this article — those which are

[2] More precisely, one can check at Bell's book on toposes and local set theories ([3], page 148) that, for any local set theory S, S satisfies internal **AC** if, and only if, the associated category $C(S)$ satisfies external **AC**. It is well-known that each $C(S)$ is a topos — 3.16, page 87 of Bell, op. cit — and, moreover, any topos is equivalent to a category $C(S)$ for some S — Equivalence Theorem, 3.37, page 109 of Bell, op.cit.

[3] In the next section, we will clarify what we mean by 'statements from Category Theory' in each context, and we will also briefly discuss about the notions of choice we are supposing for sets, classes and conglomerates.

related to internal logics. In this article, our main aim is to call attention to statements which behave like those two first statements.[4]

On the other hand, the third one, if considered valid for all categories, implies a property of the *universe of all sets*, and so we have a *meta-external* version of **AC**, which we will refer to as an *external* version of the Axiom of Choice for categories — because it describes an external feature of *all* categories.

Having established this difference — which will be highlighted several times more in this work, since the whole research of this article is based on it —, we will be interested, in this research, in the internal features of a given category with respect to (versions of) the Axiom of Choice. The external features (regarding all categories) will be phenomena to be investigated in a future work. More specifically, the authors want to understand, clarify and generalize some details of the use (as well as the notion itself) of the Axiom of Choice within Category Theory, and this article is only the first step of such programme.

This article is organized in the following manner. In Section 2, we give a background review, fix some definitions and present some statements which will be called *categorial forms of the Axiom of Choice*. We start with forms of **AC** in category theory which are inspired by the product version of **AC**; by product version, we mean the version that states that the product of a non-empty family of non-empty sets is also not empty. We introduce some new notions regarding cones and distinguish between versions of non-uniqueness of cones and non-skeletal cones. Afterwards, we introduce epimorphism versions, and some of them are immediately derived from the statement regarded as being **AC** for categories — i.e., 'Every epic arrow has a section'. There, again we investigate some notions related to cones. In Section 3, we discuss some variants of *initial* objects in Category Theory, as quasi-, nearly and strict initial objects, and we discuss under which forms of the Axiom of Choice we have the equivalence between some of these notions; under a certain sense, we realize that the presence of the Axiom of Choice within a category determines *how its initial objects are supposed to be*. In Section 4, we investigate some examples of categories, including the category **Top**, comma categories and poset categories, showing that the presented set-forms of the Axiom of Choice need not to be equivalent in categories other than **Set**: we will check that some of our categorial forms of the Axiom of Choice are fulfilled by some categories as the ones mentioned — while others forms, are not.

2 Categorial forms of the Axiom of Choice

2.1 Background: on choice for sets, classes and conglomerates

We assume the reader is familiar with the first-order language, as well as the axioms, of (Zermelo-Fraenkel, **ZF**) Set Theory — the choiceless Set Theory. This theory has only the primitive concepts of 'set' and 'membership' (between sets), but can represent syntactically the notion of 'class' as a (meta-theoretic) equivalence class of formulas ϕ, where $\phi \sim \phi'$ iff $\mathbf{ZF} \vdash \phi \leftrightarrow \phi'$. As usual, we *denote classes* by $\{x : \varphi(x)\}$, where φ is a first-order formula in the language of Set Theory with x as a free variable. Sets determine classes, since, for a given (and *properly, previously constructed*) set x, we can consider the formula $\varphi_x(z) := z \in x$ and we can identify $x = \{z : \varphi_x(z)\}$. There are classes that are not sets; they are called *proper classes*. Examples of proper classes include the *universal class*

[4] As far as our knowledge goes, we are presenting in this article a *new, original approach* to the relationship between the Axiom of Choice and the Category Theory — and we hope that such strong statement will be justified to the reader in the remaining of the article.

$V := \{x : x = x\}$, the *class of ordinals* $On := \{x : x \text{ is an ordinal}\}$, the *Russell's class* $R := \{x : x \notin x\}$ — which coincides with the universal class under the Axiom of Foundation –, among many others.

In **ZF**, if $I = \{x : \alpha(x)\}$ is a proper class, the expression '$y \in I$' is simply a short for '$\mathbf{ZF} \vdash \alpha(y)$'. On the other hand, the theory **NBG** (Von Neumann; Bernays; Gödel), has the notion of 'class' as primitive concept and the notion of 'set' is defined in this theory as 'being a member of some class'. As it is well-known, the mapping that takes arbitrary formulas in the language of **ZF** to corresponding 'bounded formulas' in **NBG** is a conservative translation from **ZFC** into **NBG**.

Still considering that I is a class (either in **ZF** or in **NBG**, it does not matter), we may also think of I as an indexer for classes, so the collection $\{x_i : i \in I\}$, where x_i is a set for every $i \in I$, is also class — being, in fact, the image of a *class function*, which is, of course, a class whose elements are ordered pairs satisfying the usual requirements for a function.

If I is a class and, for every $i \in I$, X_i is also a class, then the collection of classes $\{X_i : i \in I\}$ will be said to be a *conglomerate*.

Sets, classes and conglomerates, as it is widely known, are necessary ingredients for the foundations of Category Theory (see, e.g. [1]), so it is necessary to give, as well, a formal, proper foundation for these notions. There are many ways for giving such a foundation, within Set Theory. The most easily describable way is as follows: let κ be a strongly inaccessible cardinal[5] and consider the κ-th level V_κ of the cumulative hierarchy of the universe of all sets: this set is a model of **ZF** and that's why we cannot prove neither the existence nor the relative consistency of the existence of such a cardinal — unless **ZF** itself is inconsistent. In fact, V_κ is a 'Grothendieck's universe' (see Bourbaki's appendix to the *Préfaisceaux* of SGA4, [2]), i.e., a certain transitive set that is closed under many set-theoretic constructions; **ZF** establish an order preserving bijection between the class of strongly inaccessible cardinals (with the ordinal ordering) and the class of all Grothendieck universes (ordered by inclusion). Thus, fixed a strongly inaccessible cardinal κ, it is natural to interpret 'set' as 'be a member of V_κ', to take elements of $V_{\kappa+1}$ (all of them) to be classes, and let the elements of $V_{\kappa+2}$ which can be indexed by classes as being the conglomerates. Depending on how much of closedness under operations/constructions one may want to ensure for a particular structure — families of conglomerates indexed by classes, families of conglomerates indexed by conglomerates, functions between conglomerates, functions between families of conglomerates, etc. —, one may go up to $V_{\kappa+\omega}$, or even assume the existence of a second strongly inaccessible cardinal. In the final structure, we can either assume that the Axiom of Choice will be restricted only to families of sets, i.e., we could only consider the Axiom of Choice restricted to V_κ — or we could even not to assume any kind of choice principle at all, in any level. A structure like this is nice to be considered as starting point of a foundation for Category Theory, since from this point on we can discuss notions of choice in a general setting, including classes and conglomerates,[6] and this is precisely what we will do next.

First, let us introduce a very general statement for three 'Axioms of Choice', say. The following definition is done in the language of a first-order structure M where we can properly discuss the notions of sets, classes and conglomerates, and no choice principle is previously assumed.

Definition 2.1 (Axiom of Choice, general statement)
We say that the *Axiom of Choice for (Sets, Classes, Conglomerates)* holds if whenever $\mathcal{X}$, $\mathcal{Y}$ are (sets, classes, conglomerates, respectively) and $f : \mathcal{X} \to \mathcal{Y}$ is a surjective map, then f has a right inverse.

[5]For a treatment of inaccessible cardinals without the Axiom of Choice, we refer to [6].

[6]This strategy of starting with strongly inaccessible cardinals is not, necessarily, the only way to proceed with a set-theoretical foundation of Category Theory; as will be clear in what follows, this discussion will be more important in an upcoming continuation of this work, so we will not go further into details in this article.

From now on, the notion of *choice function* may refer to functions (between sets), class functions, maps defined in conglomerates, etc. — always with the expected meaning.[7] Similarly, we believe that notions as 'a class being an element of a conglomerate' are self-explanatory.

The following definition relates to the previous one.

Definition 2.2 (Axiom of Global Choice)
We say that the *Axiom of Global Choice* holds if there is a choice function defined in the class of all non-empty sets.

The Axiom of Global Choice is equivalent to the existence of a well-ordering of the universe (Proposition 4.1 of [13]).

It is easy to check that, in terms of choice functions, these axioms declare the following:

- The Axiom of Choice for Sets holds if, and only if, every set whose elements are non-empty sets has a choice function.
- The Axiom of Global Choice holds if, and only if, every class whose elements are non-empty sets has a choice function.
- The Axiom of Choice for Classes holds if, and only if, every conglomerate whose elements are non-empty classes has a choice function.
- The Axiom of Choice for Conglomerates holds if, and only if, every indexed-by-a-conglomerate family of non-empty conglomerates has a choice function.

In terms of systems of representatives of equivalence relations, we have the following:

- The Axiom of Choice for (Sets, Classes, Conglomerates) holds if, and only if, every equivalence relation over a (set, class, conglomerate, respectively) has a complete system of representatives.

It is easy to see that we could also produce statements very similar to the last ones, using, as the key notion, the existence of well-orderings (in each context and under the expected requirements).

One could think that, above the set level, say, we would be in the presence of a gradation of choice axioms — being, for instance, the Axiom of Choice of Conglomerates strictly stronger than the Axiom of Choice for Classes. Maybe surprisingly, this is not the case. Again, the following equivalences are done over the language of a structure M on which we can work with sets, classes and conglomerates, and none choice principle is assumed; one may think of M as being the structure obtained from inaccessible cardinals above described, or else one may assume that we are under his/her favorite definition of a set-theoretic foundation for Category Theory. The arguments are simple enough to be valid in any case.

Proposition 2.3
The following statements are equivalent:

(i) The Axiom of Global Choice.
(ii) The Axiom of Choice for Classes.
(iii) The Axiom of Choice for Conglomerates.

The previous proposition is surely well-known, but we did not find it neither stated nor proved in the literature. So, let us sketch a proof for it.

[7] In the description above, where '$\mathcal{X}$ is a conglomerate' means '$\mathcal{X} \in V_{\kappa+2}$', the possible choice functions $c: \mathcal{X} \to \bigcup \mathcal{X}$ belongs to $V_{\kappa+5}$.

PROOF. It is clear that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). We will check that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (ii) (Sketch) Let $\{X_i : i \in I\}$ be a conglomerate of non-empty classes. Using the Axiom of Foundation, for every $i \in I$, define

$$\alpha_i = \min\{\beta \in On : (\exists x \in X_i)[\text{rank}(x) = \beta]\} = \min\{\text{rank}(x) : x \in X_i\}.$$

For every $i \in I$, let x_i be the *set* given by $x_i = X_i \cap V_{\alpha_i+1}$. So, $\{x_i : i \in I\}$ is a class of non-empty sets, and the existence of such choice function follows from Global Choice.

(ii) $\Rightarrow$ (iii) (Sketch) Let $\mathcal{X} = \{X_i : i \in I\}, \mathcal{Y} = \{Y_j : i \in J\}$ be conglomerates and $f : \mathcal{X} \to \mathcal{Y}$ be a surjective map. As we are assuming the Axiom of Choice for Classes, consider a well-ordering of the class I and, for every $j \in J$, map Y_j to X_m, where m is the minimal element of $\{i \in I : f(X_i) = Y_j\}$ in the fixed well-ordering of I. This procedure clearly give us a right inverse for f. ■

Notice that we used, in a very strong way, the Axiom of Foundation in the above proof. Such essentiality of the use of Foundation was already noticed in [13], when a proof of the equivalence of the Axiom of Global Choice with the existence of a well-ordering of the universe is given. Indeed, it seems that assuming the existence of a well-ordering on the universe — and, consequently, on every class — goes a long way; after getting used to the arguments, one could check (or even foresee) that all equivalent statements presented (in terms of surjective maps, choice functions and classes of representatives of equivalence relations over classes and conglomerates) follow easily from the existence of a well-ordering of the universe, and are in fact equivalences of such assumption.

Now, we can return to our choiceless set-theoretic structure M, where we are able to work properly with the notions of sets, classes and conglomerates, and assume the Axiom of Global Choice, say. Automatically, the Axiom of Choice for Sets holds, as an immediate consequence of Global Choice, and the Axioms of Choice for Classes and Conglomerates also hold, as equivalences of Global Choice. This new structure, with this nice, right amount of choice, is usually considered as the starting point of the foundation of Category Theory (again, see [1]). This is also the point where equivalences between statements from Category Theory and versions of Axiom of Choice are able to emerge. As already commented, in [9] there is a proof of the equivalence between the Axiom of Choice and the statement 'Every category has a skeleton'; in fact, what is proved there is that the statement 'Every *small*[8] category has a skeleton' is equivalent to the Axiom of Choice *for sets*. Isbell and Wright, in the 60s (see [11]), have presented an argument which proves that the statement 'Every *locally small*[9] category has a skeleton' implies the Axiom of Global Choice.

Furthermore, it is a well-known, easily checkable fact that the Axiom of Choice for Conglomerates implies that 'Every category has a skeleton' (Proposition 4.14 of [1]). So, summing up, we have that

Choice for Conglomerates $\Rightarrow$
Skeletons for every category $\Rightarrow$
Skeletons for every locally small category $\Rightarrow$
Axiom of Global Choice $\iff$ Choice for Classes $\iff$ Choice for Conglomerates

So, we have just highlighted a statement from Category Theory ('Every category has a skeleton') which declares a property of *all* categories and constitutes a equivalence of the Axiom of Choice

[8] A category $\mathcal{C}$ is said to be *small* if both obj($\mathcal{C}$) and hom($\mathcal{C}$) are sets.

[9] A category $\mathcal{C}$ is said to be *locally small* if obj($\mathcal{C}$) is a class and for every $a, b \in$ obj($\mathcal{C}$) — which are sets — one has that hom(a, b) is also a set.

for Classes. This is an example of what we will define, in this article, as being a *class-form of the Axiom of Choice*. Incidently, we also have highlighted a statement ('Every small category has a skeleton') which declares a property of *all small categories* and which constitutes a equivalence of the Axiom of Choice *for Sets*. Statements of this last kind will not be considered in this article — although the authors believe that they deserve some further investigation. We will be more interested in statements as the *Axiom of Choice for Categories* — i.e., 'Every epimorphism has a section'. Such statement can be interpreted within any category; but, when considered in the category of sets, constitute a equivalence of the Axiom of Choice *for Sets*. This is an example of what we will define, in this article, as being a *set-form of the Axiom of Choice*.

However, in the same way we have dedicated some time for giving a formalization (as detailed as we needed for this article) of our notions of choice, we will dedicate some time in the next section to formalize (again, with the level of formalization that we need in this article) what we understand as *statements from Category Theory*.

2.2 Background: On the language for category theory

We assume the reader is familiar with the basic notions of Category Theory (objects, morphisms ('arrows'), identity arrows and composition of morphisms, monomorphisms and epimorphisms, the classes $obj(\mathcal{C})$ and $hom(\mathcal{C})$ for a given category $\mathcal{C}$, etc.). We also use freely the notion of *oriented* (or *directed*) graphs[10](needed in order to define *diagrams* in a category). Categories will be denoted as $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$; oriented graphs will be denoted as $\mathcal{G},\mathcal{H},\mathcal{I},\mathcal{J}$.

In this article, we consider statements which will be referred to as *statements from Category Theory*. Let us formalize this notion. The language for *'a'* category is usually defined as a two-sorted first-order language (where morphisms and objects are the distinct sorts), together with *source* and *target* maps (which assign, respectively, domain and codomain for every morphism), composition maps and identity maps, together with all expected axioms; one could also proceed with the trick of identifying identity morphisms with objects, and then formulate every single notion in terms of a 1-sorted theory. When it comes to deal with constructions and operations over categories, it is necessary to introduce some higher order statements; the reader could easily check that some of the statements we will introduce presently are clearly of second order. However, as the foundation of the Category Theory will be considered within a first order, set-theoretical structure M (as described in the previous subsection) on which it is clear that *every statement, even those which are not of first order*, can be 'translated' to the language of M, we will not be extremely concerned with the order of the statements we will deal with in this article. We assume the same level of freedom which is commonly accepted in the research of Category Theory — and so we freely consider statements over the usual categorial constructions; the authors believe that, for the specifical purposes of this work, there is no necessity of exaggerate into details about the usual language needed to discuss a given category.

So, consider a statement φ which is stated in the assumed language for 'a' category — which declares properties of objects, morphisms and/or constructions -, and fix a category $\mathcal{C}$. The statement which is obtained by *relativization* of φ with respect to $\mathcal{C}$ — meaning, all quantifiers are restricted to $\mathcal{C}$ (for instance, the relativization of 'Every epic arrow splits' is 'Every epic arrow *in* $\mathcal{C}$ splits', and so on) will be denoted as $\varphi_{\mathcal{C}}$. This relativized statement $\varphi_{\mathcal{C}}$ will be also freely referred to as being a *restriction* of φ to $\mathcal{C}$, or, with some considerable degree of language abuse, even as being

[10]In fact, we will always work with *multigraphs*, since there can be more than one morphism between two objects in a category.

a *translation* of φ to $\mathcal{C}$. We will be particularly interested in statements φ whose restrictions to the category **Set** constitute equivalences (in the usual, **ZF** based sense) of the Axiom of Choice for Sets.

In the authors' opinion, it should not be surprising at all that some second (or even higher) order statements from Category Theory turn out to, when restricted/translated/relativized to **Set**, be equivalences of the Axiom of Choice. In fact, one should argue that the Axiom of Choice itself, which declares, for instance, that *every set can be well-ordered*, is a statement about the *subsets* of every set ('for every set there is a linear order on it such that *every non-empty subset has a minimal element*'). However, as subsets of sets are also sets, it is possible to make, in first-order Set Theory, statements over sets, subsets of sets, subsets of subsets of sets, etc., and because of that we are assuming that it is always possible to translate *any* statement from the language of category theory — even those which are necessarily of a higher order — to a first-order statement of **ZF**, whenever we wish to consider the restriction of a particular statement to the category **Set**.

Given a statement φ of the language of a category, we can also consider quantified statements as 'For every category $\mathcal{C}$, $\varphi_{\mathcal{C}}$ holds' or 'There is a category $\mathcal{C}$ where $\varphi_{\mathcal{C}}$ holds'. As we hope it is now obvious to the reader, these statements are supposed to be considered as statements of the language of a set-theoretical structure M which satisfies the requirements described in the previous subsection. Implications and equivalences between statements like these are also supposed to be done in the language of such a M. We will give some importance to statements φ whose validity of $\varphi_{\mathcal{C}}$ for *all* categories $\mathcal{C}$ constitute equivalences of the Axiom of Choice for Classes — although a more profound investigation of these statements will be done in a future work.

In what follows, we propose a way to clarify the discussion made at the Introduction (about statements which are seen as versions and/or equivalences of **AC** in Category Theory) and we will present some definitions of various types of categorial forms of the Axiom of Choice.

From now on, **AC** will denote either the Axiom of Choice for Sets (or, even, any of its well-known equivalences) when referring specifically to the category **Set**, or will denote the statement 'Every epic arrow splits' when referring to a category $\mathcal{C}$ in general. The desired context will be always left clear.

2.3 Set- and class-forms of the axiom of choice

As we be believe it was left clear in the previous subsection, let φ be a (non necessarily first-order) statement of Category Theory — meaning that φ can be interpreted within any category. Depending on certain properties of φ, such statement could be either a *set-form* of the Axiom of Choice or a *class-form* of the Axiom of Choice. Let us introduce the precise definitions.

Definition 2.4
Let φ be a formula written in the language of the theory of categories. Then

(a) we say that the statement φ is a *Categorial Set–Form of the Axiom of Choice* if the Axiom of Choice for Sets is equivalent to the statement $\varphi_{\mathbf{Set}}$.
(b) we say that the statement φ is a *Categorial Class–Form of the Axiom of Choice* if the validity of $\varphi_{\mathcal{C}}$ for all categories $\mathcal{C}$ is equivalent to the Axiom of Choice for Classes.

Example 2.5
Remember the statement written in the language of category theory 'Every epimorphism has a section'. This is a categorial set–form of the Axiom of Choice, while the statement 'There is a skeleton in the category' is a categorial class–form of the Axiom of Choice.

As already remarked in the introduction, in this article we will deal only with set-forms of the Axiom of Choice. We have defined the class-forms to give some context to our discussion, and we will investigate such class forms, in a more detailed way, in a future work.

Now, we introduce a number of such set forms, present some applications and discuss some features of such statements. For a better understanding, we recall some basic terminology from Category Theory:

Definition 2.6
Let $\mathcal{C}$ be a category. Then,

(a) we say that a *diagram* D in $\mathcal{C}$ is a family of objects, together with morphisms in $\mathcal{C}$, or equivalently, 'arrows' between some of these objects, and,
(b) we say that a diagram is a *discrete diagram* in $\mathcal{C}$ if every arrow of the diagram is the identity arrow (i.e. from an object of the diagram to itself) and there are no other arrows.

Sometimes, diagrams are abbreviated by $D := (\{d_i\}_{i\in I};\ d_i \xrightarrow{g_{ij}} d_j)$.

Our definition of diagram was given accordingly to [10] (page 58) and, even being a little simplified, it is enough for the purposes of this article; if one looks for a more formal definition, then a diagram can be defined as functor $D\colon I \longrightarrow \mathcal{C}$, where I is an *index category* (or *scheme*) of the diagram (see e.g. [1], page 193).

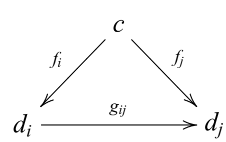
Remark 2.7

(a) Note that it is possible in a diagram D to have more than one arrow between a given pair of objects of the diagram — so, in the above notation, the morphisms g_{ij} are not supposed to be unique —, and it is also possible to have no arrows at all between them. Also, the empty diagram may be considered.
(b) Technically a diagram D in a category $\mathcal{C}$ is an oriented graph morphism $D\colon \mathcal{G} \to U(\mathcal{C})$, where $\mathcal{G}$ is an oriented graph and $U(\mathcal{C})$ is the underlying directed graph of the category $\mathcal{C}$. This notion of diagram corresponds to the 'functorial' version that we have just mentioned (right after the previous definition).

Let us recall the following definition.

Definition 2.8
Let $D := (\{d_i\}_{i\in I};\ d_i \xrightarrow{g_{ij}} d_j)$ be a diagram in a category $\mathcal{C}$.

(a) We say that a *(projective) cone* over D consists of an object $c \in ob(\mathcal{C})$ together with arrows $c \xrightarrow{f_i} d_i$ for every object d_i and every $i \in I$ such that



commutes for every arrow g_{ij} of the diagram. Such a cone will be denoted as $C := \{c,\ c \xrightarrow{f_i} d_i\}$.

(b) If $C := \{c,\ c \xrightarrow{f_i} d_i\}$ and $C' := \{c',\ c' \xrightarrow{f_i'} d_i\}$ are cones over the same diagram D, then a morphism $h\colon C \to C'$ is a $\mathcal{C}$-arrow $h\colon c \to c'$ such that $f_i = f_i' \circ h$, for every $i \in I$. The classes of all

cones over D and of all cone morphisms, endowed with obvious identities and composition, constitutes a category: $Cone(D)$.
As expected, two cones C, C' over the same diagram D will be said to be *isomorphic cones* if they are isomorphic in the category $Cone(D)$.

(c) A limit cone over the diagram D is a terminal object in the category $Cone(D)$; thus limit cones are unique up to (unique) $Cone(D)$-isomorphism.

In the next subsections we will see that our categorial 'set–versions' of **AC** (which are divided in 'product versions' and 'epimorphism versions') show that — maybe surprisingly — the Axiom of Choice primarily deals with cones and initial objects. Let us start with the following:

2.4 Product versions

In this subsection, we start introducing some equivalences in the category **Set** for the Axiom of Choice, which can be proved easily to be set-forms of **AC** by considering the Axiom of Choice in the way asserting that the product of every non-empty family of non-empty sets is a non-empty set. From the second subsection on, we will introduce some new product versions of the Axiom of Choice.

2.4.1 Products of non-initials

As the starting point, we introduce the form **PNI**, abbreviating *products of non-initials*.

DEFINITION 2.9
PNI, for *products of non-initials*, is the following statement from Category Theory:

Every product of a non-empty family of non-initial objects is a non-initial object.

It is obvious that **PNI** is a set-form of **AC**.

Next, we introduce a form which *is not* a set-form of **AC** — but it is also interesting to be considered.

DEFINITION 2.10
PFNI, for *products of finite families of non-initials*, is the following statement from Category Theory:

Every product of a non-empty, finite family of non-initial objects is a non-initial object.

PFNI is not a set-form of **AC** simply because its translation/relativization to Set Theory is a theorem of **ZF**; as the subjacent logic of **ZF** is a first-order *finitary* logic, we are always allowed to make a finite number of arbitrary choices over finite families of non-empty sets — and that is why **Set** satisfies **PFNI**, and there is no need of neither **AC** nor any extra axiom for ensuring it. As will be seen in the last section, there are elementary categories which do not satisfy **PFNI**.

Of course, as pure statements of Category Theory, for every category $\mathcal{C}$ we have that $\mathbf{PNI}_{\mathcal{C}} \Rightarrow \mathbf{PFNI}_{\mathcal{C}}$.

2.4.2 Non-uniqueness of cones

We introduce the form $\mathbf{NUC_1}$, abbreviating *Non-uniqueness of cones, first part*.

Definition 2.11
NUC$_1$, for *Non-uniqueness of cones, first part*, is the following statement from Category Theory:

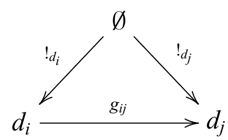
A discrete, non-empty diagram of non-initial objects has more than one cone.

Consider the

Definition 2.12
We will call *non-degenerated* a category $\mathcal{C}$ that has at least two objects.

Remark 2.13

(a) If $\mathcal{C}$ is a non-degenerated category, then the *empty* diagram (that is vacuously discrete) has more than one cone.
(b) In the category **Set**, the cone consisting of the empty set together with all due empty functions, i.e., $\varnothing := \{\emptyset,\ \emptyset \xrightarrow{!_{d_i}} d_i\}$ is always a cone for *any* diagram $D := (\{d_i\}_{i\in I};\ d_i \xrightarrow{g_{ij}} d_j)$. Observe that D can be discrete or not. Clearly the following diagram always commutes in **Set**:

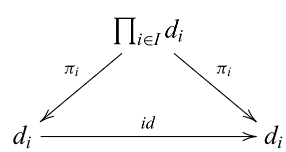


because of $g_{ij} \circ !_{d_i} = !_{d_j}$. We call this cone simply the *empty cone*, $\varnothing$.

Recall that the category **Set** has initial (and terminal) object (the empty set $\emptyset$ is the unique initial object and every singleton is a terminal object) and so we have the following

Fact 2.14
In the category **Set**, **AC** holds $\iff$ **NUC$_1$** holds.

Proof. Suppose **AC** in **Set**, given by the formulation that for every family $\{A_i\}_{i\in I}$ of non-empty sets the product $\prod_{i\in I} A_i$ is not empty — i.e. assume **PNI** holds in **Set**. Under this assumption, we have the natural projections $\pi_j : \prod_{i\in I} A_i \to A_j$ for every such family. The category **Set** is non-degenerated; for a *non-empty* discrete diagram $D := (\{d_i\}_{i\in I};\ d_i \xrightarrow{id} d_i)$ we have the following cone given by



and so **AC** implies **NUC$_1$**. For the reverse implication, we argue contrapositively: suppose that the Axiom of Choice is not valid in **Set**, and so there is a non-empty family of non-empty sets $\{A_i\}_{i\in I}$ such that $\prod_{i\in I} A_i$ is empty. Clearly – using the universal property of the product in **Set** – the only cone for the discrete diagram $D := (\{A_i\}_{i\in I};\ A_i \xrightarrow{id} A_i)$ is the empty cone. ∎

It follows that **NUC$_1$** is a set-form of **AC**.
It is clear that we can mimic the $\Rightarrow$-part of the preceding proof and prove the following:

PROPOSITION 2.15
Assume that $\mathcal{C}$ is a category with products and an initial object. Under such assumptions, if $\mathcal{C}$ satisfies **PNI** then $\mathcal{C}$ satisfies $\mathbf{NUC_1}$.

It is worthwhile remarking that, in a certain sense, the previous results show that the existential aspect of the Axiom of Choice gives us, in a certain way, *more options for cones*. Notice that — *in* **ZF** — if one of the sets of a family $\{A_i : i \in I\}$ is empty then the only cone over the corresponding discrete diagram will be given by the empty cone. So, we also have the following stronger categorial version of $\mathbf{NUC_1}$, denoted by $\mathbf{NUC_2}$ **-** *Non-uniqueness of cones, second part* and given as follows:

DEFINITION 2.16
$\mathbf{NUC_2}$, for *Non-uniqueness of cones, second part*, is the following statement from Category Theory:

If the category is non-degenerated, then a non-empty discrete diagram has only a cone if, and only if, at least one of the objects of the diagram is initial.

REMARK 2.17

(a) First observe that, for each non-degenerated category, $\mathbf{NUC_2}$ is stronger than $\mathbf{NUC_1}$ in the sense that $\mathbf{NUC_2}$ implies $\mathbf{NUC_1}$, and $\mathbf{NUC_1}$ is the $\Rightarrow$-part of $\mathbf{NUC_2}$.
(b) It is not difficult to see, considering our previous remarks, that in the category **Set**, the statements **AC**, $\mathbf{NUC_1}$ and $\mathbf{NUC_2}$ are all equivalent. In particular, $\mathbf{NUC_2}$ is a set-form of **AC**.

2.4.3 Non-skeletal cones

Inspired on the notion of *skeletal category*,[11] we introduce another categorial set-form of choice, using the notion of *skeletal cone*, which we will introduce and explain in the following definition.

DEFINITION 2.18
Let $\mathcal{C}$ be a category and $D := (\{d_i\}_{i \in I};\ d_i \xrightarrow{g_{ij}} d_j)$ a diagram. Then,

(a) we say that a cone $C = \{c,\ c \xrightarrow{f_i} d_i\}$ for the diagram D is a *skeletal cone* if every isomorphic cone (over the same diagram D) with $d \cong c$ implies that $d = c$. Otherwise, we say that C is a *non-skeletal cone* and
(b) we say that a skeletal cone is *strongly skeletal* if besides of $c = d$ the only possible isomorphism $c \xrightarrow{\sim} c$ is given by the identity morphism.

REMARK 2.19

(a) Consider the statements:
(a_1) every cone over D is skeletal;
(a_2) *Cone*(D) is an skeletal category;
(a_3) every cone over D is strongly skeletal;
then clearly: $(a_3) \Rightarrow (a_2) \Rightarrow (a_1)$.
(b) Consider a partially ordered set $(X, \leq)$ viewed as a category $\mathcal{C}_X$ (see Remark 3.4). Then, given a diagram D over $\mathcal{C}_X$, every cone C over D is strongly skeletal.
(c) If the category $\mathcal{C}$ has an initial object 0, then $C_0 = \{0,\ 0 \xrightarrow{!_{d_i}} d_i\}$ is (obviously) a cone over (any diagram) D. Moreover, if the category $\mathcal{C}$ has an *unique* initial object[12] 0 (for instance,

[11] A category where isomorphic objects are necessarily identical.
[12] Here we mean really unique, non only unique up to isomorphism !

the categories $\mathbf{Set}^{\mathcal{A}}$ — where $\mathcal{A}$ is a category — and $\mathbf{Top} \times \mathbf{Cat}$ — see Example 3.9 — have this property), then C_0 is a *strongly skeletal* cone over (any diagram) D.

Remark 2.20
Let $\{A_i : i \in I\}$ be a non-empty family of non-empty sets and A be given by $A = \prod_{i \in I} A_i \neq \emptyset$, then $C = \{A, A \xrightarrow{\pi_i} A_i\}$ is a non-skeletal cone — since any set equipotent to A (eventually including the set A itself, with non-trivial bijections) will constitute an isomorphical cone which does not satisfy the requirements of any of the previous definitions in 2.18.

We use the notions introduced in Definition 2.18 to give two other categorial forms of **AC**. Let us start with the following version. We introduce by $\mathbf{NSC_1}$, abbreviating *Non-skeletal cones, first part*.

Definition 2.21
$\mathbf{NSC_1}$, for *Non-skeletal cones, first part*, is the following statement from Category Theory:

Every non-empty discrete diagram of non-initial objects has a non-skeletal cone.

We have the following

Fact 2.22
In the category **Set**, **AC** holds $\Longleftrightarrow$ $\mathbf{NSC_1}$ holds.

Proof. The $\Rightarrow$-direction is proved by remark 2.20. For the other direction, suppose the Axiom of Choice is not valid, and with the same argument as in proof of 2.14, we have the empty cone $\varnothing$ as the only cone for this diagram, and observe that the empty cone is a (strongly) skeletal cone in **Set**. ∎

With similar arguments involving a **ZF** valid implication as we have done for $\mathbf{NUC_1}$, we can also introduce a stronger version of $\mathbf{NSC_1}$, the version $\mathbf{NSC_2}$ — *Non-skeletal cones, second part*.

Definition 2.23
$\mathbf{NSC_2}$, for *Non-skeletal cones, second part*, is the following statement from Category Theory:

A non-empty discrete diagram has a non-skeletal cone
if, and only if, all of its objects are non-initial.

Remark 2.24

(a) First observe that $\mathbf{NSC_2}$ is stronger than $\mathbf{NSC_1}$ in the sense that $\mathbf{NSC_2}$ implies $\mathbf{NSC_1}$, and $\mathbf{NSC_1}$ is the $\Leftarrow$-part of $\mathbf{NSC_2}$.
(b) It is not difficult to see that in the category **Set**, the properties **AC**, $\mathbf{NSC_1}$ and $\mathbf{NSC_2}$ are all equivalent. In particular, $\mathbf{NSC_2}$ is a set-form of the Axiom of Choice.

As we will see in the next section, the stronger versions presented have huge influence on the structure of the initial objects of any category satisfying them. In this sense, the presence of these forms of the Axiom of Choice in a category determines how the initial objects are supposed to be.

Remark 2.25
It was, indeed, quite surprising for the authors to realize that we were in the position of argue that, in **Set**, our approach of categorial forms somehow shows that *The Axiom of Choice is mainly talking about the empty set, and not about the products!*.

2.5 Epimorphism versions

In this subsection, we introduce two notions of the **AC** which are derived from epimorphisms, which are always surjective functions in the category **Set**. Let us start with the following:

2.5.1 Cones with epic morphisms

A remarkable feature of **ZFC** is that *projections are surjective*, when one deals with products of non-empty families of non-empty sets. This motivates a new categorial version of **AC**, denoted by **CEM** — *Cones with epic morphisms*. **CEM** is the following statement from category theory:

Definition 2.26
CEM, for *Cones with epic morphisms*, is the following statement from category theory:

Every non-empty discrete diagram whose objects are all non-initial
has a cone where all constituent morphisms are epic arrows.

We have the following easy result, whose proof we omit, and which asserts that **CEM** is a set-form of choice.

Fact 2.27
In the category **Set**, **AC** holds $\iff$ **CEM** holds.

2.5.2 Generating a cone from an epimorphism

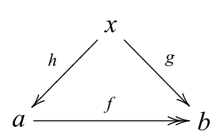
The next categorial version of **AC** we will present is, at principle, an easy consequence of the statement 'Every epimorphism has a section'. To do that, we introduce the following notion:

Definition 2.28
Let $\mathcal{C}$ be a category, $a,b \in ob(\mathcal{C})$, and $D := \{a, b,\ a \overset{f}{\twoheadrightarrow} b\}$ a diagram in $\mathcal{C}$ with $a \overset{f}{\twoheadrightarrow} b$ an epic arrow. We say that f *generates cones* if for every object x in $\mathcal{C}$ and for every morphism $x \xrightarrow{g} b$ in $\mathcal{C}$, there is a morphism $x \xrightarrow{h} a$ in $\mathcal{C}$ such that

$$\{x,\ x \xrightarrow{h} a\ ,\ x \xrightarrow{g} b\}$$

is a cone over the diagram D.

Remark 2.29
Using diagrams, $a \overset{f}{\twoheadrightarrow} b$ generates cones if for every $x \xrightarrow{g} b$ there is $x \xrightarrow{h} a$ such that the following diagram



commutes, i.e. $f \circ h = g$.

Using the last definition, we are able to introduce one more categorial version of **AC**. We introduce the form **EGC** — *Epimorphisms generate cones*.

Definition 2.30
EGC, for *Epimorphisms generate cones*, is the following statement from category theory:

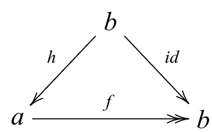
Every epic arrow generates cones.

We have the following

Fact 2.31
For each category $\mathcal{C}$, **AC** holds $\Longleftrightarrow$ **EGC** holds.

Proof. Suppose we have given the **AC** by one formulation of the introduction, that 'every epimorphism has a section'. So start with the epimorphism $a \xrightarrow{f} b$. By **AC**, let $s: b \to a$ be a section of f. Given now the morphism $x \xrightarrow{g} b$ for some object x in $\mathcal{C}$, we define $h: x \to a$, by setting, $h = s \circ g$ and see that f generates cones, i.e., **EGC** holds.

For the other direction, suppose that **EGC** is valid. Let $a \xrightarrow{f} b$ be an epimorphism in $\mathcal{C}$. Now it is easy to see, in the diagram below, that a suitable application of **EGC** is enough to give a right inverse to f.



So, the proof is complete. ∎

Remark 2.32
The statement **EGC** calls our attention to epimorphisms and cones, but, if we look at **EGC** as property of the objects of a category satisfying it, then it is easy to see that **EGC** says that *'Every object is projective'*.[13] The authors believe that it is still worth considering the form **EGC** because it calls the attention to the *morphisms*, instead of to the *objects*; in some point of the future research, we will be probably interested in *weak versions* of our set-forms, so a weak form of **AC** would state as '*Certain objects* are projective', while a weak form of **EGC** would state as '*Certain epimorphisms* generate cones'. Distinguish between those weak forms seems very interesting.

We remind the reader that, for a class of categories which include toposes, there is also a related notion — *choice objects*, see [9], 1.57 page 83, or [12] pages 990 and 991 —, and choice objects are slightly different from projective objects; in the presence of the Axiom of Choice, however — as one could suspect —, these notions are equivalent. The authors believe that investigating choice objects and projective objects under the presented set-forms approach will be worthwhile in the future research.[14]

[13] The notion of *projective object* is well-known, nevertheless the reader may find a definition in page 118 of [15].

[14] Speaking in terms of entire relations between sets — and one can generalize the notion of relation within Category Theory, see e.g. [9] 1.412 page 38 —, A is projective if every entire relation from A to B, for any B, contains a function $f: A \to B$, and B is choice if every entire relation from A to B, for any A, contains a function $f: A \to B$. Roughly speaking, A is projective if we can make choices in families of non-empty sets which are *indexed by A* and B is choice if we can make choices in families of non-empty sets which are *subsets of B*. The definitions generalize to Category Theory accordingly.

3 Applications and comparisons

In this section, we want to describe some applications of the various notions introduced in section 2. We investigate some notions of initial objects and we check that they coincide in the presence of some categorial forms of the Axiom of Choice. We begin with the following:

3.1 Nearly initial objects

In this subsection, let $\mathcal{C}$ be a category and denote — if it exists — by 0 a fixed initial object of the category $\mathcal{C}$. Recall that if $\mathcal{C}$ has an initial object 0 then there is precisely one morphism $0 \longrightarrow x$ for every object x in $\mathcal{C}$. We introduce the following notion:

Definition 3.1
Let $\mathcal{C}$ be a category with initial object 0 and let a be an object in $\mathcal{C}$.

We say that a is a *nearly initial object* of $\mathcal{C}$ if there is an epimorphism $0 \xrightarrow{f} \!\!\!\!\!\! \twoheadrightarrow a$.

Remark 3.2

(a) This notion should not be confused with the related, but more general notion of *quasi-initial object* in a category $\mathcal{C}$. These are usually defined in the following way (see e.g. [5], page 67): Let x be an object in $\mathcal{C}$. We say that x is *quasi-initial* if $|Hom(x,y)| \leq 1$, for every object y, or equivalently, for every object y in the category $\mathcal{C}$, there is *at most one* morphism from x to y. But notice that, in this last definition, the category $\mathcal{C}$ is *not supposed to have an initial object*. In Fact 3.3, we will show that under the existence of an initial object 0, these two notions are equivalent.
(b) Clearly, every initial object is *nearly* initial and also *quasi*-initial.
(c) Let a be a nearly initial object. As epic arrows are right cancellable by definition, it is clear that, if for a given object b in $\mathcal{C}$ there is a morphism $a \longrightarrow b$ then such morphism is unique.

We have the following fact, which establishes the connection between *nearly* and *quasi*-initial objects in a category $\mathcal{C}$; in fact, whenever we are able to define nearly initial objects, they will be quasi-initial, and conversely.

Fact 3.3
Let $\mathcal{C}$ be a category with initial object 0. Then the notions of *nearly* and *quasi*-initial object are equivalent.

Proof. First, let a be nearly initial object of $\mathcal{C}$, i.e., there is an epic $0 \xrightarrow{f} \!\!\!\!\!\! \twoheadrightarrow a$. If there are $g,h : a \to b$ with $g \circ f = h \circ f$, then, by the fact that f is epic, we must have $g = h$, showing that a is quasi-initial. On other side, if a is quasi-initial, there is a morphism $f : 0 \to a$ and this morphism has to be right cancellable, because there is at most one morphism from a to another object b. ∎

Remark 3.4
Recall that each pre-ordered set, $\mathcal{X} = (X, \leq)$, can be seen as a small category (called the *poset category* associated with $\mathcal{X}$), whose set of objects is the set X, and the set of arrows is the binary relation $\leq$. So, for each $x,y \in X$, when there is a morphism from x to y (and this will happen only in the case of $x \leq y$), then such morphism is unique. It turns out that, in a poset category, every arrow is monic and epic, since any of its arrows is uniquely determined by its domain and codomain.

Example 3.5

(a) Suppose a non-empty set A and the power set $\mathcal{P}(A)$ partially ordered by the inclusion relation. Clearly, $\emptyset$ is the initial object, and it is not difficult to see that the singletons $\{t\}$, for $t \in A$ are nearly initial and quasi-initial. More than this, in this category *all* objects are nearly and quasi-initial. This example also establish that pairs of nearly initial objects (respectively, quasi-initial objects) in a category are not isomorphic, in general.

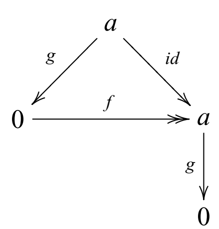
(b) Now consider the category associated with the poset under inclusion $\mathcal{P}^\star(A) := \mathcal{P}(A) \setminus \{\emptyset\}$, A with at least two elements; so, the order is as above but *without* the empty set $\emptyset$. Then all objects of $\mathcal{P}^\star(A)$ are quasi-initial, but $\mathcal{P}^\star(A)$ do not have nearly initial objects.

We are asking now, when is a nearly initial object, in fact, an initial one ? And when is a quasi-initial object an initial one?

By the preceding Remark 3.2, in order to a nearly initial object to be initial, what has to be checked is that a is a *weakly initial object* (see [15], V.6) – that is, there is a morphism $a \longrightarrow b$ for *every* object b. In the following, we will show now that, for categories satisfying **EGC**, the notions of initial object and nearly initial object are equivalent.

Theorem 3.6
Let $\mathcal{C}$ be a category with initial object 0 satisfying **EGC**. Then every nearly initial object is initial.

Proof. Let a be nearly initial in $\mathcal{C}$. By definition, there is an epimorphism $0 \overset{f}{\twoheadrightarrow} a$, and by **EGC** it generates cones. Considering the morphism $a \xrightarrow{id} a$, we have the existence of the arrow $g : a \to 0$ such that the following diagram



comutes.

So, we have that $f \circ g = id_a$ and also by the fact that 0 is initial, we have, that $g \circ f = id_0$. Therefore, a and 0 are isomorphic and so a is an initial object in $\mathcal{C}$. ∎

Corollary 3.7
Let $\mathcal{C}$ be a category with initial object 0 satisfying **EGC**. Then every quasi-initial object is initial.

3.2 Strict initial objects

We keep on arguing that, in a certain sense, the Axiom of Choice speaks about *initial* objects in category theory. In the category **Set**, the initial object exists and is given by the empty set $\emptyset$. We recall the following

Definition 3.8 ([16], page 61, [14], page 194)
Let $\mathcal{C}$ be a category and 0 an initial object in $\mathcal{C}$. We say that 0 is a *strict initial object* if for every object x of $\mathcal{C}$ and every morphism $x \xrightarrow{f} 0$ in $\mathcal{C}$, we have that f is an isomorphism.

Example 3.9

(a) In the categories **Set**, **Top**, **Cat** (the category of all small categories and functors) and $(P_{\perp}, \leq)$ a partial order with least element $\perp$, considered as a category, the initial objects $\emptyset$ and $\perp$ are *strict* initial.

(b) In the category **Grp**, with objects groups and the group morphisms as morphisms, we have that $\{e\}$, e the neutral element, is a *zero* element, i.e., initial and terminal, but not strict initial.

We have the following equivalence:

Lemma 3.10
Let $\mathcal{C}$ be a category and 0 an initial object in $\mathcal{C}$. Are equivalent:

(i) 0 is strict initial in $\mathcal{C}$; and
(ii) for every $b \in ob(\mathcal{C})$ such that there is $f : b \to 0$ a morphism, we have that b is initial.

Proof. Suppose first that (i) is satisfied, i.e., 0 is strict initial. Let now $b \in ob(\mathcal{C})$ and $f : b \to 0$ a morphism. By (i), f is an isomorphism, and so, with 0 initial, b is also initial. For the converse, suppose (ii) and $b \in ob(\mathcal{C})$ and $b \xrightarrow{f} 0$ in $\mathcal{C}$, a morphism. Using (ii), b is initial and so f is an isomorphism, showing that 0 is strict initial object. ∎

Notice that, if one tries to give a categorical model to the classical, propositional logic (with morphisms representing implications, as usual), then $\perp$ is a kind of strict initial object: $\perp$ implies all statements; and every statement implying it, is false.

If one asks when an initial object is strict initial, one of our forms — $\mathbf{NUC_2}$ — in categories gives a solution. We have the following result, which shows — in view of the preceding paragraph — that our categorial forms of the Axiom of Choice indeed carry some features intrinsecal to classical reasoning.[15]

Theorem 3.11
Let $\mathcal{C}$ be a category satisfying $\mathbf{NUC_2}$. Then every initial object of $\mathcal{C}$ is a strict initial object.

Proof. Let $\mathcal{C}$ be a category and 0 an initial object. Let x be an object in $\mathcal{C}$ and $x \xrightarrow{f} 0$ a morphism. Consider the discrete diagram $\{0, id_0\}$ — which has *one* (in fact, *all*) of its objects initial. Notice that $\{x,\ x \xrightarrow{f} 0\}$ is a cone over this diagram. By $\mathbf{NUC}_2$, there is only one such a cone, so $\{x,\ x \xrightarrow{f} 0\}$ is in fact $\{0,\ 0 \xrightarrow{id} 0\}$ — it follows that $x = 0$, as desired. ∎

Remark 3.12

(a) In the previous proof, one could prove the same result by using a weak version of $\mathbf{NUC_2}$ — say, $\mathbf{NUC_{2,weak}}$, which states that the uniqueness of the cone over a diagram with an initial object is a *uniqueness up to isomorphism*.
(b) $\mathbf{NUC_{2,weak}}$ is a categorial set-form of choice because in **Set** the only object isomorphic to the empty set is the empty set itself, i.e., the initial object is strict as seen above.

[15] The most celebrated result of this kind (relating the Axiom of Choice to Classical Logic) is, of course, the one due to Diaconescu ([7]) — which is usually quoted as ‘The Axiom of Choice implies the Excluded Middle Principle in a topos’ or ‘A topos satisfying **AC** has to be Boolean’, or even ‘An intuitionistic model of Set Theory satisfying **AC** has to be a classical one’.

(c) In any category $\mathcal{C}$ we have the following implications,

$$0 \text{ is strict initial object} \Rightarrow^a 0 \text{ is initial} \Rightarrow^b 0 \text{ is nearly initial.}$$

We have seen above that under $\mathbf{NUC_2}$ we have the converse of $\Rightarrow^a$ and under **EGC** we have the converse of $\Rightarrow^b$. In the category **Set**, all these notions are equivalent.

3.3 A few comparisons

We have defined various categorial versions of the Axiom of Choice in such a way that those statements are, by definition, equivalent in the category **Set**. However, it turns out that for many known and classical categories, the presented categorial set–versions of the Axiom of Choice *are not equivalent* — i.e., such statements, which are equivalent in **Set**, need not to be equivalent, in general, in a given category. Let us give some examples for this, and we begin with the following one.

Example 3.13
Consider the category **Top**, with objects the topological spaces and the morphisms, the continuous functions between them.

(a) It is well-known that epimorphisms in **Top** coincide with continuous surjections and that **Top** has products (they are given by the Tychonoff product). But, in **Top**, it is not true that every epic has a (continuous) section. Consider the following example: Let $f:\mathbb{R}\to S^1, t\mapsto f(t):=e^{2\pi it}$, where $S^1:=\{z\in\mathbb{C}\,|\,|z|=1\}$ is the circle with radius 1 in the complex plane. Observe that f is epic, but by a direct topological argument (or by an application of the fundamental group functor) it is easy to see that f does not have a (global) section.
(b) Take a look at the other axioms. With the same arguments as in the category **Set** we can show that in **Top**, **AC** and the **NUC** and **NSC** versions are equivalent. Also, since projections are continuous and surjective within **ZFC**, the category **Top** satisfies **CEM**. But **Top** but does not satisfy **EGC**, since **EGC** is equivalent to **AC**.

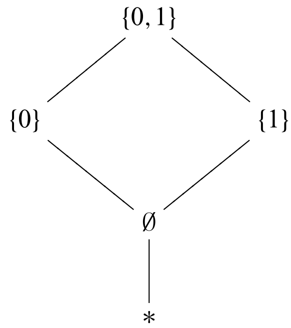
Here we consider another source of examples - categories from posets.

Example 3.14
Let $\mathcal{X}$ denote the category associated to the partially ordered set $(X,\leq)$.

(a) If $X\neq\emptyset$, at least the identity-arrows exist; by the fact that the only morphisms from objects, x to y exists iff $x\leq y$, it is clear that any $\mathcal{X}$ *does not* satisfy the axiom, that every epic has a section. By the same reason, $\mathcal{X}$ *does not* satisfy the Strong Axiom of Choice, mentioned in the introduction.
(b) Consider now a set X partially ordered by the *equality relation*. Thus the poset category $\mathcal{X}$ is *discrete* (there are no arrows other than the identity arrows) and satisfies our above mentioned versions of the Axiom of Choice.
(c) Suppose $\mathcal{X}$ be a poset category with products — this happens, for instance, when $(X,\leq)$ is a *complete* lattice — then it is easy to see that the set-form given in terms of the product, the form **PNI**, *does not* hold always. The product is given by the *infimum*, and clearly it is possible to have $\bigwedge_{i\in I} a_i=\bot$ and $a_i\neq\bot$ for all $i\in I$.
(d) Take a look now of the versions **NUC** introduced in 2.4.2. We will give two examples, in one the versions **NUC** are valid and in the other not. Let us define $X:=\mathcal{P}(A)$, with $A:=\{0,1\}$ and $\leq:=\subseteq$. Consider first the lattice $(X,\leq)$, seen as a category. It is easy to see that the diagram

$D := \{\{0\},\{1\}; id_{\{0\}}, id_{\{1\}}\}$ has a unique cone given by $C := \{\emptyset; \emptyset \leq \{0\}, \emptyset \leq \{1\}\}$. Because the diagram does not contain initial objects and there is only one cone for this diagram, this category *does not* satisfy $\mathbf{NUC_1}$ and clearly not $\mathbf{NUC_2}$. In general, if a poset has a nontrivial antichain, then its associated category does not satisfy the versions **NUC**.

But modifying a little bit this category, we will see that we have a category satisfying both of the **NUC**-versions. Define $Y := \mathcal{P}(A) \cup \{*\}$, with $A := \{0,1\}$ and for the new element $* \leq t$, for *all* elements of Y. So we have the following diagram for $(Y, \leq)$:



Considering the diagram D from above, we have now two cones for D, the one, C, mentioned above and the other is given by $C_* := \{*; * \leq \{0\}, * \leq \{1\}\}$. So, the category $\mathcal{Y}$ *does* satisfy both versions of **NUC**. Note that these two cones are *not* isomorphic.

(e) Considering the **NSC**-versions of the Axiom of Choice in a poset $(X, \leq)$ considered as a category $\mathcal{P}$, we easily see that every cone herein is strongly skeletal. Thus the **NSC**-versions *do not* hold.

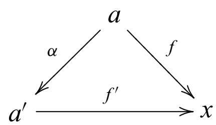
(f) What about the **CEM**-version? Notice that in a poset-category $\mathcal{X}$, initial elements do not exist always and there can exist discrete diagrams *without* morphisms (other than the identities) targeted to them, and so it is not always assured that every discrete diagram has a cone. On the other hand, as every morphism in $\mathcal{X}$ is epic, we have that

$$\mathcal{X} \text{ satisfies } \mathbf{CEM}$$
$$\Updownarrow$$
$$\text{every non-empty discrete diagram of non-initial objects has a cone.}$$

(g) Let us construct an example of a poset category such that **EGC** is valid and another such that **EGC** is not valid. Reminding that we have only an arrow from objects a to b in a poset category precisely when $a \leq b$, we consider the three element chain $X := \{0,1,2\}$, with the natural order $0 \leq 1 \leq 2$. So by $0 \leq 2$, we have an epic arrow which do not generate cones, if considering the morphism $1 \leq 2$, because there is no morphism from 1 to 0. So, in this poset category **EGC** does not hold. On the other hand, considering a non-empty set X partially ordered by the equality relation, it is easy to see that this (discrete) poset category satisfies **EGC** — because the only morphisms are the identities.

We consider now the slice category $\mathcal{C} \downarrow x$, for some object x in $\mathcal{C}$, which is a special case of a comma category, cf. [9]. Recall that the objects in this categories are given by morphisms $f, f' \in Ob(\mathcal{C} \downarrow x)$, i.e., $f : a \to x$ and $f' : a' \to x$ are morphism in the category $\mathcal{C}$. A morphism between these two objects,

$\alpha : f \to f'$ is given by a $\mathcal{C}$-arrow $\alpha : a \to a'$ such that the following diagram



commutes, i.e., $f' \circ \alpha = f$. Consider now the following

FACT 3.15

(a) If 0 is an initial object in $\mathcal{C}$, then the unique arrow $!_x : 0 \to x$ is an initial object in $\mathcal{C} \downarrow x$.
(b) An arrow $\alpha : f \to f'$ in $\mathcal{C} \downarrow x$ is an epimorphism iff $\alpha : a \to a'$ is an epimorphism in $\mathcal{C}$.
(c) Let $\alpha : f \to f'$ in $\mathcal{C} \downarrow x$ and $\beta : a' \to a$ in $\mathcal{C}$. Then $\alpha \circ \beta = id_{a'}$ in $\mathcal{C}$ iff $\beta : f' \to f$ in $\mathcal{C} \downarrow x$ and $\alpha \circ \beta = id_{f'}$ in $\mathcal{C} \downarrow x$.
(d) If $\mathcal{C}$ satisfies **AC** (and/or its equivalent **EGC**), then $\mathcal{C} \downarrow x$ satisfies **AC** (and/or **EGC**), for each object x.
(e) If $\mathcal{C}$ have terminal object 1, then the following statements are equivalent:
- $\mathcal{C}$ satisfies **AC**;
- for each object x in $\mathcal{C}$, $\mathcal{C} \downarrow x$ satisfies **AC**.

PROOF. The itens (a), (c) and ($\Leftarrow$) in (b) are straightforward. For ($\Rightarrow$) in (b), see for instance the (dual of) Lemma in page 121 of [15]. Item (d) is a direct consequence of itens (b) and (c). The equivalence in (e) follows from (d) and the (canonical) isomorphism of categories $Proj : (\mathcal{C} \downarrow 1) \xrightarrow{\cong} \mathcal{C}$. ∎

The preceding facts on slice categories will be very useful in the analysis of the upcoming Example 3.16 and Fact 3.17, which will show that a well-known, usual construction made over the category **Set** goes in the following way: in one hand, the validity of **AC** and **EGC** after the construction is equivalent to the validity of **AC** in **Set**, but in the other hand we will have after the construction, *in any case*, the failure of a considerable number of the presented set-forms of **AC**.

EXAMPLE 3.16
Let us consider the slice category $\mathbf{Set} \downarrow x$, for some object x in **Set**. Let us assume that in **Set**, the Axiom of Choice is valid thus, by the Fact 3.15.(d) that we have just proved, $\mathbf{Set} \downarrow x$ satisfies **AC** and **EGC**.

Now, if we are asking for the product of two morphisms f and f', then we must prove that there is an object in this category, such that this object satisfies the *universal* property of the product. It is not difficult to see that a product of two morphisms in $\mathbf{Set} \downarrow x$ is given by the *pullback* in the category **Set**, which we know that exists — it is the fibered product of a and a' over x, i.e. $(a \times_x a', p, p')$ where $a \times_x a' := \{(u, u') \in a \times a' : f(u) = f'(u')\}$ and p, p' are the projections. So, observe that, with this in mind, it is relatively easy to see, that in this category neither **PNI** nor **PFNI** holds; notice that the translation of **PFNI** in this category is the assertion, that the fibered product of a finite number of arrows $f_i : a_i \to x$, $i \leqslant n$, with *non-empty* domains a_i, is an induced function $f : a_1 \times_x \cdots \times_x a_n \to x$ having non-empty domain.

FACT 3.17
For each set x with at least two distinct elements, the category $\mathbf{Set} \downarrow x$ *does not* satisfy the axioms **PNI**, **PFNI**, $\mathbf{NUC_i}$ and $\mathbf{NSC_i}$ for $i = 1, 2$, and neither satisfies **CEM**.

PROOF. The proof is easy, and showing that **PFNI** does not hold clearly implies that **PNI** does not hold. For this, consider two applications $f : a \to x$ and $f' : a' \to x$, with $a, a' \neq \emptyset$ and $im(f) \cap im(f') = \emptyset$ (recall that we have assumed $|x| \geq 2$). Then the product of f and f' in $\mathbf{Set} \downarrow x$ is given by the unique function $a \times_x a' \to x$ that makes the diagram

$$\begin{array}{ccc} a \times_x a' & \xrightarrow{p'} & a' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f'} \\ a & \xrightarrow{f} & x \end{array}$$

a pullback.

But, as it is easily seen,

$$\begin{array}{ccc} \emptyset & \xrightarrow{!_a} & a' \\ \downarrow{\scriptstyle !_{a'}} & & \downarrow{\scriptstyle f'} \\ a & \xrightarrow{f} & x \end{array}$$

is a pullback — and, moreover, it is the *unique* cone over the diagram $(a \xrightarrow{f} x \xleftarrow{f'} a')$, seen as a discrete diagram in $\mathbf{Set} \downarrow x$ — and so **PFNI** does not hold. The mentioned uniqueness immediately shows also that $\mathbf{NUC_i}$, $\mathbf{NSC_i}$, $i = 1, 2$, doesn't hold. As $p = !_a : \emptyset \to a, p' = !_{a'} : \emptyset \to a'$, are not surjective then, by Fact 3.15.(c), **CEM** does not hold. ∎

REMARK 3.18

If we are asking whether the converse implications of Theorems 3.6 and 3.11 hold, we easily can have counterexamples using Example 3.14. In this example, we have constructed posets where $\mathbf{NUC_2}$ does not hold and simultaneously $\perp$ is strict initial and initial object. Thus the converse of Theorem 3.11 does *not* hold.

Also in the same Example 3.14, we have seen that a three element chain with 0 the initial object, which is nearly initial, does *not* satisfy **EGC**.

We finish this section with the following:

REMARK 3.19

(a) Many of the categorial properties considered in the previous sections are stable under each kind of construction of categories where limits and colimits are 'pointwise calculated' — for instance, arbitrary (small) produts of categories $\prod_{a \in A} \mathcal{C}_a$, and functor categories $\mathcal{C}^{\mathcal{A}}$, where $\mathcal{A}$ is small.

(b) $\mathbf{Set}^{\mathcal{A}}$ satisfies **AC**, $\mathbf{NUC_i}$ and $\mathbf{NSC_i}$ for $i = 1, 2$, **CEM** and **EGC**.
$\mathbf{Top}^{\mathcal{B}}$ satisfies $\mathbf{NUC_i}$ and $\mathbf{NSC_i}$ for $i = 1, 2$, and **CEM**.
$\mathbf{Set}^{\mathcal{A}} \times \mathbf{Top}^{\mathcal{B}}$ satisfies $\mathbf{NUC_i}$ and $\mathbf{NSC_i}$ for $i = 1, 2$, and **CEM**.

4 Final remarks

Our investigation will go on, pursuing new applications of the presented categorial versions of **AC** — and also pursuing new categorial versions of choice.

As already remarked, the authors were focused on set-forms of **AC** in this article. We intend to investigate class-forms, as well as applications, in a future work.

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