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Zip bifurcation in a competitive system with diffusion*

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Abstract We study the existence of a global attractor in a reaction-diffusion system which describes the interaction among $n+1$ species, amongst which n species of predators compete for a single prey. Also, we prove the persistence of the zip bifurcation phenomenon for the reaction-diffusion system, which was introduced by Farkas [5] for a three dimensional ODE prey-predator system.

Keywords Biomathematics · Zip Bifurcation · Population Dynamic · Prey-Predator System

Mathematics Subject Classification (2000) 34A47

1. Introduction

The model to be studied is described by the differential equation system

$$\begin{cases} \dfrac{\partial S}{\partial t}(x,t) = \delta_0 \Delta S + \gamma\left(1 - \frac{S}{K}\right) S - \displaystyle\sum_{i=1}^{n} m_i f_i(S) u_i \quad \text{in} \quad \Omega \times (0,\infty) \\ \dfrac{\partial u_i}{\partial t}(x,t) = \delta_i \Delta u_i + (m_i f_i(S) - d_i) u_i, \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.1}$$

This paper is dedicated to the memory of professor Miklós Farkas, colleague and friend.

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where $\Omega \subset I\!R^N$ $(N = 1, 2, 3)$ is an open connected bounded domain with smooth boundary $\partial\Omega$. We will assume that the functions S and u_i satisfies the Neumann boundary conditions

$$\frac{\partial S}{\partial \nu}(x,t) = 0, \ \frac{\partial u_i}{\partial \nu}(x,t) = 0, \quad i = 1, 2, \ldots, n, \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{1.2}$$

where $\nu = \boldsymbol{\nu}(x)$ denotes the outer unit normal to $\partial\Omega$ and $\Delta = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}$ is the Laplacian operator. S is the population density of the prey, and u_i, $i = 1, 2, \ldots, n$, are the population densities of n predators competing for the prey. $\delta_i > 0$, $i = 1, 2, \ldots, n$, are the diffusion rates. We admit that the functional response f_i is Holling type II for each predator i that is, $f_i(S) = S/(a_i + S)$, for each $i = 1, 2, \ldots, n$, where the constant a_1, $a_2, \ldots, a_n$ are nonnegative. The parameters compound the model represent

- S: *quantity of prey*
- γ: *intrinsic growth rate of prey*
- K: *carrying capacity of the environment*
- u_i: *quantity of predator* i
- m_i: *maximal birth rate of predator* i
- d_i: *mortality of predator* i
- a_i: *half saturation constant of predator* i.

In the second section of this work we prove that the positive cone $\mathbb{R}_+^{n+1} = \{(S, u_1, \ldots, u_n) : S \geq 0, u_i \geq 0\}$ is positively invariant for system (1.1), (1.2) in accordance with the theorems given in [9] as well that the solutions for this system there exist for all $t > 0$. In the third section, based in Alikakos [1] (see also Dung and Smith [3]), we will show the existence of a global attractor for system (1.1), (1.2). Finally, in the fourth section we prove the occurrence of the *zip bifurcation* phenomenon.

The zip bifurcation concept was introduced by Farkas [5] in connection with the study of the ordinary differential version of (1.1) modeling the interaction between two predator species competing for a single prey, that is, system (1.1) with $n = 2$ and $\delta_0 = \cdots = \delta_n = 0$. In this case, if $a_1 > a_2$ and $\frac{a_1 d_1}{m_1 - d_1} = \frac{a_2 d_2}{m_2 - d_2}$, then, aside from the 'trivial' equilibrium points $(S, u_1, u_2) = (0, 0, 0)$ and $(S, u_1, u_2) = (K, 0, 0)$, the system has also a family of equilibrium points contained in a line segment H_K in $\mathbb{R}_+^3$. Using the carrying capacity K as a parameter, Farkas [5] has shown that there exists an interval I such that, for each $K \in I$, there is a point $(\lambda, \xi_1(K), \xi_2(K)) \in H_K$ such that any equilibrium point $(\lambda, \xi_1, \xi_2) \in H_K$ is *unstable* if $\xi_1 < \xi_1(K)$ and *stable* if $\xi_1 > \xi_1(K)$. Farkas has called this situation as a *zip* or *velcro* bifurcation phenomenon. We will describe this concept more carefully in Section 4..

2. Positivity and global existence of the solutions

Since S and u_i are population densities, only non-negative solutions are of biological interest, thus we will prove the local existence and positivity of solutions for the system (1.1), (1.2). In the following, we will use some results about invariant region (cf. Smoller [11] or [2]) and another conditions given in Morgan [9] to show that the solutions there exist for all time. First, we will write (1.1), (1.2) as an evolution equation on a Banach space. Let $|v|$ be the Euclidean norm of a vector $v = (S, u_1, \dots, u_n) \in \mathbb{R}^{n+1}$:

$$|v| = \left(|S|^2 + \sum_{i=1}^{n} |u_i|^2 \right)^{1/2}.$$

Let $p > \max\{1, \frac{N}{2}\}$ and $X = L^p(\Omega, \mathbb{R}^{n+1})$ be the Banach space of the functions $v : \Omega \to \mathbb{R}^{n+1}$ such that $|v|^p$ is integrable on Ω in the norm $\|v\|_{L^p} = (\int_\Omega |v(x)|^p \, dx)^{1/p}$, with obvious changes if $p = \infty$.

Let $A : D(A) \subset X \to X$ be the self-adjoint and nonnegative linear operator defined by

$$D(A) = \left\{ v \in W^{2,p}(\Omega, \mathbb{R}^{n+1}) : \frac{\partial v}{\partial \nu}(x) = 0, x \in \partial\Omega \right\}, \tag{2.3}$$

and

$$(Av)(x) = -D\Delta v = (-\delta_0 \Delta S(x), -\delta_1 \Delta u_1(x), \dots, -\delta_n \Delta u_n(x)), \tag{2.4}$$

for $v \in D(A)$ and $x \in \Omega$.

Then, for every $\rho > 0$, the operator $A + \rho I$ is self-adjoint and positive, so the fractional power spaces $X^\alpha = D((A + \rho I)^\alpha)$, endowed with the graphic norm $\|v\|_{\alpha,\rho} = \|(A + \rho I)^\alpha v\|_{L^p}$, $\alpha \geq 0$, are well-defined. As different choice of ρ induce equivalent norms in X^α, we will omit the dependence of ρ (see Henry [8], Def. 1.4.7). It is well known (see Henry [8]) that $-A$ is a generator of a analytic semi-group $\{e^{-At} : t \geq 0\}$ on X that satisfies $\|e^{-At}\|_{L(X)} \leq 1$, for all $t \geq 0$.

In the next proposition, we collect some results whose proofs can be find in Henry [8], Theorems 1.4.8 and 1.6.1.

Proposition 2.1

(1) *For each $\alpha \geq 0$, X^α is a Banach space with the norm $\|v\|_\alpha = \|(A + \rho I)^\alpha v\|$. Furthermore, since A has compact resolvent, $D(A) \hookrightarrow X^\alpha \hookrightarrow X$ are compact for each $0 \leq \alpha < 1$.*
(2) *If $2\alpha - \frac{N}{p} > k - \frac{N}{q}$ and $q \geq p$, then $X^\alpha \hookrightarrow W^{k,q}(\Omega)$;*
(3) *If $2\alpha - \frac{N}{p} > \nu \geq 0$, then $X^\alpha \hookrightarrow C^\nu(\Omega)$.*

In sight of the Proposition 2.1, we will assume that $p > \max\{1, \frac{N}{2}\}$ and we choose $\max\{\frac{1}{2}, \frac{N}{2p}\} < \alpha < 1$, so that X^α satisfies

$$\text{(i) } X^\alpha \hookrightarrow W^{1,p}(\Omega) \tag{2.5}$$

$$\text{(ii) } X^\alpha \hookrightarrow C^\nu(\Omega), \tag{2.6}$$

with compact embedding, where $0 \leq \nu < 2\alpha - \frac{N}{p}$. From Henry [8], Theorem 1.4.3, it follows that $e^{-At}(X^\alpha) \subset X^\alpha$, for all $t \geq 0$ and $\|e^{-At}\|_{L(X^\alpha)} \leq t^{-\alpha}$, for $t > 0$.

Next result is an immediate consequence of maximum principle.

Lemma 2.1 *Let $\lambda > 0$, $f \in L^p(\Omega)$ and u be solutions of*

$$\begin{cases} -\Delta u(x) + \lambda u(x) = f(x), & x \in \Omega \\ \frac{\partial u}{\partial \nu}(x) = 0, x \in \partial\Omega. \end{cases}$$

If $f(x) \geq 0$ for $x \in \Omega$, then $u(x) \geq 0$ for all $x \in \overline{\Omega}$.

From Lemma 2.1 and (2.4), the resolvent $(\lambda + A)^{-1}$ of $-A$ is nonnegative for each $\lambda > 0$. Since

$$e^{-At}v = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} v$$

for all $v \in X$ and $t \geq 0$, e^{-At} is a semigroup of nonnegative operators, so the cone of nonnegative functions X^α_+ is closed and invariant by $\{e^{-At} : t \geq 0\}$.

Since f_i are not defined for all values of $S < 0$, let us modify f_i as a C^∞ function in the interval $(-\infty, 0)$ by taking $f_i(S) = 0$ for $S \leq -\frac{a_i}{2}$, $i = 1, \ldots, n$; we still denote this C^∞ modified functions by f_i. With these modifications, let $F : X^\alpha \to X$ be defined by

$$\begin{aligned} F(v)(x) = \Bigg(&\gamma\left(1 - \frac{S(x)}{K}\right)S(x) - \sum_{j=1}^{n} m_j f_j(S(x))u_j(x), \\ &(m_1 f_1(S(x)) - d_1)u_1(x), \ldots, (m_n f_n(S(x)) - d_n)u_n(x)\Bigg), \end{aligned} \tag{2.7}$$

for $x \in \Omega$. Then it holds the following

Proposition 2.2 *F is Lipschitz continuous in bounded sets of X^α. Furthermore, for each bounded set $B = \{v \in X^\alpha : v \geq 0$ and $\|v\|_\alpha \leq b\}$, there exists a real positive constant $\beta = \beta(B)$ such that $F(v) + \beta v \geq 0$ for all $v \in B$.*

Proof The first statement follows from $X^\alpha \hookrightarrow C^\nu(\Omega)$ and from the inequalities

$$|f_i(S_1) - f_i(S_2)| \leq \frac{1}{a_i}|S_1 - S_2|$$

and

$$|g(S_1) - g(S_2)| \leq \gamma|S_1 - S_2|\left(1 + \frac{1}{K}|S_1 + S_2|\right),$$

for all $S_1 \geq 0$, $S_2 \geq 0$, where $g(S) = \gamma(1 - \frac{S}{K})S$. To proof the next statement, let $B = \{v \in X^\alpha : v \geq 0$ and $\|v\|_\alpha \leq b\}$. If C is the immersion constant of X^α in $C^\nu(\Omega)$, then

$$0 \leq u_j(x) \leq C\|v\|_\alpha \leq Cb,$$

for all $x \in \Omega$, $0 \leq j \leq n$ and $v \in B$ [we take $u_0 = S$].

Let $F = (F_0, F_1, \ldots, F_n)$; if $1 \leq j \leq n$, then, for all $v = (S, u_1, \ldots, u_n) \in B$, we have

$$F_j(v) + \beta u_j = (m_j f_j(S) - d_j)u_j + \beta u_j \geq (\beta - d_j)u_j \geq 0,$$

for $\beta \geq d_j$. If $j = 0$, then

$$\begin{aligned} F_0(v) + \beta u_0 &= g(S) + \beta S - \sum_{j=1}^{n} m_i f_i(S)u_j \\ &\geq g(S) + \beta S - \frac{\max\{m_1, \ldots, m_n\}}{\min\{a_1, \ldots, a_n\}}S(u_1 + u_2 + \cdots + u_n) \\ &\geq \left[\gamma(1 - \frac{S}{K}) - \frac{\max\{m_1, \ldots, m_n\}}{\min\{a_1, \ldots, a_n\}}nCb + \beta\right]S \\ &\geq \left[\gamma(1 - \frac{Cb}{K}) - \frac{\max\{m_1, \ldots, m_n\}}{\min\{a_1, \ldots, a_n\}}nCb + \beta\right]S, \end{aligned}$$

since $f_i(S) \leq \frac{1}{a_i} S$, for all $S \geq 0$ and $1 \leq i \leq n$. Therefore, taking

$$\beta \geq \max\left\{d_1, \ldots, d_n, -\gamma(1 - \frac{Cb}{K}) + \frac{\max\{m_1, \ldots, m_n\}}{\min\{a_1, \ldots, a_n\}} Cnb\right\},$$

we get $F(v) + \beta v \geq 0$, for all $v \in B$ and the proof is complete.

With the previous notations (1.1), (1.2) can be written as an evolution equation

$$\begin{cases} \dot{v}(t) + Av(t) = F(v(t)), t > 0 \\ v(0) = v_0. \end{cases} \tag{2.8}$$

In the following we will study the well-posedness of (1.1), (1.2) on X_+^α. As usual, we first study *mild solutions* of (2.8) before classical solutions. The proofs will be omitted since they follow from the previous results and from Theorems 3.3.3 and 3.5.2 of Henry's book [8].

Theorem 2.3 *For each $v_0 \in X_+^\alpha$, there exist $t_1 > 0$ and an unique continuous function $v : [0, t_1) \to X_+^\alpha$ such that $v(0) = v_0$ and*

$$v(t) = e^{-At} v_0 + \int_0^t e^{-A(t-s)} F(v(s))\, ds,$$

for all $0 \leq t < t_1$. If $t_1 < \infty$, then $\limsup_{t \to t_1-} \|F(v(t))\| = \infty$. In addition, for each $t_2 < t_1$, v is C^1 on $(0, t_2)$, $v(t) \in X_+^\sigma$, for all $0 < \sigma < \alpha$ and v is a classical solution of (1.1), (1.2).

From Theorem 2.3, we conclude that solutions of (1.1), (1.2) exist in some interval $[0, t_1)$, are nonnegative and classical, so that the hypotheses in p. 199 in the Smoller book [11] are satisfied. Therefore, $\Sigma = \mathbb{R}_+^{n+1}$ is an invariant rectangle by the flux of (1.1), (1.2). To show that the solutions are globally defined, we use the following result due to Morgan [9].

Theorem 2.4 *Assume that $D = diag\,(d_1, \ldots, d_m)$ is a real matrix with entries $d_j > 0$, $f : \mathbb{R}^m \to \mathbb{R}^m$ is a C^1 function and $\mathbb{R}_+^m$ is invariant for*

$$\begin{cases} v_t(x,t) = D\Delta v(x,t) + f(v(x,t)), & x \in \Omega, \quad t > 0 \\ \dfrac{\partial v}{\partial \nu}(x,t) = 0, & x \in \partial\Omega, \quad t > 0 \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.9}$$

Suppose that v is a maximal solution of (2.9) defined on $[0, t_1)$ and that there exist C^2 functions $H : \mathbb{R}_+^m \to \mathbb{R}$ and $h_i : \mathbb{R}_+ \to \mathbb{R}$ such that

1. $H(z) = \sum_{i=1}^m h_i(z_i)$ *for all $z \in \mathbb{R}_+^m$;*
2. $h_i(z_i) \geq 0$, $h_i''(z_i) \geq 0$ *for all $z_i \in \mathbb{R}_+$ and $1 \leq i \leq m$;*
3. $H(z) \to \infty$ *if and only if $|z| \to \infty$ in $\mathbb{R}_+$;*

4. *There exists a matrix* $A \in \mathbb{R}^{m\times m}$ *satisfying* $a_{ij} \geq 0$ *and* $a_{ii} > 0$ *such that, for each* $1 \leq j \leq m$, *there exist* $r \geq 0$, $K_1 \geq 0$, $K_2 \geq 0$, *independent of* j, *such that* $\sum_{i=1}^{j} a_{ji} h_i'(z_i) f_i(z) \leq K_1 (H(z))^r + K_2$ *for all* $z \in \mathbb{R}_+^m$;
5. *There exist* $q_1 \geq 0$, $K_5 \geq 0$, $K_6 \geq 0$ *such that, for all* $1 \leq i \leq m$, *it holds* $h_i'(z_i) f_i(z) \leq K_5 (H(z))^{q_1} + K_6$ *for all* $z \in \mathbb{R}_+^m$;
6. *There exist* $K_7 \geq 0$ *e* $K_8 \geq 0$ *such that* $\nabla H(z) \cdot f(z) \leq K_7 H(z) + K_8$ *for all* $z \in \mathbb{R}_+^m$.

Then, $t_1 = \infty$.

Proposition 2.5 *The solutions of (1.1), (1.2) are defined for* $t \geq 0$.

Proof We show that (1.1), (1.2) satisfy the hypotheses of Theorem 2.4. Indeed, let $v = (S, u_1, \ldots, u_n)$ be a solution of (1.1), (1.2). Let $H(S, u_1, \ldots, u_n) = S + u_1 + \cdots + u_n$, $h_i = u_i$ and $A = (a_{ij}) \in \mathbb{R}^{(n+1)\times(n+1)}$ satisfying $a_{ij} = 0$ for $i < j$ and $a_{ij} = 1$ for $i \geq j$, for $0 \leq i, j \leq n+1$. Hypotheses (1)-(3) are trivially satisfied. Taking $r = 1$, $K_1 = \gamma$ and $K_2 = 0$, we have

$$\begin{aligned}\sum_{i=1}^{j} a_{ji} h_i'(u_i) f_i(u) &= \gamma u_0 - \frac{\gamma}{k} u_0^2 - d_1 u_1 - \cdots - d_n u_n \\ &< \gamma(u_0 + \cdots + u_n) \quad = \quad K_1 H(u) + K_2,\end{aligned}$$

so that hypothesis (4) is satisfied. Now,

$$h_1' f_1(u) = \gamma u_0 - \frac{\gamma}{K} u_0^2 - \sum_{i=1}^{n} m_i \frac{u_0 u_i}{a_i + u_0} \leq \gamma H(u)$$

and

$$h_i' f_i(u) = m_i \frac{u_0 u_i}{a_i + u_0} - d_i u_i < m_i u_i < m_i H(u).$$

Therefore, by taking $q_1 = 1$, $K_6 = 0$ and $K_5 = \max\{\gamma, m_1, \ldots, m_n\}$, hypothesis (5) is satisfied. Finally,

$$\begin{aligned}\nabla H(u) \cdot f(u) &= \gamma u_0 - \frac{\gamma}{K} u_0^2 - \sum_{i=1}^{n} m_i \frac{u_0 u_i}{a_i + u_0} + \sum_{i=1}^{n} m_i \frac{u_0 u_i}{a_i + u_0} - \sum_{i=1}^{n} d_i u_i \\ &\leq K_7 H(u) + K_8,\end{aligned}$$

so hypothesis (6) is satisfied for $K_7 = \gamma$ and $K_8 = 0$, and the proof is complete.

As a consequence, we have the following

Corollary 2.6 *The problem (1.1), (1.2) define a nonlinear semigroup* $T(t) : X_+^\alpha \to X_+^\alpha$.

3. Global attractor

In this section we show that system (1.1), (1.2) has a compact global attractor. To this end, we show that the semigroup associated to the problem is compact dissipative, that is, there exists a bounded set which attracts all bounded sets. To get the estimates, we apply the results due to Alikakos [1] (see also H. Smith [3] and Henry [8]). First, we get uniform bounds of the first component.

Proposition 3.1 *Let $u = (S, u_1, \ldots, u_n)$ be a solution of (1.1), (1.2) with initial condition $u(\cdot, 0) \geq 0$. Then, there exist $0 < d \leq \gamma$ such that*

$$0 \leq S(x,t) \leq \frac{2K\gamma}{d} + e^{-dt} \max_{x \in \overline{\Omega}} S_0(x).$$

for all $(x,t) \in \Omega \times [0, \infty)$.

Proof Let $d = \min\{\gamma, d_1, \ldots, d_n\}$. Observe that $\gamma(1 - \frac{S}{K})S \leq -dS + 2K\gamma$, for all $S \in \mathbb{R}$. Therefore, S satisfies

$$\begin{cases} S_t(x,t) \leq \delta_0 \Delta S - dS + 2k\gamma, & (x,t) \in \Omega \times (0, \infty) \\ \dfrac{\partial S}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0, \infty) \\ S(x,0) = S_0(x), & x \in \Omega, \quad S_0(x) \geq 0, \end{cases} \tag{3.10}$$

so, by comparison, $S(x,t) \leq w(x,t)$, where w is solution of

$$\begin{cases} w_t(x,t) = \delta_0 \Delta w - dw + 2k\gamma, & (x,t) \in \Omega \times [0, \infty) \\ \dfrac{\partial w}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0, \infty) \\ w(x,0) = S_0(x), & x \in \Omega. \end{cases} \tag{3.11}$$

Writing $w(x,t) = z(x,t)e^{-dt} + \frac{2K\gamma}{d}(1 - e^{-dt})$, we conclude that z satisfies

$$\begin{cases} z_t(x,t) = \delta_0 \Delta z & (x,t) \in \Omega \times (0, \infty) \\ \dfrac{\partial z}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0, \infty) \\ c[.3pc]z(x,0) = S_0(x), & x \in \Omega, \quad S_0(x) \geq 0. \end{cases} \tag{3.12}$$

By the Strong Maximum Principle (see [10] p. 174), $z(x,t)$ assume its maximum at $t = 0$. Hence, $0 \leq z(x,t) \leq \max_{x \in \Omega} z(x,0)$ for all $(x,t) \in \Omega \times [0, \infty)$ and therefore

$$0 \leq S(x,t) \leq \frac{2K\gamma}{d}(1 - e^{-dt}) + e^{-dt} \max_{x \in \Omega} S_0(x),$$

for $(x,t) \in \Omega \times [0, \infty)$.

Next, we get uniform bounds in the L^1-norm.

Proposition 3.2 *Let $d = \min\{\gamma, d_1, \ldots, d_n\}$ and $u(\cdot, t)$ be a solution of (1.1), (1.2). Then*

$$\sum_{i=0}^{n} \|u_i(\cdot, t)\|_{L^1} \leq e^{-dt} \sum_{i=0}^{n} \|u_i(\cdot, 0)\|_{L^1} + \frac{|\Omega| K\gamma}{d}(1 - e^{-dt}),$$

for all $t \geq 0$.

Proof Let $S = u_0$. Adding the equations of (1.1), integrating on Ω and using the Green's Identities we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega \sum_{i=0}^n u_i(x,t)\,dx &= \int_\Omega \sum_{i=0}^n \delta_i \Delta u_i + \int_\Omega \gamma(1-\frac{u_0}{K})u_0 - \int_\Omega \sum_{i=1}^n d_i u_i \\ &\le \int_\Omega \left[-du_0(x,t) + 2K\gamma - \sum_{i=1}^n du_i(x,t)\right] dx \\ &= -d\int_\Omega \sum_{i=0}^n u_i(x,t)\,dx + |\Omega|K\gamma.\end{aligned}$$

By Gronwall's inequality we have

$$\sum_{i=0}^n \|u_i(\cdot,t)\|_{L^1} \le e^{-dt}\sum_{i=0}^n \|u_i(\cdot,0)\|_{L^1} + \frac{|\Omega|K\gamma}{d}(1-e^{-dt})$$

for all $t \ge 0$.

Remark 3.1 According to the previous Proposition, if $\|u(\cdot,0)\|_{L^1}$ is bounded then $\|u(\cdot,t)\|_{L^1}$ is bounded for all $t \ge 0$.

Proposition 3.3 *Suppose that $u_i(x,t) \ge 0$, $0 \le i \le n$, is a solution of (1.1), (1.2) and assume that exist constant $C > 0$ such that sup $_{t\ge 0}\int_\Omega u_i(x,t)\,dx < C$, for $1 \le i \le n$. Then, there exists a constant C^*, depending on C and $\|u_i(\cdot,0)\|_{L^\infty}$, $1 \le i \le n$, such that sup $_{t\ge 0}\|u_i(\cdot,t)\|_{L^\infty} \le C^*$.*

Proof We employ the ideas of Alikakos [1]. To simplify the notation, we write, for $1 \le i \le n$, an equation of (1.1) as

$$u_t = \delta\Delta u + F(u),$$

where $F(u) = m\frac{u_0}{a+u_0}u - du$ and $u_0 = S$. Multiplying the previous equation by u^{2^k-1}, integrating, using the Green's Identity and the fact that $F(u) \le mu$, for $u \ge 0$, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2^k}\int_\Omega u^{2^k}\,dx\right) &= \delta\int_\Omega (\Delta u)u^{2^k-1}\,dx + \int_\Omega u^{2^k-1}F(u)\,dx \\ &\le -\frac{2^k-1}{2^{2k-2}}\ \delta\int_\Omega |\nabla(u^{2^{k-1}})|^2\,dx + m\int_\Omega u^{2^k}\,dx. \qquad (3.13)\end{aligned}$$

On the other hand, $\nabla u\nabla(u^{2^k-1}) = (2^k-1)u^{2^k-2}|\nabla u|^2$ and therefore

$$\nabla u\nabla(u^{2^k-1}) = \frac{2^k-1}{2^{2k-2}}|\nabla(u^{2^{k-1}})|^2.$$

Since

$$\frac{d}{dt}\left(\frac{1}{2^k}\int_\Omega u^{2^k}\,dx\right) = \frac{1}{2^{k-1}}\frac{d}{dt}\left(\frac{1}{2}\int_\Omega (u^{2^{k-1}})^2\,dx\right),$$

we have

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega (u^{2^{k-1}})^2\,dx\right) \le -\frac{2^k-1}{2^{k-1}}\delta\int_\Omega |\nabla(u^{2^{k-1}})|^2\,dx + 2^{k-1}m\int_\Omega (u^{2^{k-1}})^2\,dx. \tag{3.14}$$

Let

$$w = u^{2^{k-1}}\quad,\quad \delta_k = \delta\frac{2^k-1}{2^{k-1}}\ \text{ and }\ m_k = m2^{k-1}. \tag{3.15}$$

Thus, (3.14) can be written as

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega w^2\,dx\right) \le -\delta_k\int_\Omega |\nabla w|^2\,dx + m_k\int_\Omega w^2\,dx. \tag{3.16}$$

By Nirenberg-Gagliardo and Young inequalities, for any $\epsilon > 0$, there exists $C > 0$ such that

$$\int_\Omega w^2\,dx \le \epsilon\int_\Omega |\nabla w|^2\,dx + C_\epsilon\left(\int_\Omega w\,dx\right)^2, \tag{3.17}$$

where $C_\epsilon = \epsilon + C\epsilon^{-\frac{N}{2}}$. Indeed, since

$$\|w\|_{W^{k,p}} \le C\|w\|^{\theta}_{W^{m,q}}\|w\|^{(1-\theta)}_{L^r}$$

where $p \ge q,\quad p \ge r,\quad 0 \le \theta \le 1\quad$ and $\quad \theta \ge \frac{N}{2+N},\quad$ then

$$\|w\|_2^2 = \int_\Omega w^2\,dx = \|w\|^2_{W^{0,2}} \le C\|w\|^{2\theta}_{W^{1,2}}\|w\|^{2(1-\theta)}_{L^1}.$$

Changing $\|w\|^{2\theta}_{W^{1,2}}$ by $(\frac{\epsilon}{C})^{\frac{1}{p}}\|w\|^{2\theta}_{W^{1,2}}$, $\|w\|^{2(1-\theta)}_{L^1}$ by $\frac{\epsilon}{C}^{-\frac{1}{p}}\|w\|^{2(1-\theta)}_{L^1}$ and using the Young inequality we obtain

$$\begin{aligned}
C\|w\|^{2\theta}_{W^{1,2}}\|w\|^{2(1-\theta)}_{L^1} &\le C\left[\left(\left(\frac{\epsilon}{c}\right)^{\frac{1}{p}}\|w\|^{2\theta}_{W^{1,2}}\right)^p + \left(\left(\frac{\epsilon}{c}\right)^{-\frac{1}{p}}\|w\|^{2(1-\theta)}_{L^1}\right)^q\right]\\
&= \epsilon\|w\|^{2\theta\frac{1}{\theta}}_{W^{1,2}} + C^{\frac{q}{p}}\epsilon^{-\frac{q}{p}}\|w\|^{2(1-\theta)\frac{1}{1-\theta}}_{L^1}\\
&= \epsilon\|w\|^2_{W^{1,2}} + C\epsilon^{-\frac{N}{2}}\|w\|^2_{L^1}\\
&= \epsilon[\|\nabla w\|^2_{L^2} + \|w\|^2_{L^1}] + C\epsilon^{-\frac{N}{2}}\|w\|^2_{L^1}\\
&= \epsilon\|\nabla w\|^2_{L^2} + \left(\epsilon + C\epsilon^{-\frac{N}{2}}\right)\|w\|^2_{L^1}
\end{aligned} \tag{3.18}$$

where $p = \frac{1}{\theta}$, $q = \frac{1}{1-\theta}$, $C_\epsilon = C\epsilon^{-\frac{N}{2}}$, $\theta = \frac{N}{2+N}$ and the equivalent norm $\|w\|_{W^{1,2}(\Omega)} = [\int_\Omega |\nabla w|^2\,dx + (\int_\Omega w\,dx)^2]^{\frac{1}{2}}$ on $W^{1,2}(\Omega)$ was used. Hence,

$$\|w\|_2 = \epsilon\|\nabla w\|^2_{L^2} + C_\epsilon\|w\|^2_{L^1}$$

where $C_\epsilon = \epsilon + C\epsilon^{-\frac{N}{2}}$, as claimed.

Replacing ϵ by ϵ_k in (3.17), multiplying each side of (3.17) by $(m_k + \epsilon_k)$ we have

$$-(m_k+\epsilon_k)\epsilon_k\int_\Omega |\nabla w|^2\,dx \le -(m_k+\epsilon_k)\int_\Omega w^2\,dx + (m_k+\epsilon_k)C_{\epsilon_k}\left(\int_\Omega w\,dx\right)^2. \tag{3.19}$$

Choosing ϵ_k such that $m_k\epsilon_k + \epsilon_k^2 \leq \delta_k$, it follows from (3.16) and (3.19) that

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_\Omega w^2\,dx\right) &\leq -\delta_k\int_\Omega |\nabla w|^2\,dx + m_k\int_\Omega w^2\,dx\\ &\leq -(m_k+\epsilon_k)\int_\Omega w^2\,dx + (m_k+\epsilon_k)C_{\epsilon_k}\left(\int_\Omega w\,dx\right)^2\\ &\quad + m_k\int_\Omega w^2\,dx\\ &\leq -\epsilon_k\int_\Omega w^2\,dx + (m_k+\epsilon_k)C_{\epsilon_k}\left[\mathrm{bound}_{t\geq 0}\int_\Omega w\,dx\right]^2,\end{aligned}$$

where $\mathrm{bound}_{t\geq 0}\int_\Omega w\,dx$ is an upper bound for $\int_\Omega w\,dx$ for all $t \geq 0$. By Gronwall's inequality, we have

$$\int_\Omega w^2\,dx \leq e^{-2\epsilon_k t}|\Omega|\,\|w(\cdot,0)\|^2_{L^\infty(\Omega)} + \frac{(m_k+\epsilon_k)C_{\epsilon_k}}{\epsilon_k}\left[\mathrm{bound}_{t\geq 0}\int_\Omega w\,dx\right]^2(1-e^{-2\epsilon_k t}),$$

for all $t \geq 0$, where $|\Omega|$ denotes the volume of Ω.

Replacing w by $u^{2^{k-1}}$ and assuming that $|\Omega| = 1$ for simplicity, we obtain

$$\begin{aligned}\int_\Omega u^{2^k}\,dx &\leq e^{-2\epsilon_k t}\|u^{2^{k-1}}(\cdot,0)\|^2_{L^\infty(\Omega)} + \frac{(m_k+\epsilon_k)C_{\epsilon_k}}{\epsilon_k}\left[\mathrm{bound}_{t\geq 0}\int_\Omega u^{2^{k-1}}\,dx\right]^2\\ &\leq \max\left\{\frac{(m_k+\epsilon_k)C_{\epsilon_k}}{\epsilon_k}\left[\mathrm{bound}_{t\geq 0}\int_\Omega u^{2^{k-1}}\,dx\right]^2,\quad \|u(\cdot,0)\|^{2^k}_{L^\infty(\Omega)}\right\},\end{aligned} \tag{3.20}$$

for all $t \geq 0$.

We can assume that $\frac{(m_k+\epsilon_k)C_{\epsilon_k}}{\epsilon_k} \geq 1$, for $k = 1, 2, \ldots$, and that the constant C bounding $\int_\Omega u(x,t)\,dx$ for $t \geq 0$ is also an upper bound for $\|u(\cdot,0)\|_{L^\infty(\Omega)}$. We claim that (3.20) implies

$$\int_\Omega u^{2^k}\,dx \leq \left(\frac{(m_k+\epsilon_k)C_{\epsilon_k}}{\epsilon_k}\right)^{2^0}\left(\frac{(m_{k-1}+\epsilon_{k-1})C_{\epsilon_{k-1}}}{\epsilon_{k-1}}\right)^{2^1}\cdots\left(\frac{(m_1+\epsilon_1)C_{\epsilon_1}}{\epsilon_1}\right)^{2^{k-1}}C^{2^k}, \tag{3.21}$$

for any $t \geq 0$. Indeed, introducing the notations $\int_\Omega u^{2^k}\,dx = X_k$ and $C_k = \frac{(m_k+\epsilon_k)C_{\epsilon_k}}{\epsilon_k} \geq 1$, we have $X_0 = \|u(\cdot,0)\|_{L^\infty(\Omega)} \leq C$ and inequalities (3.20) are equivalent to

$$\begin{aligned}X_1 &\leq \max\{C_1X_0^2,\quad C^2\}\\ X_2 &\leq \max\{C_2X_1^2,\quad C^{2^2}\} \leq \max\{C_2\max\{C_1^2X_0^{2^2}, C^{2^2}\},\quad C^{2^2}\}\\ &= \max\{\max\{C_2C_1^2X_0^{2^2}, C_2C^{2^2}\},\quad C^{2^2}\} = \max\{C_2C_1^2X_0^{2^2}, C_2C^{2^2}\}\end{aligned}$$

$$
\begin{aligned}
X_3 &\le \max\{C_3X_2^2,\ \ C^{2^3}\} \le \max\{\max\{C_3C_2^2C_1^{2^2}X_0^{2^3}, C_3C_2^2C^{2^3}\},\ \ C^{2^3}\}\\
&= \max\{C_3C_2^2C_1^{2^2}X_0^{2^3}, C_3C_2^2C^{2^3}\}\\
&\cdot\\
&\cdot\\
&\cdot\\
X_k &\le \max\{C_kC_{k-1}^2C_{k-2}^{2^2}\dots C_1^{2^{k-1}}X_0^{2^k},\ \ C_kC_{k-1}^2C_{k-2}^{2^2}\dots C_2^{2^{k-1}}C^{2^k}\}.
\end{aligned}
$$

Since $X_0 \le C$, we have $X_0^{2^k} \le C^{2^k}$, and therefore

$$X_k \le C_kC_{k-1}^2C_{k-2}^{2^2}\dots C_1^{2^{k-1}}C^{2^k},$$

as claimed.

We now choose ϵ_k as $\epsilon_k = \frac{2\delta}{m2^k}$. It is a simple matter to verify that $\epsilon_k(m_k+\epsilon_k) \le \delta_k$. With this choice, we have

$$
\begin{aligned}
\frac{(m_k+\epsilon_k)C_{\epsilon_k}}{\epsilon_k} &= \left(1+\frac{m_k}{\epsilon_k}\right)\left(\epsilon_k^{(N+2)/2}+C\right)\epsilon_k^{-N/2}\\
&= \left(1+\frac{m^2}{4\delta}2^{2k}\right)+\left(\epsilon_k^{(N+2)/2}+C\right)\epsilon_k^{-N/2}\\
&\le \frac{m^2}{2\delta}\ 2^{2k}\ 2C\ \left(\frac{m}{2\delta}\right)^{N/2}\ 2^{kN/2} \qquad (3.22)\\
&= \frac{m^2C}{\delta}\ \left(\frac{m}{2\delta}\right)^{N/2}\ 2^{k(N/2+2)}\\
&= 2^{k(\lambda+2)}\bar{m},
\end{aligned}
$$

where $\lambda = \frac{N}{2}$ and $\bar{m} = \frac{m^2C}{\delta}\ (\frac{m}{2\delta})^{N/2}$.

Hence, the right hand side of (3.21) becomes

$$
\begin{aligned}
\int_\Omega u^{2^k}\,dx &\le (2^{k(\lambda+2)}\overline{m})^{2^0}(2^{(k-1)(\lambda+2)}\overline{m})^{2^1}(2^{(k-2)(\lambda+2)}\overline{m})^{2^2}\\
&\quad\times\dots\times(2^{(k-(k-1))(\lambda+2)}\overline{m})^{2^{k-1}}\cdot C^{2^k} \qquad (3.23)\\
&= \bar{m}^{2^k-1}\cdot 2^{(\lambda+2)(k+2(k-1)+2^2(k-2)+\dots+2^{k-1}(k-(k-1)))}\cdot C^{2^k}\\
&= \bar{m}^{2^k-1}\cdot 2^{(\lambda+2)(-k+2^{k+1}-2)}\cdot C^{2^k}.
\end{aligned}
$$

Taking $\frac{1}{2^k}$ – power on both sides of (3.23) and taking limit for $k \to +\infty$, we obtain

$$\|u\|_{L^\infty(\Omega)} \le 2^{2(\lambda+2)}\bar{m}C, \qquad (3.24)$$

and the proof is complete.

Theorem 3.4 *Assume that the hypotheses of Proposition 3.3 hold. Then, there exists a constant $B > 0$ such that*

$$\limsup_{t\to\infty}\|u_i(t,\cdot)\|_{X^\alpha} < B.$$

Proof Let $u(t) = (u_0(t), u_1(t), \dots, u_n(t))$ be a solution of (2.8). By the variation of constant formula

$$u(t) = e^{-At}u(0) + \int_0^t e^{-A(t-s)}F(u(s))ds, \tag{3.25}$$

where A and F are as in section 2. Applying A^α on both sides of (3.25) we have

$$A^\alpha(u(t)) = A^\alpha e^{-At}u(0) + \int_0^t A^\alpha e^{-A(t-s)}F(u(s))ds.$$

From estimates gotten from Proposition (3.3), there exist constant C such that

$$\limsup_{t\to\infty} \|F(u(s))\|_p < C,$$

for all $p \geq 1$. Hence,

$$\begin{aligned}
\|u(t)\|_{X^\alpha} &= \|A^\alpha(u(t))\| \\
&\leq \|A^\alpha e^{-At}u(0)\|_p + \int_0^t \|A^\alpha e^{-A(t-s)}\|_X \|F(u(s))\|_p \, ds \qquad (3.26)\\
&\leq C_\alpha t^{-\alpha}e^{-\delta t}\|u(0)\|_p + \int_0^t CC_\alpha(t-s)^{-\alpha}e^{-\delta(t-s)} \, ds \\
&\leq C_\alpha t^{-\alpha}e^{-\delta t}\|u(0)\|_p + \int_0^\infty CC_\alpha r^{-\alpha}e^{-\delta r} \, dr,
\end{aligned}$$

which implies the result.

Corollary 3.5 *System (1.1), (1.2) generates a nonlinear semigroup $T(t) : X^\alpha_+ \to X^\alpha_+$ satisfying:*

i) *$T(t)$ is compact for $t > 0$;*
ii) *$T(t)$ is point dissipative;*
iii) *Orbit of bounded sets are bounded.*

As a consequence, system (1.1), (1.2) has a compact global attractor.

Proof Note that by the embedding

$$X^\alpha \quad \hookrightarrow \quad C^\nu(\Omega) \times \cdots \times C^\nu(\Omega),$$

we can replace the uniform norm of the Proposition 3.3 by the norm C^ν for some $\nu > 0$. On the other hand, C^ν is compactly embedded into $C(\overline{\Omega})$, so we have (i). Assertions (ii) and (iii) follow from Theorem 3.4. The last conclusion follows from results given in J. Hale's book [7].

4. Zip bifurcation in a reaction diffusion system

In this section we will examine the stability or instability of homogeneous equilibrium for the problem (1.1), (1.2), tat is, solutions independent on x. We show that system (1.1), (1.2) exhibits a *zip bifurcation* when the carrying capacity K is varied.

We consider the system (1.1), (1.2) and we assume that the nonnegative constants $a_1, a_2, \ldots, a_n$ satisfy

$$0 < a_n < a_{n-1} < \cdots < a_2 < a_1. \tag{4.27}$$

Except by the 'trivial' solutions $(S, u_1, \ldots, u_n) = (0, 0, \ldots, 0)$ and $(S, u_1, \ldots, u_n) = (K, 0, \ldots, 0)$, the system (1.1), (1.2) have solutions independent of x if and only if $m_i > d_i$ and the solutions of $f_i(S) = d_i/m_i$, $i = 1, 2, \ldots, n$, coincide, that is,

$$\frac{a_1 d_1}{m_1 - d_1} = \frac{a_2 d_2}{m_2 - d_2} = \cdots = \frac{a_n d_n}{m_n - d_n}. \tag{4.28}$$

Let $\lambda_i = \frac{a_i d_i}{m_i - d_i}$, for $i = 1, 2, \ldots, n$. From now on, we assume that $m_i > d_i$ and $\lambda_1 = \cdots = \lambda_n$. The common value of λ_i's will be denoted by λ: $\lambda_1 = \cdots = \lambda_n = \lambda$. It follows that 'non-trivial' homogeneous equilibrium points of (1.1), (1.2) belong to the $(n-1)$ dimensional manifolds H_K defined by

$$H_K = \left\{ (\lambda, \xi_1, \ldots, \xi_n) \in \mathbb{R}_+^{n+1} : \sum_{i=1}^{n} \frac{m_i \xi_i}{a_i + \lambda} = \gamma \left(1 - \frac{\lambda}{K}\right) \right\}. \tag{4.29}$$

The linearization of system (1.1), (1.2) at a point $(\lambda, \xi_1, \ldots, \xi_n) \in H_K$ is

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = \delta_0 \Delta S + \left(-\dfrac{\gamma\lambda}{K} + \lambda \displaystyle\sum_{i=1}^{n} \dfrac{m_i \xi_i}{(a_i + \lambda)^2} \right) S - \displaystyle\sum_{i=1}^{n} \dfrac{m_i \lambda}{a_i + \lambda} u_i & \text{in } \Omega \times (0, \infty) \\ \dfrac{\partial u_i(x,t)}{\partial t} = \delta_i \Delta u_i + \dfrac{\beta_i \xi_i}{a_i + \lambda} S, \quad i = 1, 2, \ldots, n, & \text{in } \Omega \times (0, \infty) \end{cases} \tag{4.30}$$

with boundary conditions (1.2), where $\beta_i = m_i - d_i$, $i = 1, 2, \ldots, n$.

Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots \to \infty$ and $\{\psi_k\}_{k=0}^{\infty}$ be the eigenvalues and eigenfunctions of the Laplacian operator in Ω with Neumann boundary on $\partial\Omega$:

$$\begin{cases} \Delta \psi_k = \lambda_k \psi_k & \text{in } \Omega \\ \dfrac{\partial \psi_k}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.31}$$

We can assume that $\{\psi_k\}_{k=0}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$.

Let $w = (S, u_1, \ldots, u_n)$, $D = \operatorname{diag}(\delta_0, \delta_1, \ldots, \delta_n)$ and J be the matrix

$$J = \begin{pmatrix} \lambda \displaystyle\sum_{i=1}^{n} \dfrac{m_i \xi_i}{(a_i + \lambda)^2} - \dfrac{\gamma\lambda}{K} & -\dfrac{m_1 \lambda}{a_1 + \lambda} & -\frac{m_2 \lambda}{a_2 + \lambda} & \ldots & -\dfrac{m_{n-1} \lambda}{a_{n-1} + \lambda} & -\dfrac{m_n \lambda}{a_n + \lambda} \\ \dfrac{\beta_1 \xi_1}{a_1 + \lambda} & 0 & 0 & \ldots & 0 & 0 \\ \dfrac{\beta_2 \xi_2}{a_2 + \lambda} & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \dfrac{\beta_{n-1} \xi_{n-1}}{a_{n-1} + \lambda} & 0 & 0 & \ldots & 0 & 0 \\ \dfrac{\beta_n \xi_n}{a_n + \lambda} & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Then, (4.30), (1.2) can be written as

$$\frac{\partial w}{\partial t} = D \Delta w + J w,$$

so, the solution of (4.30), (1.2) with initial condition $w(\cdot, 0) = w_0$ is given by

$$w(x,t) = \sum_{k=0}^{\infty} e^{(J-\mu_k D)t} \langle w_0, \psi_k \rangle \psi_k(x), \tag{4.32}$$

where $\langle w_0, \psi_k \rangle = \int_\Omega w_0(x)\psi_k(x)\,dx$.

It follows from the linearization principle that a 'non-trivial' homogeneous solution of (1.1), (1.2) is asymptotically stable if the eigenvalues of all matrix $J - \mu_k D$ have negative real part; if there exists a $k \geq 1$ such that $J - \mu_k D$ has an eigenvalue with positive real part then the solution is unstable. Note that for $\mu_0 = 0$ and the matrix J has zero as an eigenvalue of multiplicity $(n-1)$ and two eigenvalues with positive (resp. negative) real part according to

$$\sum_{i=1}^{n} \frac{m_i \xi_i}{(a_i + \lambda)^2} > \frac{\gamma}{K} \quad \text{or} \quad < \frac{\gamma}{K}. \tag{4.33}$$

The following result is an immediate consequence of this note and the proof can be found in [6].

Proposition 4.1 *If $K > a_1 + 2\lambda$, then all equilibrium on H_K are unstable.*

4.1. The case $n = 2$

To simplify computations, from now on we consider $n = 2$ to study the persistence of the zip bifurcation phenomenon in the reaction diffusion system (1.1), (1.2). The corresponding ODE model was considered first by Farkas [5], where he studied the competition between two predators for a single prey, given by

$$\begin{cases} \dot{S} &= \gamma(1 - \frac{S}{K})S - m_1 f_1(S) x_1 - m_2 f_2(S) x_2 \\ \dot{x_1} &= m_1 f_1(S) x_1 - d_1 x_1 \\ \dot{x_2} &= m_2 f_2(S) x_2 - d_2 x_2, \end{cases} \tag{4.34}$$

where the functional response and the parameters have the same meaning as those of model (1.1) for the case $n = 2$. We keep the same notations and hypotheses as before. In the next result, we collect the main results of [4, 5].

Proposition 4.2 *If $\lambda < K < a_2 + 2\lambda$, each equilibrium $(\lambda, \xi_1, \xi_2) \in H_K$ of (4.34) is asymptotically stable. If $K > a_1 + 2\lambda$, (λ, ξ_1, ξ_2) is unstable.*

When $a_2 + 2\lambda < K < a_1 + 2\lambda$, it also follows from [4] (see also [6]) that there exist a point $(\lambda, \xi_1(K), \xi_2(K)) \in H_K$ such that the equilibrium on $H_U = \{(\lambda, \xi_1, \xi_2) \in H_K : \xi_1 < \xi_1(K)\}$ are unstable for the flow of (4.34) and the equilibrium on $H_S = \{(\lambda, \xi_1, \xi_2) \in H_K : \xi_1 > \xi_1(K)\}$ are stable. The point $(\xi_1(K), \xi_2(K))$ is obtained solving the system

$$\begin{cases} \dfrac{m_1 \xi_1}{a_1 + \lambda} + \dfrac{m_2 \xi_2}{a_2 + \lambda} &= \dfrac{\gamma(K - \lambda)}{K} \\[2ex] \dfrac{m_1 \xi_1}{(a_1 + \lambda)^2} + \dfrac{m_2 \xi_2}{(a_2 + \lambda)^2} &= \dfrac{\gamma}{K}. \end{cases} \tag{4.35}$$

Now, if E is an unstable equilibrium for the flow of (4.34) then E is also an unstable homogeneous equilibrium for the flow of (1.1), (1.2), since the subspace of the function independent of x is invariant for the flow of (1.1), (1.2). We will show that if E is stable for the flow of (4.34) then E is also stable for the flow of (1.1), (1.2) *independently* of the diffusion diagonal matrix $D = \mathrm{diag}(\delta_0, \delta_1, \delta_2)$. This fact is surprisingly and shows that the zip phenomenon is preserved by the introduction of diffusion in system (4.34).

Theorem 4.3 *Suppose that E is an equilibrium of (1.1), (1.2) independent of x. If E is stable for the flow of (4.34), then E is stable for the flow of (1.1), (1.2), independently of $D = \mathrm{diag}(\delta_0, \delta_1, \delta_2)$.*

Proof Let $E = (\lambda, \xi_1, \xi_2)$ be an equilibrium of (1.1), (1.2) independent of x. Then $\lambda < K$ and (ξ_1, ξ_2) satisfies

$$\frac{m_1\xi_1}{a_1+\lambda} + \frac{m_2\xi_2}{a_2+\lambda} = \frac{\gamma(K-\lambda)}{K}.$$

The hypothesis that E is stable for (4.34) is equivalent to

$$\frac{m_1\xi_1}{(a_1+\lambda)^2} + \frac{m_2\xi_2}{(a_2+\lambda)^2} < \frac{\gamma}{K} \tag{4.36}$$

(see [4] or [6] for detail).

Let us to write the characteristic polynomial $P_k(\nu)$ of $J - \mu_k D$, where J is the Jacobian matrix of (4.34) evaluated at E. To simplify the computations, we introduce the notations

$$a = \lambda \sum_{i=1}^{2} \frac{m_i\xi_i}{(a_i+\lambda)^2} - \frac{\gamma\lambda}{K}, \quad b = \frac{\beta_1\xi_1}{a_1+\lambda}, \quad c = \frac{\beta_2\xi_2}{a_2+\lambda}, \quad d = -\frac{m_1\lambda}{a_1+\lambda}$$

$e = -\frac{m_2\lambda}{a_2+\lambda}$, so that the condition (4.36) means that $a < 0$. Therefore, for each $k \geq 0$, the eigenvalues of $J - \mu_k D$ are the roots of the equation

$$P_k(\nu) = \nu^3 + A_k\nu^2 + B_k\nu + C_k, \tag{4.37}$$

where

$$\begin{aligned}
A_k &= \mu_k(\delta_0+\delta_1+\delta_2) - a,\\
B_k &= \mu_k^2(\delta_0\delta_1+\delta_0\delta_2+\delta_1\delta_2) - a\mu_k(\delta_1+\delta_2) - bd - ec\\
C_k &= \mu_k^3\delta_0\delta_1\delta_2 - \mu_k^2 a\delta_1\delta_2 - \mu_k(ec\delta_1 + bd\delta_2).
\end{aligned}$$

Since $a < 0$, $bd < 0$ and $ec < 0$, we have $A_k > 0$, $B_k > 0$ and $C_k > 0$, for all $k \geq 1$. At $k = 0$, the polynomial

$$P_0(\nu) = \nu^3 - a\nu^2 - (bd+ec)\nu,$$

has zero as a single eigenvalue and two complex conjugate with negative real part. We claim that, for any $k \geq 1$, all eigenvalues of $J - \mu_k D$ have negative real part. This will follows from Routh-Hurwitz's criterion if we show that $A_k B_k > C_k$, for any $k \geq 1$. To prove this, we consider the function $AB - C = (AB - C)(\mu)$ given by

$$\begin{aligned}(AB-C)(\mu) = \mu^3&[2\delta_0\delta_1\delta_2 + \delta_0^2(\delta_1+\delta_2) + \delta_1^2(\delta_0+\delta_2) + \delta_2^2(\delta_0+\delta_1)] \\ &- a\mu^2[\delta_0\delta_1 + \delta_0\delta_2 + (\delta_0+\delta_1+\delta_2)(\delta_1+\delta_2)] \\ &- \mu[\delta_0(bd+ec) + \delta_1 bd + \delta_2 ec \\ &- a^2(\delta_1+\delta_2)] + a(bd+ec),\end{aligned} \tag{4.38}$$

obtained from the expression of $A_k B_k - C_k$, changing μ_k by μ.

Since $a < 0$, $ec < 0$ and $bd < 0$, the coefficients of (4.38) are all positive, and therefore $(AB - C)(\mu)$ is positive for any nonnegative values of μ. Taking into account that $\mu_k > 0$ for any $k \geq 1$, the claim is proved. Thus, for any $k \geq 1$, we have $A_k B_k - C_k > 0$ therefore, the roots of $P_k(\nu)$ has negative real part.

On the other hand, taking $\nu = \mu_k z$, the roots of $P_k(\nu) = 0$ are roots of the equation

$$z^3 + \frac{A_k}{\mu_k} z^2 + \frac{B_k}{\mu_k^2} z + \frac{C_k}{\mu_k^3} = 0. \tag{4.39}$$

When $k \to \infty$, the coefficients of (4.39) have limit and the limiting equation is given by

$$z^3 + (\delta_0+\delta_1+\delta_2)z^2 + (\delta_0\delta_1 + \delta_0\delta_2 + \delta_1\delta_2)z + \delta_0\delta_1\delta_2 = 0. \tag{4.40}$$

If z_1, z_2 e z_3 are roots of (4.40), then $\mathrm{Re} z_j < 0$, $j = 1, 2, 3$. By Rouché's Theorem, there exist neighborhoods V_1, V_2 and V_3 of z_1, z_2 and z_3, respectively, mutually disjoint, and $k_0 \geq 1$ such that if $k \geq k_0$, the polynomial (4.39) has a root $z_k^{(j)}$ in each neighborhood V_j. Since zero is a simple eigenvalue of J and the roots of (4.37) are $\nu_{k,j} = \mu_k z_k^{(j)}$, for $j = 1, 2, 3$, $k \geq 1$ and $\mu_k \geq 0$, there exists $\alpha > 0$ such that the remainder of the spectrum of the operator in (4.30) belongs to $\{\nu : \mathrm{Re}\,\nu < -\alpha\}$. Therefore, all the hypotheses of Henry (see [8] p. 108) are satisfied, and therefore E is stable.

As a consequence, if $\lambda < K < a_2 + 2\lambda$, the hyperplane of equilibrium points H_K of (4.34) is asymptotically stable for the flow of (1.1), (1.2). If $K > a_1 + 2\lambda$, then H_K is unstable. If $a_2 + 2\lambda < K < a_1 + 2\lambda$, let $(\xi_1(K), \xi_2(K))$ be the unique solution of system (4.35). Then it holds

Theorem 4.4 *For any K satisfying $a_2+2\lambda \leq K \leq a_1+2\lambda$, the point $(\lambda, \xi_1(K), \xi_2(K))$ split H_K in two parts H_K^u and H_K^s; the equilibrium of (1.1), (1.2) on the set*

$$H_K^u = \{(\lambda, \xi_1, \xi_2) \in H_K : \xi_1 < \xi_1(K)\}$$

are unstable and on the set

$$H_K^s = \{(\lambda, \xi_1, \xi_2) \in H_K : \xi_1 > \xi_1(K)\}$$

are stable, independently of the diffusion matrix $D = \mathrm{diag}(\delta_0, \delta_1, \delta_2)$.

This final result shows that the zip bifurcation phenomenon can be preserved by the introduction of a diagonal diffusion matrix D in the model (4.34), independent of the domain Ω. Moreover, Theorem 4.3 shows that Turing instability may occurs only in models with crossed diffusion and/or very special domains Ω.

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