

## RESEARCH ARTICLE

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# Time-scale analysis nonlocal diffusion systems, applied to disease models

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The objective of the present paper is to use the well-known Ross–Macdonald models as a prototype, incorporating spatial movements, identifying different time scales and proving a singular perturbation result using a system of local and nonlocal diffusion. This results can be applied to the prototype model, where the vector has a fast dynamics, local in space, and the host has a slow dynamics, nonlocal in space.

**KEYWORDS**

epidemiology models, Neumann problem, nonlocal systems, reaction–diffusion equations

**MSC CLASSIFICATION**

35K57; 92B05

## 1 | INTRODUCTION

In this work, we are interested in the dynamics of vector-borne diseases. These dynamics, due to the interaction between hosts and vectors, behave quite different from direct diseases. In fact, they behave, as we will show, more likely to direct diseases with nonlinear incidence rates (see<sup>1</sup>). It has been a challenge for scientists and public health officers to predict outbreaks of such diseases and, in some countries, diseases like dengue and malaria are a leading cause of serious illness and death among children. Another point is that such illness, due to globalization, are spreading all over the world. For instance, dengue is currently the human viral disease with the highest number of cases, being an arbovirus of the family *Flaviviridae*, genus *Flavivirus*, is transmitted through the bite of female mosquitoes of the genus *Aedes* infected with the virus, which are also responsible for the transmission of Zika, Chikungunya, and Yellow Fever virus.<sup>2</sup>

Dengue is estimated to be endemic in more than 100 countries, where climate favors the proliferation of vectors and approximately half of the world's population is at risk of contracting the disease.<sup>3–6</sup> Due to lack of vaccination, basic sanitation, climate changes, and with increasing human mobility, such diseases are spreading and appearing in new regions. The host population can become infected in environments that are not their places of residence, and because mosquitoes do not travel great distances, humans can carry the disease to different locations where there are susceptible mosquitoes, and this may also lead to increased population heterogeneity and consequently in changes of the disease dynamics.<sup>7–9</sup> Thus, it is of interest that we have reliable models that can predict the spread of outbreaks trying to incorporate space heterogeneity, human, and vector dynamics.

For dengue, the current control measures are the control of the vector population and its breeding sites, with the use of insecticides, adulticides, and population awareness campaigns. More recently, it also been successfully tested control measures with *Wolbachia* bacteria, which prevents the vector from transmitting.<sup>10,11</sup> Some vaccines have been tested, and others are in the testing phase, but the great difficulty is that such vaccines should be tetravalent, that is, be effective against the four existing serotypes of the disease.<sup>12,13</sup>

Mathematical models were improved to analyze the dynamics of dengue propagation and to evaluate the best strategies of control. Studies with spatial networks, or meta-populations, provide a way to understand the interactions between individuals in different scales, being a powerful tool to understand the characteristics of transmission in communities, regions, and countries incorporating spatial heterogeneity.<sup>7,14,15</sup> In addition, as in modeling the dynamics of several vector-borne diseases, if the goal is to fit the model to real data, one has to deal with the asymptomatic cases, reliable data, in particular for the mosquitoes population, besides having to take into account the different time scale of vectors and hosts, which makes it difficult to study and understand the dynamics of the disease.<sup>16,17</sup>

Our approach will be deterministic, we will not take into account stochastic effects or incorporate the element of chance in the models. The prototype model here is continuous in the space domain, but a lot of work has been done considering discrete networks in space, which will provide system of ordinary differential equations (ODEs). There are advantages and disadvantages to both approaches. From the mathematical point of view, there are several theoretical challenges in the continuous model.

One of the main questions that public health officers, and thus modeling in epidemiology, concern is the global stability of equilibria, because this characterizes if a disease will become endemic. This will, from the epidemiology side, be characterized by the basic reproduction number,  $R_0$ , being larger or smaller than one. On the other hand, from the mathematical point of view, this is characterized by the existence of a stable equilibrium point, with positive number (or density) of infected individuals. This characterization is crucial for predicting outbreaks. In practice, once we propose a model to predict the outbreak, one has to fit the parameters for a specific disease, this is a real challenge when vectors are involved because it is very difficult to estimate its population, a crucial parameter in most problems.<sup>17</sup>

We will propose a host-vector disease compartmental model that will include space and time dynamics, thus capturing the heterogeneity of hosts and vectors in space. This is quite a challenge because hosts dynamics, in medium scales, are very difficult to model. The purpose is to follow the ideas of multiscale dynamics, in order to simplify the system modeling the vectors and hosts dynamics. We will consider that vector population dynamics is much faster than the hosts dynamics. This model can be applied, for instance, for dengue (see<sup>18</sup>).

There are several results dealing with singular perturbation using several different diffusion operators, one can refer to other studies.<sup>19–21</sup> But here, we will couple a nonlocal operator to a local one. We will consider a vector-borne disease modeled by the Ross–Macdonald model incorporating spatial movements, both for host population and for vector population. We consider that hosts can move nonlocally and vectors can move locally, this will lead us to the following prototype that mix different infinite dimensional operators in the same system.

## 2 | SETTING THE MODEL

Being more precise, the model we employed to describe the dynamics of the disease transmission considers that the total host population ( $N_h$ ) is divided in susceptible ( $S$ ) and infected ( $I$ ), and it is coupled with the compartments of susceptible  $S_m$  and infected  $I_m$  vectors with total population given by  $N_m$ ; the model is named  $SIS_mI_m$ . Thus, we describe the interaction dynamics between the compartments through a system of ODEs:

$$\begin{aligned} dS/dt &= \mu_h(N_h - S) - \beta SI_m/N_m \\ dI/dt &= \beta SI_m/N_m - (\gamma + \mu_h)I \\ dS_m/dt &= \mu_m(N_m - S_m) - \omega S_m I/N_h \\ dI_m/dt &= \omega S_m I/N_h - \mu_m I_m, \end{aligned}$$

where  $\mu_h$  is the birth/mortality rate of hosts,  $\beta$  and  $\omega$  are the transmission rates from vectors to hosts and from hosts to vectors, respectively,  $\gamma$  is the recovery rate of hosts, and  $\mu_m$  is the vector birth/mortality rate.

Assuming that birth and mortality rates are equal, we have that populations remain constant over time, that is,  $N_h(t) = S(t) + I(t)$  and  $N_m(t) = S_m(t) + I_m(t)$ , consequently, we can easily obtain  $S(t) = N_h(t) - I(t)$  and  $S_m(t) = N_m(t) - I_m(t)$  and then work with an equivalent reduced system:

$$\begin{aligned} dI/dt &= \beta(N_h - I)I_m/N_m - (\gamma + \mu_h)I \\ dI_m/dt &= \omega(N_m - I_m)I/N_h - \mu_m I_m. \end{aligned}$$

We also consider that the host and vector dynamics are in different scales given by the order of the birth/mortality and transmission rates. Hence, to describe this time-scale separation, we add the singular term  $1/\varepsilon$ , which leads us to the following system:

$$\begin{aligned} dI/dt &= \beta(N_h - I)I_m/N_m - (\gamma + \mu_h)I \\ dI_m/dt &= \frac{1}{\varepsilon} (\bar{\omega}(N_m - I_m)I/N_h - \bar{\mu}_m I_m). \end{aligned}$$

Moreover, letting  $i = I/N_h$ ,  $j = I_m/N_m$ ,  $\alpha_h = \beta$ ,  $\beta_h = \gamma + \mu_h$ ,  $\alpha_v = \bar{\omega}$ ,  $\beta_v = \bar{\mu}_m$  and rewriting the parameters, we get

$$\begin{aligned} di/dt &= \alpha_h(1 - i)j - \beta_h i \\ dj/dt &= \frac{\alpha_v}{\varepsilon}(1 - j)i - \frac{\beta_v}{\varepsilon} j. \end{aligned} \quad (1)$$

In this way, see for instance Rocha et al.,<sup>16</sup> we set a system where the vector population dynamics is much faster than the hosts one as  $\varepsilon \approx 0$ .

As for the spatial mobility, we will consider a regular bounded space domain  $\Omega \subset \mathbb{R}^n$  with exterior unit normal  $\vec{n}$ . The spatial movement for the vector will be modeled by the usual Laplacian operator with Neumann boundary condition  $(\Delta)$ , and the hosts spatial dynamics will be modeled by a nonlocal operator  $K_J$  defined as follows:

$$K_J i(x) = \int_{\Omega} J(x - y)(i(y) - i(x))dy, \quad x \in \Omega.$$

Along the whole paper, we assume that the kernel  $J$  satisfies the hypotheses

$$\begin{aligned} (H_J) \quad & J \in C(\mathbb{R}^n, \mathbb{R}) \text{ is nonnegative with } J(0) > 0, \\ & J(-x) = J(x) \text{ for every } x \in \mathbb{R}^n \text{ and} \\ & \int_{\mathbb{R}^n} J(x) dx = 1. \end{aligned}$$

Under these conditions, the  $K_J$  is known as a nonlocal operator with nonsingular kernel and Neumann condition.<sup>22</sup> Putting the local disease dynamics (1) with the spatial dynamics, we get our main model with Neumann boundary condition and  $d_1, d_2 > 0$ ,

$$\begin{cases} \frac{\partial i}{\partial t} = \alpha_h(1 - i)j - \beta_h i + d_1 K_J i, \\ \frac{\partial j}{\partial t} = \frac{\alpha_v}{\varepsilon}(1 - j)i - \frac{\beta_v}{\varepsilon} j + d_2 \Delta j, \end{cases} \quad x \in \Omega, t > 0 \quad (2)$$

$$\frac{\partial j}{\partial \vec{n}} = 0, \quad x \in \partial\Omega. \quad (3)$$

The paper is organized as follows. In Section 3, we use asymptotic expansion approach to find a limit equation to (2). Indeed, we obtain a limit model, which represents the original system in an effective way as  $\varepsilon$  goes to zero. We also discuss some properties to the limit equation, such as, conditions to guarantee the existence of a positive stationary solution globally stable. Next, in Section 4, we consider a more general system, which includes our prototype model (2), showing convergence at  $\varepsilon = 0$ . Assuming appropriated assumptions, we prove convergence in  $L^2(\Omega)$  spaces in finite intervals of time. Finally, we make some comments about the dynamics of the prototype model as a consequence of our estimates in Section 5.

### 3 | ASYMPTOTIC EXPANSION

In this section, we use power series expansion to analyze in a formal way the asymptotic behavior of the singular perturbed system (2) with respect to parameter  $\varepsilon > 0$ . We assume that functions  $i$  and  $j$  satisfy

$$i = i_0 + \varepsilon i_1 + \dots \quad \text{and} \quad j = j_0 + \varepsilon j_1 + \dots$$

Thus, the time derivatives themselves set

$$\frac{di}{dt} = \frac{di_0}{dt} + \varepsilon \frac{di_1}{dt} + \dots \quad \text{and} \quad \frac{dj}{dt} = \frac{dj_0}{dt} + \varepsilon \frac{dj_1}{dt} + \dots ,$$

which gives us from the right-hand side of (2) that

$$\begin{aligned} \frac{di}{dt} &= [\alpha_h(1 - i_0)j_0 - \beta_h i_0 + d_1 K_J i_0] \\ &\quad + \varepsilon [\alpha_h(j_1 - i_0 j_1 - i_1 j_0) - \beta_h i_1 + d_1 K_J i_1] + O(\varepsilon^2) \\ \frac{dj}{dt} &= [\alpha_v(1 - j_0)i_0 - \beta_v j_0] \\ &\quad + \varepsilon [\alpha_v(i_1 - j_0 i_1 - j_1 i_0) - \beta_v i_1 + d_2 \Delta j_0] + O(\varepsilon^2). \end{aligned}$$

Hence, if we plug these expressions in the system (2), we get at  $\varepsilon = 0$

$$\begin{cases} \frac{\partial i_0}{\partial t} = \alpha_h(1 - i_0)j_0 - \beta_h i_0 + d_1 K_J i_0, \\ 0 = \alpha_v(1 - j_0)i_0 - \beta_v j_0. \end{cases}$$

Consequently, we obtain

$$j_0 = m(i_0) = \frac{\alpha_v i_0}{\alpha_v i_0 + \beta_v}$$

and then, we deduce the reduced equation

$$\frac{\partial i_0}{\partial t} = \alpha_h(1 - i_0)m(i_0) - \beta_h i_0 + d_1 K_J i_0 \quad (4)$$

with initial condition  $i_0(t, x) = i_0(0, x)$ .

It will be seen in Section 4 that the solutions  $i$  of (2) can be approximated to the functions  $i_0$  given by (4). Indeed, at  $\varepsilon = 0$ , we have

$$i \approx i_0$$

under appropriated functional spaces, initial conditions, and finite time.

### 3.1 | The limit problem

Let us now discuss a little the limit Equation (4). First, we notice that Hutson et al.<sup>23</sup>, Theorem 3.2 implies that the strong maximum principle works to (4) in the space of nonnegative continuous functions in  $\bar{\Omega}$ , which we denote here by  $C(\Omega)$ . Hence, because  $K_J$  is zero in any constant function, we have that the nonlocal Equation (4) defines a dynamical system, which behaves as the solutions of the ordinary differential equation

$$\frac{dz}{dt} = \alpha_h(1 - z)m(z) - \beta_h z. \quad (5)$$

Indeed, problem (4) possesses two constant equilibria, the null constant and

$$i_0^* = \frac{\alpha_h \alpha_v - \beta_h \beta_v}{\alpha_h + \alpha_v \beta_h}.$$

Thus, under the additional condition

$$(\mathbf{H}_C) \quad \alpha_h \alpha_v > \beta_h \beta_v,$$

we conclude that the null function is an unstable equilibrium to (4) and  $i_0^*$  is globally stable. In fact, we have the following result:

**Proposition 3.1.** *Let us assume condition  $(\mathbf{H}_J)$ .*

- a) Suppose that  $i_0(0, x)$  is continuous and nonnegative. Then, (4) possesses a continuous, nonnegative solution for all  $t > 0$ .
- b) Assume still condition  $(H_C)$ . Then, the positive constant  $i_0^*$  is the unique stationary and positive solution to (4), which is globally stable in  $L^\infty(\Omega)$  for any solution with nontrivial and nonnegative initial condition in  $C(\Omega)$ .

*Proof.* The existence and uniqueness of the globally stable equilibrium is a consequence of the maximum principle arguments shown at Hutson et al.<sup>23</sup>, Theorem 3.2 being minor amendments of the proofs of the classical theory discussed in Fife and Pao.<sup>24,25</sup> See also Hutson<sup>23</sup>, Theorem 3.6 and Henry.<sup>20</sup>, Exercise 8 On the other hand, the convergence in  $L^\infty(\Omega)$  of the solutions follows from comparison with the solutions of the ODE (5).  $\square$

*Remark 1.* It is not difficult to see that if  $\alpha_h \alpha_v \leq \beta_h \beta_v$ , then the null function is the unique stationary and nonnegative solution of (4), which is globally stable for any solution with nonnegative initial condition.

## 4 | CONVERGENCE RESULTS

In this section, we estimate the convergence of the solutions in a more general framework. We analyze the following singularly perturbed system:

$$\begin{cases} \dot{x} = f(x, y) + K_J x \\ \varepsilon \dot{y} = g(x, y) + \varepsilon \Delta y \end{cases} \quad \text{in } \Omega, \quad \varepsilon > 0, \quad (6)$$

with homogeneous Neumann boundary condition

$$\frac{\partial y}{\partial N} = 0 \quad \text{on } \partial\Omega. \quad (7)$$

As before, we suppose  $\Omega \subset \mathbb{R}^n$  is a regular bounded domain,  $\Delta$  is the Laplacian differential operator, and  $K_J$  is the nonlocal one

$$K_J x(u) = \int_{\Omega} J(u - v)(x(v) - x(u)) dv, \quad u \in \Omega.$$

The nonlinearities  $f$  and  $g : \mathbb{R}^2 \mapsto \mathbb{R}$  are smooth functions and will include the class of those ones discussed in the previous sections.

We show that in the limit  $\varepsilon \rightarrow 0$ , the slow component  $x(t)$  converges to a function  $X(t)$ , which is governed by the effective equation

$$\dot{X} = f(X, m(X)) + K_J X, \quad \text{in } \Omega, \quad (8)$$

where  $y = m(x)$  is the graph representation of a set given by

$$g(x, m(x)) = 0. \quad (9)$$

Under  $f$  and  $g$ , we set the following conditions:

$(H_{fg})$  There exist positive constants  $M, N, \gamma, \alpha, \beta$  and  $\delta$  such that for all  $x$  and  $y$  satisfying  $0 \leq x \leq N$  and  $0 \leq y \leq M$ , we have

(i)  $|f(x, y)|$  and  $|g(x, y)|$  uniformly bounded for a constant  $k > 0$ ;

(ii)

$$\frac{\partial f}{\partial x}(x, y) \leq -\alpha \quad \text{and} \quad 0 \leq \frac{\partial f}{\partial y}(x, y) \leq \beta N;$$

(iii)

$$\frac{\partial g}{\partial y}(x, y) \leq -\delta \quad \text{and} \quad 0 \leq \frac{\partial g}{\partial x}(x, y) \leq \gamma M.$$

$(H_\infty)$  Next, we assume the nonlinearities  $f$  and  $g$  are such that

(i) there exists a constant  $\hat{\rho} \in \mathbb{R}$  in such way that  $(f(x, y), g(x, y)) + \hat{\rho}(x, y)$  is an increasing function;

(ii)  $(f(x, y), \varepsilon^{-1}g(x, y)) \cdot (x, y) < 0$  wherever  $(x, y)$  does not belong to the rectangle

$$\mathcal{R} = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq N \quad \text{and} \quad 0 \leq v \leq M\}.$$

Finally, we suppose function  $m$  given by (9) satisfies

$$(\mathbf{H}_m) \quad 0 \leq m(x) \leq \frac{\gamma}{\delta} MN \quad \text{and} \quad 0 \leq m'(x) \leq \frac{\gamma M}{\delta}$$

whenever  $0 \leq x \leq N$ .

*Remark 2.* Due to condition  $(\mathbf{H}_\infty)$ , it follows from maximum principle arguments, discussed for instance in Andreu-Vaillio et al.<sup>22</sup>, Lemma 3.11 and Henry<sup>20</sup>, Exercise 8, the global existence and uniformly boundedness of solutions to (6) in  $L^\infty(\Omega)$  for all  $\epsilon > 0$  wherever the initial condition  $(\|x_0\|_{L^\infty}, \|y_0\|_{L^\infty}) \in \mathcal{R}$ . Indeed, we can use comparison to show that problem (6) defines a dynamical system as the ODE

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = \epsilon^{-1}g(x, y) \end{cases}$$

because their solutions also satisfy the boundary condition (7).

*Remark 3.* We notice that nonlinearities  $f(x, y) = \beta(N - x)y - \alpha x$  and  $g(x, y) = \gamma(M - y)x - \delta y$  with  $M, N, \alpha, \beta, \gamma$ , and  $\delta$  positive constants satisfy conditions  $(\mathbf{H}_{fg})$ ,  $(\mathbf{H}_\infty)$ , and  $(\mathbf{H}_m)$ . In this way, (6) can be seen as a generalization of problem (2).

#### 4.1 | Estimates and convergence

Let  $(x, y)$  be the solution of (2), and let us introduce the deviation of the fast variable from the invariant set

$$y = m(x) + \eta.$$

First, we estimate the rate at which  $\eta$  goes to zero. Because

$$\dot{\eta} = \dot{y} - m'(x)\dot{x},$$

we can rewrite system (2) as

$$\begin{cases} \dot{x} = f(x, m(x) + \eta) + K_J x \\ \dot{\eta} = \epsilon^{-1}g(x, m(x) + \eta) + \Delta(m(x) + \eta) \\ \quad - m'(x)(f(x, m(x) + \eta) + K_J x) \\ \dot{y} = \epsilon^{-1}g(x, m(x) + \eta) + \Delta(m(x) + \eta). \end{cases} \quad (10)$$

Thus,

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} \eta^2 du \right) &= \int_{\Omega} \eta \dot{\eta} du \\ &= \int_{\Omega} \frac{\eta}{\epsilon} (g(x, m(x) + \eta) - g(x, m(x))) du \\ &\quad + \int_{\Omega} \eta \Delta(m(x) + \eta) du - \int_{\Omega} \eta m'(x) (f(x, m(x) + \eta) + K_J x) du \\ &= I_1 + I_2 - I_3. \end{aligned}$$

Now, let us evaluate each one of the integrals  $I_i$  for  $i = 1, 2, 3$ . Because  $g$  is smooth and satisfies (iii) at  $(\mathbf{H}_{fg})$ , there exists  $b = b(u)$  such that

$$I_1 = \int_{\Omega} \frac{\eta^2}{\epsilon} \frac{\partial g}{\partial y}(x, b) du \leq -\frac{\delta}{\epsilon} \int_{\Omega} \eta^2 du \leq -\frac{\delta}{\epsilon} \|\eta\|_{L^2(\Omega)}^2.$$

Next, integrating by parts, and using boundary condition (7) and hypotheses  $(\mathbf{H}_m)$ , we obtain by Young's inequality that

$$\begin{aligned}
 I_2 &= \int_{\Omega} \eta \Delta(m(x) + \eta) \, du \\
 &= - \int_{\Omega} |\nabla \eta|^2 \, du - \int_{\Omega} \nabla \eta \cdot \nabla(m(x)) \, du \\
 &\leq - \int_{\Omega} |\nabla \eta|^2 \left(1 - \frac{\xi^2}{2}\right) \, du + \frac{1}{2\xi^2} \int_{\Omega} |\nabla(m(x))|^2 \, du \\
 &\leq \frac{1}{2\xi^2} \int_{\Omega} |m'(x)|^2 |\nabla x|^2 \, du \\
 &\leq \frac{1}{2} \left(\frac{\gamma M}{\xi \delta}\right)^2 \|\nabla x\|_{L^2(\Omega)}^2,
 \end{aligned}$$

if  $\xi > \sqrt{2}$ .

Also, because  $f$  is a smooth function and satisfies (ii) at  $(\mathbf{H}_{fg})$ , we have that there exists  $b$  with  $m(x) \leq b \leq m(x) + \eta$ , without loss of generality, such that

$$\begin{aligned}
 I_3 &= \int_{\Omega} \eta m'(x) [f(x, m(x) + \eta) - f(x, m(x)) + f(x, m(x)) + K_J x] \, du \\
 &= \int_{\Omega} \eta m'(x) \left[ \frac{\partial f}{\partial y}(x, b) \eta + f(x, m(x)) + K_J x \right] \, du \\
 &\geq \int_{\Omega} \eta m'(x) [f(x, m(x)) + K_J x] \, du.
 \end{aligned}$$

Hence, because  $|f(x, y)| \leq K$  for all  $0 \leq x \leq N$  and  $0 \leq y \leq M$ , and

$$\begin{aligned}
 \left| \int_{\Omega} y K_J x \, du \right| &= \left| \int_{\Omega} y(u) \int_{\Omega} J(u-v)(x(v) - x(u)) \, dv \, du \right| \\
 &\leq \left| \int_{\Omega} y(u) \int_{\Omega} J(u-v) x(v) \, dv \, du \right| + \left| \int_{\Omega} y(u) x(u) \int_{\Omega} J(u-v) \, dv \, du \right| \\
 &\leq \|x\|_{L^2(\Omega)} \|y\|_{L^2(\Omega)} (|\Omega| \|J\|_{\infty} + 1),
 \end{aligned}$$

we obtain from  $(\mathbf{H}_m)$

$$\begin{aligned}
 -I_3 &\leq \frac{\gamma M}{\delta} \int_{\Omega} |\eta| [|f(x, m(x))| + |K_J x|] \, du \\
 &\leq \frac{\gamma M}{\delta} \|\eta\|_{L^2(\Omega)} [k|\Omega|^{1/2} + \|x\|_{L^2(\Omega)} (|\Omega| \|J\|_{\infty} + 1)].
 \end{aligned}$$

Thus, we get that

$$\begin{aligned}
 \frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} \eta^2 \, du \right) &= I_1 + I_2 - I_3 \\
 &\leq -\frac{\delta}{\epsilon} \|\eta\|_{L^2(\Omega)}^2 + \frac{1}{2} \left(\frac{\gamma M}{\xi \delta}\right)^2 \|\nabla x\|_{L^2(\Omega)}^2 \\
 &\quad + \frac{\gamma M}{\delta} \|\eta\|_{L^2(\Omega)} [k|\Omega|^{1/2} + \|x\|_{L^2(\Omega)} (|\Omega| \|J\|_{\infty} + 1)].
 \end{aligned}$$

Now, it follows from Lemma 6.1 that norms  $\|x\|_{L^2(\Omega)}$  and  $\|\nabla x\|_{L^2(\Omega)}$  are uniformly bounded in any bounded interval of time as  $[0, T]$ . Thus, there exist positive constants  $D_0$  and  $D_1$  such that

$$\frac{d}{dt} \left( \|\eta\|_{L^2(\Omega)}^2 \right) \leq -\frac{2\delta}{\epsilon} \|\eta\|_{L^2(\Omega)}^2 + D_0 \|\eta\|_{L^2(\Omega)} + D_1 \quad \forall t \in [0, T].$$

Consequently, by Young's inequality, we get

$$\frac{d}{dt} \left( \|\eta\|_{L^2(\Omega)}^2 \right) \leq -\frac{\delta}{\varepsilon} \|\eta\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2\delta} D_0^2 + D_1.$$

Hence, if we integrate this inequality in  $[0, t]$ , we can conclude that

$$\|\eta\|_{L^2(\Omega)}^2 \leq \frac{\varepsilon}{\delta} \left( \frac{\varepsilon}{2\delta} D_0^2 + D_1 \right) (1 - e^{-t\delta/\varepsilon}) + e^{-t\delta/\varepsilon} \|\eta(0)\|_{L^2(\Omega)}^2 \quad (11)$$

for all  $t \in [0, T]$ .

Let us estimate now the convergence of the functions  $x$  given by the system (2) and (10) to the solution  $X$  of the limit Equation (8) as  $\varepsilon$  goes to zero. We proceed as before, considering

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} |x - X|^2 du \right) &= \int_{\Omega} (x - X)(x - X)' du \\ &= \int_{\Omega} (x - X)(f(x, m(x) + \eta) - f(X, m(X))) du \\ &\quad + \int_{\Omega} (x - X)K_J(x - X) du. \\ &= I_1 + I_2. \end{aligned}$$

First, we evaluate  $I_1$  by

$$\begin{aligned} I_1 &= \int_{\Omega} (x - X)(f(x, m(x) + \eta) - f(X, m(X))) du \\ &= \int_{\Omega} (x - X)(f(x, m(x) + \eta) - f(x, m(x)) + f(x, m(x)) - f(X, m(x)) \\ &\quad + f(X, m(x)) - f(X, m(X))) du \\ &= \int_{\Omega} (x - X) \left( \frac{\partial f}{\partial y}(x, b_1)\eta + \frac{\partial f}{\partial x}(a_1, m(x))(x - X) + \frac{\partial f}{\partial y}(X, b_2)(m(x) - m(X)) \right) du \\ &= \int_{\Omega} \frac{\partial f}{\partial y}(x, b_1)(x - X)\eta du + \int_{\Omega} \left( \frac{\partial f}{\partial x}(a_1, m(x)) + \frac{\partial f}{\partial y}(X, b_2)m'(a_2) \right) (x - X)^2 du \end{aligned}$$

for some  $b_1 \in [m(x), m(x) + \eta]$ ,  $b_2 \in [m(x), m(X)]$  and  $a_1, a_2 \in [x, X]$  (without loss of generality).

Thus, by Young's Inequality and conditions  $(\mathbf{H}_{fg})$ , we have for an appropriated  $\rho > 0$  that

$$I_1 \leq \left[ \beta N \left( \frac{\gamma M}{\delta} + \rho^2 \right) - \alpha \right] \int_{\Omega} (x - X)^2 du + \frac{\beta N}{\rho^2} \int_{\Omega} \eta^2 du.$$

On the other hand, the nonlocal operator satisfies

$$\begin{aligned} I_2 &= \int_{\Omega} (x - X)K_J(x - X) du \\ &= \int_{\Omega} (x - X)(u) \int_{\Omega} J(u - v)((x - X)(v) - (x - X)(u)) dv du \\ &= - \int_{\Omega} \int_{\Omega} J(u - v)((x - X)(v) - (x - X)(u))^2 dv du \\ &\leq 0. \end{aligned}$$

Therefore, if we set the constants

$$C_1 = \beta N \left( \frac{\gamma M}{\delta} + \rho^2 \right) - \alpha \quad \text{and} \quad C_2 = \frac{\beta N}{\rho^2}, \quad (12)$$



we get

$$\frac{d}{dt} \left( \int_{\Omega} |x - X|^2 du \right) \leq 2C_1 \int_{\Omega} (x - X)^2 du + 2C_2 \int_{\Omega} \eta^2 du. \quad (13)$$

Consequently, we get after some integrations that

$$\|x - X\|_{L^2(\Omega)}^2 \leq e^{2C_1 t} \|(x - X)(0)\|_{L^2(\Omega)}^2 + 2C_2 \int_0^t e^{2C_1(t-s)} \|\eta\|_{L^2(\Omega)}^2 ds. \quad (14)$$

Thus, due to (11), we can conclude that

$$\begin{aligned} \|x - X\|_{L^2(\Omega)}^2 &\leq e^{2C_1 t} \|(x - X)(0)\|_{L^2(\Omega)}^2 \\ &\quad + 2\epsilon C_2 e^{2C_1 t} \left[ \left( \frac{\epsilon D_0^2 + 2\delta D_1}{2\delta^2} \right) \left( \frac{1 - e^{-2C_1 t}}{2C_1} \right) \right. \\ &\quad \left. + \frac{\|\eta(0)\|_{L^2(\Omega)}^2}{2C_1 \epsilon + \delta} \left( 1 - e^{-\frac{t}{\epsilon}(2C_1 \epsilon + \delta)} \right) \right]. \end{aligned}$$

Therefore, because we are taking  $\epsilon > 0$  small, we obtain

$$\begin{aligned} \|x - X\|_{L^2(\Omega)}^2 &\leq e^{2C_1 t} \|(x - X)(0)\|_{L^2(\Omega)}^2 \\ &\quad + 2\epsilon C_2 e^{2C_1 t} \left[ \left( \frac{\epsilon D_0^2 + 2\delta D_1}{4\delta^2 |C_1|} \right) + \frac{\|\eta(0)\|_{L^2(\Omega)}^2}{2C_1 \epsilon + \delta} \right]. \end{aligned} \quad (15)$$

As a consequence, we have the following results:

**Theorem 4.1.** *Let us suppose assumptions  $(H_{fg})$ ,  $(H_{\infty})$ ,  $(H_m)$ , and  $(H_J)$  with the additional condition  $J$  of class  $C^1$ .*

*Then, for any  $T > 0$  and initial condition  $(x_0, y_0) \in C^1(\Omega) \times H^1(\Omega)$  satisfying  $0 \leq \|x_0\|_{L^\infty(\Omega)} \leq N$  and  $0 \leq \|y_0\|_{L^\infty(\Omega)} \leq M$ , there exist positive constants  $\epsilon_0$ ,  $M_1$  and  $M_2$  such that solution  $x$  of (6) satisfies*

$$\|x - X\|_{L^2(\Omega)} \leq M_1 \|(x - X)(0)\|_{L^2(\Omega)} + \epsilon M_2$$

*for all  $t \in [0, T]$  and  $\epsilon \in (0, \epsilon_0)$  where  $X$  is the solution of the limit problem (8) with initial condition  $X(0)$ .*

*In particular, if  $x(0) = X(0)$ , we have*

$$\sup_{t \in [0, T]} \|x - X\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

*Proof.* It is a direct consequence of estimate (15). □

**Remark 4.** Because the constants  $D_0$  and  $D_1$  given by Lemma 6.1 depend on  $T$ , we can not guarantee convergence at  $(0, +\infty)$ .

**Corollary 4.1.** *Under the conditions of Theorem 4.1 and the additional assumption*

$$\frac{\alpha}{\beta N} - \frac{\gamma M}{\delta} > 0, \quad (16)$$

*there exist positive constants  $\epsilon_0$ ,  $m$ , and  $M_1$  such that*

$$\|x - X\|_{L^2(\Omega)}^2 \leq e^{-mt} \left( \|(x - X)(0)\|_{L^2(\Omega)}^2 + \epsilon M_2 \right) \quad (17)$$

*for all  $t \in [0, T]$  and  $\epsilon \in (0, \epsilon_0)$ .*

In particular, if  $x(0) = X(0)$ , we have

$$\sup_{t \in [0, T]} \|x - X\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

*Proof.* Due to the additional condition (16), we can choose  $\rho$  small enough in order to set  $C_1 < 0$  in expression (12). Hence, we obtain estimate (17) from inequality (15) concluding the proof.  $\square$

*Remark 5.* Notice that, due to Remark 1, the condition (16) implies that zero is the globally stable equilibrium for the nonnegative solutions of the limit Equation (8). Thus, we are just giving an order of decaying at Corollary 4.1.

## 5 | APPLICATION

Coming back to (2),(3) it is easy to check that Theorem 4.1 and Corollary 4.1 can be applied. So, we have uniform convergence in finite intervals of time. However, both systems, namely, (2),(3), and (4) have the same equilibria which are always constant in space, the disease-free ( $i = j = 0$ ) and a nonzero equilibria, which will be the endemic equilibrium (for  $R_0 > 1$ , the basic reproduction number). In this later case, the constant endemic equilibrium is locally stable, that can be check through linealization, similar to the ODE case.

Furthermore, because we have comparison, we always can compare the constant in space solutions (which will satisfy the ODE system) with the solutions of (2),(3). But the ODE system have an equivalent singular perturbation result, with the difference that the convergence can be done, globally in time. Therefore, we can extend this results for all times.

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## APPENDIX A:

Here, we show that the solutions  $(x, y)$  of (2) are uniformly bounded in  $H^1(\Omega)$  for any finite interval of time. Notice that the boundedness in  $L^\infty(\Omega)$  follows from Remark 2, and then,  $x$  and  $y$  are uniformly bounded in  $L^2(\Omega)$ . It remains us to estimate  $\nabla x$  and  $\nabla y$  in  $L^2(\Omega)$ .

In order to estimate  $\|\nabla y\|_{L^2(\Omega)}$ , we first perform the change of variable  $t = \epsilon \tau$  in (6) obtaining

$$\dot{w} = g(z, w) + \epsilon \Delta w \quad \text{in } \Omega \quad \text{with} \quad \frac{\partial w}{\partial N} = 0 \quad \text{on } \partial \Omega$$

where  $w(\tau) = y(\tau \epsilon)$  and  $z(\tau) = x(\tau \epsilon)$ .

Hence, if we define the norm  $\|w\|_s = \|(\epsilon \Delta + I)^s w\|_{L^2(\Omega)}$  for any  $s \geq 0$  and  $\epsilon > 0$ , we get from Henry<sup>20</sup>, Theorem 1.4.3 that

$$\|e^{(\epsilon \Delta - I)\tau}\|_s \leq M e^{-\tau} \tau^{-s} \quad \text{wherever } \tau > 0 \text{ and } 0 < s \leq 1,$$

because the first eigenvalue of  $\epsilon \Delta - I$  is equal to 1 for any  $\epsilon > 0$ . Thus, due to Henry's Theorem 3.3.6<sup>20</sup> and assumptions on nonlinearity  $g$ , we have for any  $0 < s < r \leq 1$  that  $\|w\|_r$  is uniformly bounded for  $\tau > 1$  and any  $\epsilon > 0$ . Therefore, we get that  $\|y(t)\|_r = \|w(t/\epsilon)\|_r$  is uniformly bounded wherever  $t > \epsilon$ . As we are taking  $\epsilon \rightarrow 0$ , we can conclude that  $\|y\|_{H^1(\Omega)}$  is uniformly bounded for any  $t > 1$  and  $\epsilon \in (0, 1)$ .

Now, let us estimate  $\|\nabla x\|_{L^2(\Omega)}$ . We use constant variation formula at the first equation of (2) getting

$$x(t) = e^{-At} x_0 + \int_0^t e^{-A(t-s)} \left( f(x, y) + \int_\Omega J(u - v)x(s) dv \right) ds \quad \text{in } \Omega,$$

with  $A(u) = \int_\Omega J(u - v) dv$  for  $u \in \Omega$ . Hence, under the conditions  $x_0$  and  $J$  of class  $C^1$ , we get

$$\begin{aligned} \partial_i x(t) &= e^{-At} (\partial_i x_0 - \partial_i A x_0) \\ &\quad - \partial_i A \int_0^t (t-s) e^{-A(t-s)} \left( f(x, y) + \int_\Omega J(u - v)x(s) dv \right) ds \\ &\quad + \int_0^t e^{-A(t-s)} \left( \nabla f \cdot (\partial_i x, \partial_i y) + \int_\Omega \partial_i J(u - v)x(s) dv \right) ds, \end{aligned}$$

where  $\partial_i$  denotes the  $i$ th partial derivative for  $i = 1, 2, \dots, n$ .

Thus, because  $m = \min_{u \in \Omega} A(u) > 0$  with  $x$  and  $y$  uniformly bounded in  $L^\infty(\Omega)$ , we obtain that there exist positive constants  $C_j$  such that

$$\begin{aligned} e^{2mt} |\partial_i x|^2 &\leq C_0 + C_1 t \int_0^t (t-s)^2 e^{2ms} ds + C_2 t \int_0^t e^{2ms} ds \\ &\quad + C_3 t \int_0^t e^{2ms} |\partial_i y|^2 ds + C_4 t \int_0^t e^{2ms} |\partial_i x|^2 ds. \end{aligned}$$

Hence, as  $\|y\|_{H^1(\Omega)}$  is uniformly bounded, we obtain for all  $t \geq 1$  that

$$\begin{aligned} e^{2mt} \|\partial_i x\|_{L^2(\Omega)}^2 &\leq \tilde{C}_0 + \tilde{C}_1 t \left[ \int_0^t (t-s)^2 e^{2ms} ds \right. \\ &\quad \left. + \int_0^t e^{2ms} ds + \int_0^t e^{2ms} \|\partial_i x\|_{L^2(\Omega)}^2 ds \right] \end{aligned}$$

for some positive constants  $\tilde{C}_0$  and  $\tilde{C}_1$ . From Gronwall inequality, we conclude  $\|\partial_i x\|_{L^2(\Omega)}^2$  is uniformly bounded in  $[0, T]$ , which leads us to the following result.

**Lemma 6.1.** *Under assumptions  $(H_{fg})$  and  $(H_\infty)$  with  $J$  of class  $C^1$  satisfying  $(H_J)$ , we have that, for any given  $T > 0$ , there exists  $M > 0$  such that the solutions  $(x, y)$  of (6) with initial conditions  $x_0 \in C^1$  and  $y_0 \in L^\infty(\Omega) \cap H^1(\Omega)$  satisfy*

$$\|j\|_{H^1(\Omega)} \leq M \quad \text{for all } t \in [0, T],$$

where  $j = x$  or  $y$ .