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THE w_{EF} TOPOLOGY FOR A BANACH SPACE E

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Abstract. Let E and F be Banach spaces. We present some results about the topology induced on E by the strong operator topology on the space $L(L(E,F),F)$ of continuous linear operators from $L(E,F)$ into F, and study several classes of homogeneous polynomials from E into F enjoying different types of w_{EF} continuity.

1. Introduction

Throughout this paper, unless when stated otherwise, E and F will be complex Banach spaces. As usual, $L(E,F)$ will denote the space of continuous operators from E into F. The three most commonly considered topologies on $L(E,F)$ are the uniform operator topology, the strong operator topology (SOT) and the weak operator topology (WOT). The uniform operator topology is the metric topology of $L(E,F)$ induced by its operator norm

$$\|T\| = \sup_{\|x\|\leq 1} \|T(x)\| \quad \text{for all } T \in L(E,F).$$

It is well known that $L(E,F)$ is a Banach space when endowed with the uniform operator topology. From now on, unless when said explicitly the contrary, we will denote by $L(E,F)$ the Banach space $(L(E,F), \|\cdot\|)$. If $F=\mathbb{C}$, we will write E' instead of $L(E,\mathbb{C})$ and E'' instead of $L(E',\mathbb{C})$. The strong operator topology of $L(E,F)$ is the locally convex topology defined by the

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family of semi-norms $\{p_x : x \in E\}$ where, for each $x \in E$, $p_x(T) := \|T(x)\|$ for all $T \in L(E,F)$ and the weak operator topology of $L(E,F)$ is the locally convex topology defined by the family of semi-norms $\{q_{x,y^*} : x \in E,\ y^* \in F'\}$ where, for each $(x,y^*) \in E \times F'$, $q_{x,y^*}(T) := |y^* \circ T(x)|$ for all $T \in L(E,F)$.

In particular, the space $L(L(E,F),F)$ of continuous operators from $L(E,F)$ into F will be denoted by G_{EF}. If $(T_\iota)_{\iota\in\Lambda} \subset G_{EF}$ and $T \in G_{EF}$ we have that $T_\iota \xrightarrow{\text{SOT}} T$ in G_{EF} (i.e. (T_ι) converges to T in (G_{EF}, SOT)) if $\|(T_\iota - T)(\varphi)\| \to 0$ for all $\varphi \in L(E,F)$ and $T_\iota \xrightarrow{\text{WOT}} T$ in G_{EF} (i.e. (T_ι) converges to T in (G_{EF}, WOT)) if $|y^* \circ (T_\iota - T)(\varphi)| \to 0$ for all $\varphi \in L(E,F)$ and for all $y^* \in F'$. As it is well known that $(L(E,F), \text{WOT})' = (L(E,F), \text{SOT})'$ for all Banach spaces E and F, we have $(G_{EF}, \text{WOT})' = (G_{EF}, \text{SOT})'$ and consequently the WOT-closure $\bar{A}^{\text{WOT}}$ of A coincides with the SOT-closure $\bar{A}^{\text{SOT}}$ of A whenever A is a convex subset of G_{EF}.

Note that G_{EF} is a Banach space when endowed with the usual operator norm. Throughout we will write just G_{EF} to denote the space G_{EF} endowed with the uniform operator topology and, as usual, we will write (G_{EF}, τ) to denote the vector space G_{EF} endowed with any other topology τ.

The space G_{EF} was considered by Zalduendo in [17]. He identified E with a closed subspace of G_{EF} via the isometric monomorphism $\alpha\colon E \to G_{EF}$ given by $\alpha(x)(\varphi) := \varphi(x)$ for all $x \in E$ and for all $\varphi \in L(E,F)$ and showed that every element $f \in H_b(E,F)$ has an extension $\tilde{f} \in H_b(G_{EF},F)$ (where, as usual, $H_b(E,F)$ denotes the space of holomorphic mappings from E into F which are bounded on bounded sets). Note that, in the case $F = \mathbb{C}$, we have $G_{EF} = E''$ and Zalduendo's extension coincides with that of Aron and Berner (see [5]). It is well known that $H_b(E,F)$ with the usual topology of uniform convergence on bounded subsets of E (denoted by τ_b) is a Fréchet algebra whenever F is a Banach algebra. We note that we assume that our Fréchet algebras (and, consequently, the Banach algebras) are always complex, unital and commutative. The spectrum of $H_b(E,F)$, that is, the set of non-zero continuous homomorphisms from $H_b(E,F)$ into $\mathbb{C}$, was studied in [6] for $F = \mathbb{C}$ and in [15] for uniform Banach algebras F. In both cases, the spectrum was characterised when the space of the weakly continuous n-homogeneous polynomials from E into F is dense in the space $P(^nE,F)$ of continuous n-homogeneous polynomials from E into F for very n. In this situation, the Aron Berner extension of the elements of $P(^nE,F)$ to E'' is multiplicative and the density in $P(^nE,F)$ of the space of the weakly continuous n-homogeneous polynomials from E into F plays an important role. Aron and Prolla showed in [9] that an n-homogeneous polynomial from E into F is of finite type if and only if it is weakly continuous. So, $P_f(^nE,F)$ stands for the space of finite type n-homogeneous polynomials from E into F or, equivalently, for the space of weakly continuous n-homogeneous polynomials from E into F. Note that the presentation of the definition of $P_f({}^nE,F)$ as the space of weakly continuous n-homogeneous polynomials suits better our goals in this paper.

In case $F=\mathbb{C}$, the relationship between the Arens regularity of E and the analytic structure of the spectrum of $H_b(E)$ has been studied in [7] but difficulties have been encountered in working with vector valued mappings. It seems more natural to consider the case when F is a Banach algebra and study the non-zero continuous homomorphisms from $H_b(E,F)$ into the Banach algebra F instead of the complex valued homomorphisms on $H_b(E,F)$. In this case Zalduendo's extension from elements of $H_b(E,F)$ to elements of $H_b(G_{EF},F)$ will probably play the same role as the Aron Berner extension in the study of the spectrum of $H_b(E,F)$. In [14], the set of non-zero continuous homomorphisms from $H_b(E,F)$ into F (where F is a Banach algebra) was studied. It was shown that, in general, the set G_{EF} does not coincide with the set of non-zero continuous homomorphisms from $H_b(E,F)$ into F and a subset $\mathcal{U}_{EF}$ of G_{EF} was constructed on which the Zalduendo's extension is multiplicative. As a consequence this set coincides with the set of non-zero continuous homomorphisms from $H_b(E,F)$ into F. The weak operator topology on G_{EF} played an important role in [14].

Since E is a closed subspace of the Banach space G_{EF} via the linear isometry defined by $\alpha(x)(\varphi):=\varphi(x)$ for all $x\in E$ and for all $\varphi\in L(E,F)$, we may consider the restriction to E of the strong operator topology on G_{EF} and the restriction to E of the weak operator topology on G_{EF}. It is easy to verify that the weak operator topology always induces on E the weak topology $\sigma(E,E')$. But the restriction to E of the strong operator topology on G_{EF} depends strongly on F. This topology, denoted by w_{EF}, was considered for the first time in [13], under the hypothesis that F is a Banach algebra, in connection with the study of the space

$$\mathbb{P}_f({}^nE,F)=\left\{\sum_{i=1}^{k}\varphi_i^n \text{ where } \varphi_1,\dots,\varphi_k\in L(E,F) \text{ and } k\in\mathbb{N}\right\}.$$

At some stage we will need to provide the bidual F'' of the Banach algebra F with a convenient product under which F'' has a Banach algebra structure. In 1951, A. Arens defined a natural product on the bidual $\mathcal{A}''$ of any Banach algebra $\mathcal{A}$ which extends the original product considered on $\mathcal{A}$ and turns $\mathcal{A}''$ into another Banach algebra (see [2–4] for the definition and basic results). Since its definition, the Arens' product has become a very useful tool in studies related to the bidual of a Banach algebra.

In [13] it was also showed that the Aron-Berner extension of w_{EF}-continuous polynomials is $w_{E''F''}$-continuous and also that it extends polynomials in $\mathbb{P}_f({}^nE,F)$ to polynomials in $\mathbb{P}_f({}^nE'',F'')$ where F'' is endowed with the Arens product.

In the second section of this paper we are going to study the topology w_{EF} on E, motivated by the fact that this topology has played an important role in the study of problems related to certain classes homogeneous polynomials from E into F. And in the third section we are going to study several classes of homogeneous polynomials from E into F enjoying different types of w_{EF}-continuity.

Throughout $K(E,F)$ and $L_{\mathrm{wsc}}(E,F)$ denote, respectively, the set of compact (linear) operators from E into F and the set of (linear) operators from E into F that take weakly convergent sequences into norm convergent sequences. It is well known that $\varphi \in K(E,F)$ if and only if the restriction of φ to each bounded subset of E is uniformly weakly continuous. Consequently, $K(E,F) \subset L_{\mathrm{wsc}}(E,F)$.

Given any $\varphi \in K(E,F)$, it is easy to verify that Goldstein's Theorem leads to the existence of a unique $\tilde{\varphi} \in L(E'',F)$ which is weak-star continuous on the bounded subsets of E'' and satisfies $\tilde{\varphi}|E = \varphi$. In particular, if $\varphi \in E'$ we have $\tilde{\varphi}(z) = z(\varphi)$ for all $z \in E''$. In order to be able to relate the spaces E'' and G_{EF}, we will be led to pay special attention to pairs of spaces E and F for which the equality $L(E,F) = K(E,F)$ holds.

2. Properties of the topology w_{EF}

Since E is a closed subspace of G_{EF} via the linear isometry defined by $\alpha(x)(\varphi) := \varphi(x)$ for all $x \in E$ and for all $\varphi \in L(E,F)$, we may consider on E the topology w_{EF} defined as the restriction to E of the strong operator topology on G_{EF}. Clearly given a net $(x_\iota)_{\iota\in\Lambda} \subset E$ and $x \in E$ we have that $x_\iota \xrightarrow{w_{EF}} x$ in E if $\|\varphi(x_\iota - x)\| \to 0$ for all $\varphi \in L(E,F)$.

As usual, the weak topology and the strong topology on E will be denoted, respectively, by $\sigma(E,E')$ and $\beta(E,E')$ and the weak star topology on E'' will be denoted by $\sigma(E'',E')$. For simplicity, we will often write w and w^* instead of $\sigma(E,E')$ and $\sigma(E'',E)$, respectively. As E is a Banach space, the strong topology on E coincides with the topology of the norm. We will indicate $(x_\iota)_{\iota\in\Lambda}$ converges to x in $(E,\sigma(E,E'))$ by $x_\iota \xrightarrow{w} x$ and $(x_\iota)_{\iota\in\Lambda}$ converges to x in $(E,\beta(E,E'))$ by $x_\iota \xrightarrow{\|\cdot\|} x$

It is easy to verify that $\sigma(E,E') \preceq w_{EF} \preceq \beta(E,E')$ for all Banach spaces F and consequently, w_{EF} is compatible with the dual pair (E,E'). It is natural to look for pairs of Banach spaces E and F such that w_{EF} coincides with some well-known topologies. Clearly we have $w_{EF} = \sigma(E,E')$ in the case when $F = \mathbb{C}$ and $w_{EF} = \beta(E,E')$ in the case when E is a subspace of F. But in general we may have $\sigma(E,E') \neq w_{EF} \neq \beta(E,E')$. Indeed, by [13, Proposition 2.5] we have that $w_{E\ell_1} \neq \beta(E,E')$ whenever E is a reflexive infinite-dimensional Banach space. Our next result will show that, in fact, $\sigma(E,E') \neq w_{EF}$ whenever E and F are both infinite-dimensional Banach spaces.

PROPOSITION 2.1. *Given any infinite-dimensional Banach space F, we have that w_{EF} coincides with the weak topology on E if and only if E is finite-dimensional.*

PROOF. First we claim that if E and F are infinite dimensional Banach spaces, then there exists $T \in L(E,F)$ with infinite dimensional range. Indeed, if E is an infinite dimensional Banach space, it is well known that E contains a normalized Schauder basic sequence $(x_n)_{n=1}^{\infty}$ that is, E contains a closed infinite dimensional subspace $X = \overline{[\{x_n : n \in \mathbb{N}\}]}$ with a Schauder basis $(x_n)_{n=1}^{\infty}$ such that $\|x_n\| = 1$ for every $n \in \mathbb{N}$. For each positive integer m, let x_m^* be the m^{th}-coordinate functional for $(x_n)_{n=1}^{\infty}$. In particular, $x_m^*(x_m) = 1$ for all $m \in \mathbb{N}$ and $x_m^*(x_n) = 0$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. It is known that there exists a positive constant M satisfying

$$1 \leq \|x_n\| \, \|x_n^*\| = \|x_n^*\| \leq M \quad \text{for all } n \in \mathbb{N}.$$

Note that, by the Hahn–Banach Theorem, we can associate to each x_m^* $(m \in \mathbb{N})$ a continuous linear extension of x_m^* to E whose norm coincides with the norm of x_m^*. For the sake of simplicity, let us denote this continuous linear extension by x_m^* as well. Now, take a linearly independent sequence $(b_n)_{n=1}^{\infty}$ in F with $\|b_n\| = 1$ for all $n \in \mathbb{N}$ and define $T \colon E \to F$ by

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n^*(x) \cdot b_n.$$

It is easy to check that $T \in L(E,F)$ and, since clearly $b_n = T(2^n x_n) \in T(E)$ for all $n \in \mathbb{N}$, we have that T has infinite dimensional range. This completes the proof of our claim.

Now, suppose that w_{EF} coincides with the weak topology on E (where F is an arbitrary infinite dimensional Banach space). By definition of w_{EF}, we have that the mapping $x \in E \mapsto \|A(x)\| \in [0,\infty)$ is a w_{EF}-continuous seminorm in E for each $A \in L(E,F)$. Consequently, fixed $A \in L(E,F)$ arbitrarily, our hypothesis leads to the existence of $\varphi_1, \dots, \varphi_k \in E'$ such that

$$\big\{ x \in E : |\varphi_j(x)| \leq 1 \text{ for all } j = 1, \dots, k \big\} \subset \big\{ x \in E : \|A(x)\| \leq 1 \big\}.$$

Thus, $\bigcap_{j=1}^{k} \ker(\varphi_j) \subset \ker(A)$ and consequently the range of A must be finite dimensional. Since A was chosen arbitrarily in $L(E,F)$, this means that every element of $L(E,F)$ has finite dimensional range. But, by the above proved claim, there exists $T \in L(E,F)$ with infinite dimensional range whenever E and F are infinite dimensional Banach spaces. Hence, E must be finite dimensional whenever w_{EF} coincides with the weak topology on E. The converse is trivial as all the locally convex topologies in a finite dimensional space coincide. □

Remark 2.2. Given any $\varphi \in K(E,F)$, let $\tilde{\varphi} \in L(E'',F)$ denote the unique linear extension of φ to E'' which is weak-star continuous on the bounded subsets of E''. Clearly we have $\|\tilde{\varphi}\| = \|\varphi\|$ for all $\varphi \in L(E,F)$ and, if F is a Banach algebra, $\widetilde{\varphi \cdot \psi} = \tilde{\varphi} \cdot \tilde{\psi}$ for all $\varphi, \psi \in L(E,F)$. Moreover, given any $\varphi \in E'$ and $b \in F$, the mapping $\varphi \otimes b \in L(E,F)$ defined by $(\varphi \otimes b)(x) = \varphi(x)b$ satisfies $\tilde{\varphi} \otimes b = \widetilde{\varphi \otimes b}$.

Proposition 2.3. *If $L(E,F) = K(E,F)$, the mapping $\tilde{\alpha}\colon E'' \to G_{EF}$ defined by $\tilde{\alpha}(z)(\varphi) = \tilde{\varphi}(z)$ for all $z \in E''$ and for all $\varphi \in L(E,F)$ is a continuous isomorphism between E'' and the subspace $\tilde{\alpha}(E'')$ of G_{EF} satisfying the equality $\|\tilde{\alpha}\| = \|\alpha\|$.*

Proof. It is clear that $\tilde{\alpha}$ is well defined and linear. Moreover, given $z_1, z_2 \in E''$ such that $z_1 \neq z_2$, there exists $\phi \in E'$ such that $z_1(\phi) \neq z_2(\phi)$ and consequently if we take $b \in F \setminus \{0\}$ we have that $\phi \otimes b \in L(E,F)$ satisfies $\widetilde{\phi \otimes b}(z_1) = \tilde{\phi}(z_1) \cdot b \neq \tilde{\phi}(z_2) \cdot b = \widetilde{\phi \otimes b}(z_2)$ and so $\tilde{\alpha}(z_1) \neq \tilde{\alpha}(z_2)$. Hence, $\tilde{\alpha}$ is injective. Finally, the fact that for every $z \in E''$ we have $\|\tilde{\alpha}(z)\| = \sup_{\|\varphi\| \leq 1} \|\tilde{\varphi}(z)\| \leq \|z\|$ leads to the continuity of $\tilde{\alpha}\colon E'' \to G_{EF}$ and to the inequality $\|\tilde{\alpha}\| \leq 1$. Moreover, it is well known that $\|\alpha\| = 1$ and, since clearly the $\tilde{\alpha}|_E = \alpha$, we have $1 = \|\alpha\| \leq \|\tilde{\alpha}\| \leq 1$ and consequently $\|\tilde{\alpha}\| = \|\alpha\|$. $\square$

Remark 2.4. From the continuity of $\tilde{\alpha}$ we infer that $\tilde{\alpha}(B)$ is bounded in G_{EF} whenever B is a bounded subset of E''.

From now on $\tilde{\alpha}$ will denote always the injective linear mapping defined by $\tilde{\alpha}(z)(\varphi) = \tilde{\varphi}(z)$ for all $z \in E''$ and for all $\varphi \in L(E,F)$.

Proposition 2.5. *If $L(E,F) = K(E,F)$, the mapping*

$$\tilde{\alpha}\colon E'' \to (G_{EF}, \mathrm{SOT})$$

is w^*-continuous on the bounded subsets of E''. Consequently, $\tilde{\alpha}(B)$ is SOT-*relatively compact whenever B is a bounded subset of E'' or, equivalently, $\tilde{\alpha}$ is compact.*

Proof. Fix arbitrarily a bounded subset B of E''. Take any $z \in B$ and any net $(z_\iota)_{\iota \in \Lambda} \subset B$ such that $z_\iota \xrightarrow{w^*} z$. Since, for every $\varphi \in L(E,F)$, $\tilde{\varphi}$ is w^*-continuous on the bounded subsets of E'', clearly $\tilde{\varphi}(z_\iota) \to \tilde{\varphi}(z)$ for every $\varphi \in L(E,F)$ and hence $\tilde{\alpha}(z_\iota) \xrightarrow{\mathrm{SOT}} \tilde{\alpha}(z)$. So, $\tilde{\alpha}\colon E'' \to (G_{EF}, \mathrm{SOT})$ is w^*-continuous on the bounded subsets of E''. The compactness of $\tilde{\alpha}$ follows immediately from this continuity since the unit ball of E'' is w^*-compact. $\square$

We are now interested in obtaining examples of spaces E and F such that w_{EF} is metrizable on the bounded subsets of E or, equivalently, (B_E, w_{EF}) is metrizable. Since w_{EF} is the norm topology if E is a subspace of F, clearly w_{EF} is metrizable in this case. Besides that, it is also well known that $(B_{E''}, w^*)$ and, consequently, $(B_E, w_{E\mathbb{C}})$ are metrizable whenever E' is separable. Let us give some other examples. The next auxiliary result will be useful in obtaining some of these examples.

PROPOSITION 2.6. *Let E and F be Banach spaces. If E' is separable and $L(E,F) = K(E,F)$, then (B, w_{EF}) is metrizable for all bounded subset B of E.*

PROOF. The hypothesis $L(E,F) = K(E,F)$ guarantees that for any bounded net (x_β) in E we have that (x_β) converges to x in (E,w) if, and only if, (x_β) converges to x in (E, w_{EF}) (see Proposition 2.4 of [13]). So (B_E, w_{EF}) is metrizable, since (B_E, w) is metrizable. $\square$

The above result leads us to the following examples.

EXAMPLE 2.7. By [1, Proposition 1.1], $L(E,F) = K(E,F)$ whenever E is an infinite dimensional subspace of c_0 and F does not contain a copy of c_0. Thus, as a result of Proposition 2.6, we have that (B, w_{EF}) is metrizable for all bounded subset B of E whenever E is an infinite dimensional subspace of c_0 and F does not contain a copy of c_0. In particular, this is the case when $E = c_0$ and $F = l_p$ for every $1 \le p < \infty$.

EXAMPLE 2.8. If $E = l_2$ and $F = l_1$ we have that $L(E,F) = K(E,F)$. Since E' is separable, as a result of Proposition 2.6 we have that (B, w_{l_2,l_1}) is metrizable for all bounded subsets B of l_2.

The next result will be used and, since we could not find it proved in the literature, we decided to include the proof here for the sake of completeness.

PROPOSITION 2.9. *Let E and F be Banach spaces. Then the following are equivalent:*

1. *E is separable.*
2. *SOT is metrizable on bounded subsets of $L(E,F)$.*

PROOF. (1) ⇒ (2). It is enough to show that the unit ball $B = \{\varphi \in L(E,F) : \|\varphi\| \le 1\}$ is SOT-metrizable. Let $(x_n)_{n=1}^\infty$ be any countable subset of B_E dense in B_E. Let

$$\mathrm{d}(\varphi,\psi) := \sum_{n=0}^{\infty} \frac{1}{2^n} \|\varphi(x_n) - \psi(x_n)\|$$

for all $\varphi, \psi \in B$. It is easy to check that d is a metric on B. (If $\mathrm{d}(\varphi,\psi) = 0$, from the density of $(x_n)_{n=1}^\infty$ in B_E we get that $\varphi|_{B_E} = \psi|_{B_E}$ and so $\phi = \psi$.)

We are going to show that the topology τ_{d} defined on $L(E,F)$ by d in B coincides with the topology induced by SOT on B. Given $\varphi \in B$ and $r > 0$, take any $0 < r' < \frac{r}{4}$ and $m \in \mathbb{N}^*$ so that $\frac{1}{2^{m-1}} \le \frac{r}{2}$ and consider the set

$$U = \left\{ \psi \in B : \|\psi(x_i) - \varphi(x_i)\| \le r' \text{ for all } i = 1, \ldots, m \right\}.$$

Clearly U is a SOT-neighborhood of φ and the inclusion $U \subset \{\psi \in B : \mathrm{d}(\varphi, \psi) \leq \mathrm{r}\}$ follows from

$$\mathrm{d}(\varphi, \psi) \leq \mathrm{r}' \sum_{\mathrm{n}=0}^{\mathrm{m}} \frac{1}{2^{\mathrm{n}}} + \sum_{\mathrm{n}=\mathrm{m}+1}^{\infty} \frac{1}{2^{\mathrm{n}}} \|\varphi(\mathrm{x_n}) - \psi(\mathrm{x_n})\|$$

$$\leq r' \sum_{n=0}^{m} \frac{1}{2^n} + 2 \sum_{n=m+1}^{\infty} \frac{1}{2^n} < \frac{r}{2} + \frac{1}{2^{m-1}} \leq r$$

for every $\psi \in U$. Now, consider the basic SOT- neighborhood of φ given by

$$V = \big\{ \psi \in B : \|\psi(z_i) - \varphi(z_i)\| \leq s \text{ for all } i = 1, \dots, k \big\},$$

where we may suppose $z_1, \dots, z_k \in B_E$ and $s > 0$. Clearly, for each $1 \leq i \leq k$ there exists n_i such that $\|z_i - x_{n_i}\| \leq \frac{s}{4}$. If we take $s' > 0$ such that $2^{1+n_i} s' \leq s$ for all $i = 1, \dots, k$, we get that $\{\psi \in B : \mathrm{d}(\varphi, \psi) \leq \mathrm{s}'\} \subset \mathrm{V}$ since

$$\|\psi(z_i) - \varphi(z_i)\| \leq \|\psi - \varphi\| \, \|z_i - x_{n_i}\| + 2^{n_i} s' \leq 2\frac{s}{4} + \frac{s}{2} = s$$

for all ψ satisfying $\mathrm{d}(\varphi, \psi) \leq \mathrm{s}'$. This completes the proof that the metric d induces in B the SOT-topology. For the converse, assume that $\mathrm{SOT}|_B$ is metrizable. Let d be a metric in B whose associated topology τ_{d} induces in B the SOT-topology. Given any integer $n \geq 1$, take the τ_{d}-neighborhood of zero $V_n = \{\psi \in B : \mathrm{d}(\psi, 0) \leq \frac{1}{\mathrm{n}}\}$. Since $\tau_{\mathrm{d}} = \mathrm{SOT}|_B$, for every n there are a finite subset D_n of E and $r_n > 0$ such that

$$U_n = \big\{ \psi \in B : \|\psi(x)\| \leq r_n \text{ for all } x \in D_n \big\} \subset V_n.$$

We are going to show that $\overline{N} = E$ where $N = \big[\bigcup_{n \geq 1} D_n \big]$ is the vector space generated by $D = \bigcup_{n \geq 1} D_n$. Indeed, if $\psi \in L(E, F)$ is such that $\psi|_N \equiv 0$, then $\psi \equiv 0$ since $\psi \in U_n$ for every $n \geq 1$ and so $\mathrm{d}(\psi, 0) < \frac{1}{\mathrm{n}}$ for all $n \geq 1$. Now, if $\varphi_0 \in E'$ satisfies $\varphi_0|_N \equiv 0$, it is enough to consider $\varphi = \varphi_0 \otimes b \in L(E, F)$ (where $b \in F \setminus \{0\}$), to obtain that $\varphi|_N \equiv 0$. So, $\varphi \equiv 0$ and consequently $\varphi_0 \equiv 0$. Thus, by the Hahn–Banach theorem we get $\overline{N} = E$. ☐

Corollary 2.10. *If E and F are Banach spaces such that $L(E, F)$ is separable, the bounded subsets of E are w_{EF}-metrizable.*

Proof. Since E is a closed subspace of $G_{EF} = L(L(E, F), F)$ via the isometric monomorphism α, every bounded subset of E can be considered as a bounded subset of G_{EF} and by the above proposition is SOT-metrizable. The result follows from $\mathrm{SOT}|_E = w_{EF}$. ☐

Remark 2.11. The converse of the above corollary is not true. For instance, if E is any non separable Banach space and F is any Banach space such that $E \subset F$, then the bounded subsets of E are w_{EF}-metrizable (since w_{EF} is the norm topology) but $L(E,F)$ is not separable (since $L(E,F)$ separable would imply E separable).

Definition 2.12. For each pair of Banach spaces E and F, we say that E is a w_{EF}-Schur space if given any $(x_k) \in E$ we have that $x_k \xrightarrow{\|\,\|} x$ whenever $x_k \xrightarrow{w_{EF}} x$.

Note that the definition of w_{EF}-Schur space coincides with the definition of Schur space in case $F = \mathbb{C}$. Moreover, E is a w_{EF}-Schur space for all F such that $E \subset F$. So l_2 is a w_{l_2,l_2}-Schur space, but it is not a Schur space.

Proposition 2.13. *The following statements are true.*

1. *If E is Schur then E is a w_{EF}-Schur space for every Banach space F.*
2. *If E is a w_{EF}-Schur space and F is a Schur space then E is a Schur space.*

Proof. (1) Suppose that E is a Schur space and take $(x_k) \subset E$ such that $x_k \xrightarrow{w_{EF}} x$. In particular $\psi(x_k) \to \psi(x)$ for every $\psi \in E'$ and consequently $x_k \xrightarrow{\|\,\|} x$ as E a is Schur space.

(2) Suppose that E is a w_{EF}-Schur space and take $(x_k) \subset E$ such that $x_k \xrightarrow{w} x$. From the fact that $L(E,F) = L(E_w,F_w)$ we get that $\varphi(x_k) \xrightarrow{\sigma(F,F')} \varphi(x)$ for every $\varphi \in L(E,F)$ and, since F is a Schur space, this implies $\varphi(x_k) \xrightarrow{\|\,\|} \varphi(x)$ for every $\varphi \in L(E,F)$ i.e., $x_k \xrightarrow{w_{EF}} x$. So, $x_k \xrightarrow{\|\,\|} x$ as E is a w_{EF}-Schur space. ☐

Corollary 2.14. *Let E and F be Banach spaces. If F is a Schur space, then E is a w_{EF}-Schur space if and only if E is a Schur space.*

The following definitions will be used in the some examples in the next section.

Definition 2.15. A sequence (x_n) in E is weakly p-summable (or an l_w^p-sequence) if $\sum_{n=1}^{\infty} |\varphi(x_n)|^p < \infty$ for all $\varphi \in E'$.

Definition 2.16. A Banach space E is said to have the Schur property of order p (briefly, E has the SP_p) if every weakly p-summable sequence (ℓ_w^p-sequence) is norm convergent to 0.

Now we give some examples of spaces with the property defined above.

Example 2.17. Suppose $1 < p < \infty$ and p' is the conjugate of p. If $1 < q < p'$ (i.e. $\frac{1}{p} + \frac{1}{q} > 1$) then by Pitt's Theorem each bounded linear operator $T : l_{p'} \to l_q$ is compact. As a consequence all the spaces l_q have SP_p (see [12, Example 3.9(i)]).

Example 2.18. If $E = \ell_p$ $(1 < p < \infty)$ and F is a Banach space which has SP_q for q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, by [1, Proposition 2.2(a)] we have that $L(l_p, F) = K(l_p, F)$. So, as $E' = l_q$ is separable, as a result of Proposition 2.6 we have that (B, w_{l_pF}) is metrizable for all bounded subset B of l_p if $1 < p < \infty$ and Banach space F with SP_q.

3. w_{EF}-continuity of polynomials

Let $\mathcal{C}_{w_{EF}}(E, F)$ be the space of the w_{EF}-continuous mappings from E into F, $\mathcal{C}_{w_{EF}b}(E, F)$ be the space of the mappings from E into F whose restrictions to all bounded subsets of E are w_{EF}-continuous and $\mathcal{C}_{w_{EF}u}(E, F)$ be the space of the mappings from E into F whose restriction to all bounded subsets of E are uniformly w_{EF}-continuous.

Moreover, let $\mathcal{C}_{w_{EF}sc}(E, F)$ be the space of the mappings from E into F which are sequentially w_{EF}-continuous that is, the space of all $f\colon E \to F$ such that $(f(x_n))$ converges to $f(x)$ in F whenever a sequence $(x_n) \subset E$, satisfies $\varphi(x_n) \to \varphi(x)$ for every $\varphi \in L(E, F)$ (where $x \in E$).

For $\theta = w_{EF}$, $w_{EF}b$, $w_{EF}u$, $w_{EF}sc$, we let

$$\mathcal{P}_\theta({}^nE, F) = \mathcal{P}({}^nE, F) \cap \mathcal{C}_\theta(E, F).$$

It is clear that, for all $n \in \mathbb{N}$ and for all F,

$$\mathcal{P}_{w_{EF}u}({}^nE, F) \subset \mathcal{P}_{w_{EF}b}({}^nE, F) \subset \mathcal{P}_{w_{EF}sc}({}^nE, F) \subset \mathcal{P}({}^nE, F).$$

Moreover, obviously we always have $\mathcal{P}_{w_{EF}}({}^nE, F) \subset \mathcal{P}_{w_{EF}b}({}^nE, F)$ for all $n \in \mathbb{N}$.

In many cases, these new spaces of polynomials coincide with some well known spaces of polynomials. For instance, if E is a subspace of F, the fact that w_{EF} coincides with the norm topology guarantees the equality $\mathcal{P}_{w_{EF}}({}^nE, F) = \mathcal{P}({}^nE, F)$. Consequently, we have also

$$\mathcal{P}_{w_{EF}}({}^nE, F) = \mathcal{P}_{w_{EF}b}({}^nE, F) = \mathcal{P}_{w_{EF}u}({}^nE, F) = \mathcal{P}_{w_{EF}sc}({}^nE, F)$$

whenever E is a subspace of F. Moreover, since in case $F = \mathbb{C}$ we have $w_{E\mathbb{C}} = w$, the spaces defined above are the subspaces of $\mathcal{P}({}^nE)$ usually denoted by $\mathcal{P}_f({}^nE)$, $\mathcal{P}_{wu}({}^nE)$, $\mathcal{P}_{wb}({}^nE)$ and $\mathcal{P}_{\mathrm{wsc}}({}^nE)$. Note that these spaces are particular cases of the spaces $\mathcal{P}_f({}^nE, F)$, $\mathcal{P}_{wu}({}^nE, F)$, $\mathcal{P}_{wb}({}^nE, F)$ and $\mathcal{P}_{\mathrm{wsc}}({}^nE, F)$ widely studied by Aron, Hervés and Valdivia in [8]. We remark that the space $\mathcal{P}_{wu}({}^nE, F)$ had previously been considered, with a different notation, by Aron and Prolla in [9]. On the other hand, if F is a Schur space, $L(E, F) = L_{\mathrm{wsc}}(E, F)$ and this guarantees that for any sequence (x_n) in E we have that (x_n) converges to x in (E, w) if, and only if, (x_n) converges to x in (E, w_{EF}) (see [13, Proposition 2.3]). Consequently we have that $\mathcal{C}_{\rm wsc}(E,F) = \mathcal{C}_{w_{EF}sc}(E,F)$. From this we get that $\mathcal{P}_{\rm wsc}(^nE,F) = \mathcal{P}_{w_{EF}sc}(^nE,F)$ for all Banach space E and for all $n \in \mathbb{N}$ whenever F is a Schur space.

In [16], under the additional hypothesis that F is a Banach algebra, the class $\mathbb{P}_f(^nE,F)$ of n-homogeneous polynomials generated by $\{\varphi^n : \varphi \in L(E,F)\}$ was defined and studied. Note that, in case $F = \mathbb{C}$, the class $\mathbb{P}_f(^nE,\mathbb{C})$ is the class of all complex n-homogeneous polynomials generated by the set $\{\varphi^n : \varphi \in E'\}$ and it is well known that this class coincides with the class $\mathcal{P}_f(^nE)$ of the weakly continuous n-homogeneous polynomials from E into $\mathbb{C}$. It was showed in [16] that we have $\mathcal{P}_f(^nE,F) \subset \mathbb{P}_f(^nE,F)$ for every Banach algebra F with identity. Besides, it is easy to verify that every element of $\mathbb{P}_f(^nE,F)$ is w_{EF}-continuous, that is, $\mathbb{P}_f(^nE,F) \subset \mathcal{P}_{w_{EF}}(^nE,F)$. However, examples and results presented in [16] and in [13] show that this inclusion may be strict. Later in this section we will show that $\overline{\mathbb{P}_f(^nE,F)} \subset \mathcal{P}_{w_{EF}u}(^nE,F)$ for all $n \in \mathbb{N}$ whenever E is a Banach space and F is a Banach algebra (cf. Proposition 3.5).

In what follows, we will proceed with the study of the w_{EF}-continuity of polynomials in connection with their weak continuity.

The next example shows that we may have $\mathcal{P}_{\rm wsc}(^nE,F) \neq \mathcal{P}_{w_{EF}sc}(^nE,F)$.

EXAMPLE 3.1. Let $P\colon \ell_2 \to \ell_2$ be defined by $P(x) = (x_k^n)$ for each $x = (x_k) \in \ell_2$. If $\{e_m : m \in \mathbb{N}\}$ is the canonical basis of ℓ_2, it is easy to verify that $e_m \xrightarrow{w} 0$ while $\|P(e_m)\|_2 = 1$ for all $m \geq 1$. So, $P \notin \mathcal{P}_{\rm wsc}(^n\ell_2,\ell_2)$. On the other hand, it is clear that $P \in \mathcal{P}_{w_{\ell_2\ell_2}sc}(^n\ell_2,\ell_2) = \mathcal{P}(^n\ell_2,\ell_2)$.

For every Banach space E and every Banach algebra F we clearly have $\mathbb{P}_f(^1E,F) = \mathcal{P}_{w_{EF}}(^1E,F)$. The next proposition provides a way to obtain, from any w_{EF}-continuous (n-1)-homogeneous polynomial from E to F and any continuous linear functional in E, an n-homogeneous polynomial from E to F which is w_{EF}-continuous. Consequently, starting from any $Q \in \mathcal{P}_{w_{EF}}(^1E,F) = L(E,F)$ we can always obtain in this way an element of $\mathcal{P}_{w_{EF}}(^nE,F)$ for all $n \in N$.

PROPOSITION 3.2. *If E and F are Banach spaces, for every $Q \in \mathcal{P}_{w_{EF}}(^{n-1}E,F)$ and $\varphi \in E'$ we have $\varphi \otimes Q \in \mathcal{P}_{w_{EF}}(^nE,F)$.*

PROOF. First note that an n-homogeneous polynomial P from E into F is w_{EF}-continuous if and only if it is w_{EF}-continuous in zero. Now, choose $Q \in \mathcal{P}_{w_{EF}}(^{n-1}E,F)$ arbitrarily. By the definition of w_{EF} we have that, given $\varepsilon > 0$, there are $\psi_1, \psi_2, \dots, \psi_k \in L(E,F)$ such that $\|Q(x)\| < \varepsilon$ whenever $\sup_{1 \leq i \leq k} \|\psi_i(x)\| < 1$. Moreover, if $\varphi \in E'$ just take $b \in F$ such that $\|b\| = 1$ to get $\varphi \otimes b \in L(E,F)$ such that $\|\varphi \otimes b(x)\| = |\varphi(x)|$ and therefore, given $\varepsilon > 0$, there are $\psi_1, \psi_2, \dots, \psi_k, \varphi \otimes b \in L(E,F)$ such that $\|\psi_i(x)\| < 1$ for all $i = 1, 2, \dots, k$ and $\|\varphi \otimes b(x)\| < 1$ implies $\|\varphi \otimes Q\| < \varepsilon$. Hence, $\varphi \otimes Q \in \mathcal{P}_{w_{EF}}(^nE,F)$. $\square$

Proposition 3.3. *Let E and F be Banach spaces. For every $n \in \mathbb{N}$ we have $\mathcal{P}_{w_{EF}}({}^nE, F) \subset \mathcal{P}_{w_{EF}u}({}^nE, F)$.*

Proof. Take any $P \in \mathcal{P}_{w_{EF}}({}^nE, F)$. By Proposition 1.11 in [11] there exists a w_{EF}-neighborhood of zero V such that $P|_V$ is w_{EF}-uniformly continuous. Let B be a bounded subset of E. As $\sigma(E, E') \preceq w_{EF} \preceq \beta(E, E')$, B is w_{EF}-bounded and so there exists $\lambda > 0$ such that $\lambda B \subset V$. So, $P|_{\lambda B}$ is uniformly w_{EF}-continuous and we have that given $\varepsilon > 0$ there exist $\delta > 0$ and $\varphi_1, \varphi_2, \dots, \varphi_p \in L(E, F)$, where $p \in \mathbb{N}$, such that $\|P(x) - P(y)\| < \lambda^n \varepsilon$ whenever $x, y \in \lambda B$ satisfy $\|\varphi_k(x - y)\| < \delta$ for all $1 \leq k \leq p$. Now, if we take $u, v \in B$ such that $\|\varphi_k(u - v)\| < \frac{\delta}{\lambda}$ for all $1 \leq k \leq p$ we have $\|P(u) - P(v)\| < \varepsilon$. Thus, $P \in \mathcal{P}_{w_{EF}u}({}^nE, F)$. $\square$

Note that by using the fact that for every $\varphi \in E'$ and $b \in F$ we have $\varphi \otimes b \in L(E, F)$, it is easy to verify the inclusions $\mathcal{P}_w({}^nE, F) \subset \mathcal{P}_{w_{EF}}({}^nE, F)$, $\mathcal{P}_{wu}({}^nE, F) \subset \mathcal{P}_{w_{EF}u}({}^nE, F)$, $\mathcal{P}_{wb}({}^nE, F) \subset \mathcal{P}_{w_{EF}b}({}^nE, F)$ and $\mathcal{P}_{\mathrm{wsc}}({}^nE, F) \subset \mathcal{P}_{w_{EF}sc}({}^nE, F)$ for all $n \in \mathbb{N}$.

Next we are going to study properties of $\mathcal{P}_\theta({}^nE, F)$ for $\theta = w_{EF}u$ and $\theta = w_{EF}sc$.

A straightforward argument yields the following proposition.

Proposition 3.4. *If E and F are arbitrary Banach spaces and $n \in \mathbb{N}$, then*

1. *$\mathcal{P}_{w_{EF}u}({}^nE, F)$ is a closed subspace of $\mathcal{P}({}^nE, F)$ in the norm topology.*
2. *$\mathcal{P}_{w_{EF}sc}({}^nE, F)$ is a closed subspace of $\mathcal{P}({}^nE, F)$ in the norm topology.*

Proposition 3.5. *If E is a Banach space and F is a Banach algebra then $\overline{\mathbb{P}_f({}^nE, F)} \subset \mathcal{P}_{w_{EF}u}({}^nE, F)$, for all $n \in \mathbb{N}$.*

Proof. Take $P \in \mathbb{P}_f({}^nE, F)$. By definition, there exists $T_1, \dots, T_p \in L(E, F)$ such that $P(x) = \sum_{k=1}^p T_k^n(x)$ for each $x \in E$. So it is enough to show that $T^n \in \mathcal{P}_{w_{EF}u}({}^nE, F)$. Since F is a Banach algebra, for every $x, y \in B_E$ we have

$$\|T^n(x) - T^n(y)\| \leq \|T(x) - T(y)\| \cdot \left(\sum_{j=0}^{n-1} \|T^j(x) T^{n-1-j}(y)\| \right)$$

$$\leq \|T(x) - T(y)\| \cdot \left(\sum_{j=1}^{n-1} \|T(x)\|^j \|T(y)\|^{n-1-j} \right) \leq c\|T(x) - T(y)\|.$$

So, given $\varepsilon > 0$, for $\delta = \varepsilon / c$ we have $\|T^n(x) - T^n(y)\| < \varepsilon$ whenever $x, y \in B_E$ and $\|T(x) - T(y)\| < \delta$ and, consequently, $T^n \in \mathcal{P}_{w_{EF}u}({}^nE, F)$. Now $\overline{\mathbb{P}_f({}^nE, F)} \subset \mathcal{P}_{w_{EF}u}({}^nE, F)$ follows by using Proposition 3.4(1). $\square$

We remark that Proposition 3.4 and Proposition 3.5 are related with [9, Proposition 2.4].

Aron, Hervés and Valdivia showed that if $\ell_1 \not\subset E$, then $\mathcal{P}_{\mathrm{wsc}}(^nE,F) = \mathcal{P}_{wu}(^nE,F)$ (see [8, Proposition 2.12]). If we consider the w_{EF}-topology instead of the weak topology we get the following result:

Proposition 3.6. *If E and F are Banach spaces such that $\ell_1 \not\subset E$ and $L_{\mathrm{wsc}}(E,F) = L(E,F)$, then $\mathcal{P}_{w_{EF}sc}(^nE,F) = \mathcal{P}_{w_{EF}u}(^nE,F)$ for every $n \in \mathbb{N}$.*

Proof. Fix any $n \in \mathbb{N}$. It is clear that the inclusion

$$\mathcal{P}_{w_{EF}u}(^nE,F) \subset \mathcal{P}_{w_{EF}sc}(^nE,F)$$

is always true and hence, all we have to prove is the inclusion $\mathcal{P}_{w_{EF}sc}(^nE,F) \subset \mathcal{P}_{w_{EF}u}(^nE,F)$. Moreover, clearly the inclusion $\mathcal{P}_{wu}(^nE,F) \subset \mathcal{P}_{w_{EF}u}(^nE,F)$ is also always valid. Since $L(E,F) = L_{\mathrm{wsc}}(E,F)$, for any sequence (x_n) in E we have that (x_n) converges to x in (E,w) if, and only if, (x_n) converges to x in (E,w_{EF}) (see [13, Proposition 2.3]). Thus, the hypothesis $L(E,F) = L_{\mathrm{wsc}}(E,F)$ guarantees that $\mathcal{P}_{w_{EF}sc}(^nE,F) = \mathcal{P}_{\mathrm{wsc}}(^nE,F)$ for every F. Moreover, [8, Proposition 2.12] implies $\mathcal{P}_{\mathrm{wsc}}(^nE,F) = \mathcal{P}_{wu}(^nE,F)$ for every Banach space F whenever $\ell_1 \not\subset E$. Hence, under our hypothesis we have

$$\mathcal{P}_{w_{EF}sc}(^nE,F) = \mathcal{P}_{\mathrm{wsc}}(^nE,F) = \mathcal{P}_{wu}(^nE,F) \subset \mathcal{P}_{w_{EF}u}(^nE,F)$$

and this completes the proof. □

Examples 2.7, 2.8 and 2.18 provide examples of pairs of spaces E and F which satisfy the hypothesis of Proposition 3.6. For instance, in particular we have $\mathcal{P}_{w_{c_0F}sc}(^nc_0,F) = \mathcal{P}_{w_{c_0F}u}(^nc_0,F)$ for every $n \in \mathbb{N}$ and for every reflexive Banach space F.

We note that none of the assumptions in Propositions 3.6 are essential. For example, $\mathcal{P}_{w_{EF}sc}(^nE,F) = \mathcal{P}_{w_{EF}u}(^nE,F)$ for every $n \in \mathbb{N}$ whenever E is a subspace of F.

We remark that $L(E,F) = \mathcal{P}(^1E,F) = \mathcal{P}_{w_{EF}sc}(^1E,F)$ for all Banach spaces E and F. The next proposition gives a condition on E under which the equality $\mathcal{P}_{w_{EF}sc}(^nE,F) = \mathcal{P}(^nE,F)$ is satisfied for all $n \in \mathbb{N}$. The definition of Λ-space will be used.

Following Carne, Cole and Gamelin [10], we say that E is a Λ-space if $\|x_k\|$ converges to 0 whenever $(x_k)_k$ is a sequence in E such that $P(x_k)$ converges to 0 for every $P \in \mathcal{P}(^nE)$ and for all $n \in \mathbb{N}$. By Theorem 7.5 in [10], a Banach space E is a Schur space if and only if E is a Λ-space and has the Dunford–Pettis Property (DPP). This means that if the Banach space E is a Λ-space then E has DPP if and only if E is a Schur space.

Proposition 3.7. *Let E and F be Banach spaces such that E is a Λ-space. Then the following conditions are equivalent:*

1. $\mathcal{P}_{w_{EF}sc}(^nE,F) = \mathcal{P}(^nE,F)$ *for all* $n \in \mathbb{N}$.
2. $x_k \xrightarrow{w_{EF}} x$ *if and only if* $x_k \xrightarrow{\|\,\|} x$.

PROOF. If $\mathcal{P}_{w_{EF}sc}(^nE,F)=\mathcal{P}(^nE,F)$ for all $n\in\mathbb{N}$, clearly, we have that $P\otimes b\in\mathcal{P}(^nE,F)=\mathcal{P}_{w_{EF}sc}(^nE,F)$ whenever $P\in\mathcal{P}(^nE)$, $b\in F$ and $n\in\mathbb{N}$. Hence, if $x_n\xrightarrow{w_{EF}}0$ then $P\otimes b(x_k)\to 0$ and consequently $P(x_k)\to 0$ for all $P\in\mathcal{P}(^nE)$ and this is true for all $n\in\mathbb{N}$. As E is a Λ-space, this implies $x_k\xrightarrow{\|\,\|}0$. The converse in (2) is trivial since w_{EF} is weaker than the norm topology.

Furthermore, if we assume (2), the equality $\mathcal{P}_{w_{EF}sc}(^nE,F)=\mathcal{P}(^nE,F)$ for all $n\in\mathbb{N}$ follows as a result of the continuity of every $P\in\mathcal{P}(^nE,F)$. $\square$

Note that Proposition 3.7 says that given any Banach spaces E and F, if E is a Λ-space then E is a w_{EF}-Schur space if and only if $\mathcal{P}_{w_{EF}sc}(^nE,F)=\mathcal{P}(^nE,F)$ for all $n\in\mathbb{N}$.

EXAMPLE 3.8. Let $E=\ell_p$ $(1<p<\infty)$ and $F=\ell_1$. Since E is not a Schur space and F is a Schur space, by Proposition 2.13(2) we have that E is not a w_{EF}-Schur space. Besides, by [10, Theorem 6.3], $E=\ell_p$ $(1<p<\infty)$ is a Λ-space. Furthermore, as E is a Λ-space, by Proposition 3.7 the fact that E is not a w_{EF}-Schur space leads to $\mathcal{P}_{w_{EF}sc}(^nE,F)\neq\mathcal{P}(^nE,F)$ for some $n\in\mathbb{N}$. Moreover, clearly, $\mathcal{P}(^1E,F)=\mathcal{P}_{w_{EF}sc}(^1E,F)$ for all Banach spaces E and F. Hence, as a result of these facts we get $\mathcal{P}_{w_{EF}sc}(^n\ell_p,l_1)\neq\mathcal{P}(^n\ell_p,l_1)$ for some natural number $n\geq 2$ whenever $1<p<\infty$.

PROPOSITION 3.9. *Let E be a Λ-space such that $\ell_1\not\subset E$. If F is a Banach space such that $L(E,F)=K(E,F)$, then $\mathcal{P}_{w_{EF}sc}(^nE,F)\neq\mathcal{P}(^nE,F)$ for some $n\geq 2$.*

PROOF. Suppose by absurd that $\mathcal{P}_{w_{EF}sc}(^nE,F)=\mathcal{P}(^nE,F)$ for all $n\geq 2$. So, given $Q\in\mathcal{P}(^nE)$ and $b\in F\setminus\{0\}$ we have $Q\otimes b\in\mathcal{P}_{w_{EF}sc}(^nE,F)=\mathcal{P}(^nE,F)$. Consequently, for any sequence (x_k) in E such that $x_k\xrightarrow{w}0$, as $L(E,F)=K(E,F)$, we have that,(x_n) converges to x in (E,w_{EF}) (see [13, Proposition 2.4]), so $Q\otimes b(x_k)\longrightarrow 0$ and this implies $Q(x_k)\longrightarrow 0$ for all $Q\in\mathcal{P}(^nE)$ and for all $n\in\mathbb{N}$. Since E is a Λ-space, we infer from this that $\|x_k\|\longrightarrow 0$. So $\|x_k\|\longrightarrow 0$ in E whenever $x_k\xrightarrow{w}0$ and this means that E is a Schur space. This contradicts the hypothesis that $\ell_1\not\subset E$. $\square$

Next we will present another example of spaces E and F for which we have $\mathcal{P}_{w_{EF}sc}(^nE,F)\neq\mathcal{P}(^nE,F)$ for some n.

EXAMPLE 3.10. If $E=\ell_p$ $(1<p<\infty)$ and F is a Banach space which has SP_q for q satisfying $\frac{1}{p}+\frac{1}{q}=1$, we have $\mathcal{P}_{w_{EF}sc}(^n\ell_p,F)\neq\mathcal{P}(^n\ell_p,F)$ for every $n\geq p$. Indeed, if there exists $n_1\geq p$ such that $\mathcal{P}_{w_{EF}sc}(^{n_1}\ell_p,F)=\mathcal{P}(^{n_1}\ell_p,F)$ then, for every $P\in\mathcal{P}(^{n_1}\ell_p)$ and $b\in F$ such that $\|b\|\neq 0$ we have that $P\otimes b\in\mathcal{P}(^{n_1}\ell_p,F)=\mathcal{P}_{w_{EF}sc}(^{n_1}\ell_p,F)$. Now, take any sequence (x_k) in ℓ_p such that $x_k\xrightarrow{w}0$. By [1, Proposition 2.2(a)] we have $L(\ell_p,F)=K(\ell_p,F)$ and hence (x_n) converges to x in (E,w_{EF}) (see [13, Proposition 2.4]), then we have that $P(x_k)b = P \otimes b(x_k) \to P \otimes b(0) = 0$. So, if there exists $n_1 \geq p$ such that $\mathcal{P}_{w_{EF}sc}({}^{n_1}\ell_p, F) = \mathcal{P}({}^{n_1}\ell_p, F)$, then we have that $P(x_k) \to P(0) = 0$ for all $P \in \mathcal{P}({}^{n_1}\ell_p)$ and, by [10, Theorem 6.3], we get that $\|x_k\| \to 0$. This means that ℓ_p $(1 < p < \infty)$ is a Schur space which contradicts the fact that ℓ_p is reflexive. Hence, $\mathcal{P}_{w_{EF}sc}({}^{n}\ell_p, F) \neq \mathcal{P}({}^{n}\ell_p, F)$ for every $n \geq p$ whenever F is a Banach space with SP_q for q satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Note that, since every separable infinite dimensional Hilbert space is isometrically isomorphic to ℓ_2, the fact that $\mathcal{P}_{w_{EF}sc}({}^{n}H, F) \neq \mathcal{P}({}^{n}H, F)$ for every $n \geq 2$ whenever H be a separable infinite dimensional Hilbert space and F is a Banach space with SP_2 follows as a particular case of the above example.

For each $n \in \mathbb{N}$, let $\mathcal{P}_K({}^{n}E, F)$ denote the set of compact n-homogeneous polynomials $P\colon E \to F$, that is the set of all $P \in \mathcal{P}({}^{n}E, F)$ that take bounded subsets of E into relatively compact subsets of F. It is well known that the inclusion $\mathcal{P}_{wu}({}^{n}E, F) \subset \mathcal{P}_K({}^{n}E, F)$ is always true. On the other hand, the inclusion $\mathcal{P}_{w_{EF}u}({}^{n}E, F) \subset \mathcal{P}_K({}^{n}E, F)$ is not always true even for n=1. For instance, if E is an infinite dimensional closed subspace of c_0 and $c_0 \subset F$, we clearly have $\mathcal{P}_{w_{EF}u}({}^{1}E, F) = \mathcal{P}({}^{1}E, F)$ and $\mathcal{P}_K({}^{1}E, F) \subsetneq \mathcal{P}({}^{1}E, F)$.

So, it is natural to ask under which assumptions on E or F and for which values of n the inclusion $\mathcal{P}_{w_{EF}sc}({}^{n}E, F) \subset \mathcal{P}_K({}^{n}E, F)$ is true. The next proposition establishes sufficient conditions under which this inclusion is true for every $n \in \mathbb{N}$.

PROPOSITION 3.11. *Let E and F be Banach spaces such that $\ell_1 \not\subset E$ and $L(E, F) = L_{\mathrm{wsc}}(E, F)$. Then $\mathcal{P}_{w_{EF}sc}({}^{n}E, F) \subset \mathcal{P}_K({}^{n}E, F)$ and consequently $\mathcal{P}_{w_{EF}u}({}^{n}E, F) \subset \mathcal{P}_K({}^{n}E, F)$ for every $n \in \mathbb{N}$.*

PROOF. Since $L(E, F) = L_{\mathrm{wsc}}(E, F)$, for any sequence (x_n) in E we have that (x_n) converges to x in (E, w) if, and only if, (x_n) converges to x in (E, w_{EF}) (see [13, Proposition 2.3]) then, the equalities $\mathcal{P}_{w_{EF}sc}({}^{n}E, F) = \mathcal{P}_{\mathrm{wsc}}({}^{n}E, F)$ and $\mathcal{P}_{\mathrm{wsc}}({}^{n}E, F) = \mathcal{P}_{wu}({}^{n}E, F)$ follow and $\ell_1 \not\subset E$, respectively. On the other hand, the inclusion $\mathcal{P}_{wu}({}^{n}E, F) \subset \mathcal{P}_K({}^{n}E, F)$ is always true and hence the first inclusion is true. The last inclusion follows immediately from the fact that $\mathcal{P}_{w_{EF}u}({}^{n}E, F) \subset \mathcal{P}_{w_{EF}sc}({}^{n}E, F)$. □

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References

[1] S. I. Ansari, On Banach spaces Y for which $B(\mathcal{C}(\Omega), Y) = K(\mathcal{C}(\Omega), Y)$, *Pacific J. Math.*, **169** (1995), 201–218.

[2] A. Arens, Operations induced in function classes, *Monatsh. Math.*, **55** (1951), 1–19.

[3] A. Arens, The adjoint of a bilinear operation, *Proc. Amer. Math. Soc.*, **2** (1951), 839–848.

[4] P. Civin and B. Yood, The second conjugate space of a Banach algebra as an algebra, *Pacific J. Math*, **11** (1961), 847–870.

[5] R. M. Aron and P. D. Berner, A Hahn–Banach extension theorem for analytic mappings, *Bull. Soc. Math. France*, **106** (1978), 3–24.

[6] R. M. Aron, B. J. Cole and T. W. Gamelin, Weak-star continuous analytic functions, *Canad. J. Math.*, **47** (1995), 673–683.

[7] R. M. Aron, P. Galindo, D. Garcia and M. Maestre, Regularity and algebras of analytic functions in infinite dimensions, *Trans. Amer. Math. Soc.*, **348** (1996), 543–559.

[8] R. M. Aron, C. Hervés and M. Valdivia, Weakly continuous mappings on Banach Spaces, *J. Funct. Anal.*, **52** (1983), 189–204.

[9] R. M. Aron and J. B. Prolla, Polynomial approximation of differentiable functions on Banach spaces, *J. Reine Angew. Math.*, **313** (1980), 195–216.

[10] T. K. Carne, B. Cole and T. W. Gamelin, A uniform algebra of analytic functions on a Banach space, *Trans. Amer. Math. Soc.*, **314** (1989), 639–659.

[11] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces,* Springer Monographs in Mathematics, Springer-Verlag (London, 1999).

[12] J. H. Fourie and E. D. Zeekoei, On p-convergent operators on Banach lattices, *Acta Math. Sin. (Engl. Ser.)*, **34** (2018), 873–890.

[13] P. Galindo, M. L. Lourenço and L. A. Moraes, Polynomials generated by linear operators, *Proc. Amer. Math. Soc.* **132** (2004), 2917–2927.

[14] D. Garcia, M. L. Lourenço, M. Maestre and L. A. Moraes, The spectrum of analytic mappings of bounded type, *J. Math. Anal. Appl.*, **245** (2000), 447–470.

[15] D. Garcia, M. L. Lourenço, L. A. Moraes and O. Paques, The spectra of some algebras of analytic mappings, *Indag. Math. (N.S.)*, **10** ((1999), 393–406.

[16] M. L. Lourenço and L. A. Moraes, A class on polynomials from Banach space into Banach algebra, *Publ. Res. Inst. Math. Sci.*, **37** (2001), 519–527.

[17] I. Zalduendo, A canonical extension for analytic functions on Banach spaces, *Trans. Amer. Math. Soc.*, **320** (1990), 747–763.