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ORIGINAL PAPER

An Approach Between the Multiplicative and Additive Structure of a Jordan Ring

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Abstract

Let $\mathfrak{J}$ and $\mathfrak{J}^{'}$ be Jordan rings. In this paper we study the additivity of n-multiplicative isomorphisms from $\mathfrak{J}$ onto $\mathfrak{J}^{'}$ and of n-multiplicative derivations of $\mathfrak{J}$. Suppose that $\mathfrak{J}$ contains a nontrivial idempotent; we prove that if $\mathfrak{J}$ satisfying certain conditions, then n-multiplicative maps and n-multiplicative derivations from $\mathfrak{J}$ to $\mathfrak{J}^{'}$ are additive maps.

Keywords n-Multiplicative maps · n-Multiplicative derivations · Additivity · Jordan rings

Mathematics Subject Classification 17D99 · 17A36

1 Introduction

We will use $(x, y, z) = (xy)z - x(yz)$ and $[x, y] = xy - yx$ to denote the associator of elements x, y, z and the commutator of elements x, y in a not necessarily associative ring $\mathfrak{J}$.

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For those readers who are not familiar with this language, we recommend [15]. According to [3], "let $X = \{x_i\}_{i\in\mathbb{N}}$ be an arbitrary set of variables. A *nonassociative monomial of degree* 1 is any element of X. Given a natural number $n > 1$, a *nonassociative monomial of degree* n is an expression of the form $(u)(v)$, where u is a nonassociative monomial of some degree i and v a nonassociative monomial of degree $n - i$.

Let $\mathfrak{J}$ and $\mathfrak{J}'$ be two rings and $\varphi : \mathfrak{J} \longrightarrow \mathfrak{J}'$ a bijective map of $\mathfrak{J}$ onto $\mathfrak{J}'$. We call φ an $n-$*multiplicative map* of $\mathfrak{J}$ onto $\mathfrak{J}'$ if for all nonassociative monomials $m = m(x_1, \ldots, x_n)$ of degree n:

$$\varphi(m(x_1, \ldots, x_n)) = m(\varphi(x_1), \ldots, \varphi(x_n))$$

for all $x_1, \ldots, x_n \in \mathfrak{J}$. If $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathfrak{J}$, we just say that φ is a *multiplicative map*. And if $\varphi(xyx) = \varphi(x)\varphi(y)\varphi(x)$ for all $x, y \in \mathfrak{J}$, then we call φ a *Jordan semi-triple multiplicative map*.

Similarly, a map $d : \mathfrak{J} \longrightarrow \mathfrak{J}$ is called a *n-multiplicative derivation* of $\mathfrak{J}$ if

$$d(m(x_1, \ldots, x_n)) = \sum_{i=1}^{n} m(x_1, \ldots, d(x_i), \ldots, x_n)$$

for all nonassociative monomials $m = m(x_1, \ldots, x_n)$ of degree n and arbitrary elements $x_1, \ldots, x_n \in \mathfrak{J}$.

If $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathfrak{J}$, we just say that d is a *multiplicative derivation* of $\mathfrak{J}$". And if $d(xyx) = d(x)(yx) + xd(y)x + (xy)d(x)$ for all $x, y \in \mathfrak{J}$, then we call d a *Jordan triple multiplicative derivation*.

A ring $\mathfrak{J}$ is said to be *Jordan* if $(x^2, y, x) = 0$ and $[x, y] = 0$ for all $x, y \in \mathfrak{J}$. A Jordan ring $\mathfrak{J}$ is called *k-torsion free* if $kx = 0$ implies $x = 0$, for any $x \in \mathfrak{J}$, where k is a positive integer.

A nonzero element $e \in \mathfrak{J}$ is called an *idempotent* if $e^2 = ee = e$ and a *nontrivial idempotent* if it is a nonzero idempotent and different from the multiplicative identity element of $\mathfrak{J}$.

Let us consider a Jordan ring $\mathfrak{J}$ with a nontrivial idempotent e.

Let $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_{\frac{1}{2}} \oplus \mathfrak{J}_0$ be the Peirce decomposition of $\mathfrak{J}$ with respect to e, where $\mathfrak{J}_i = \{x_i \mid ex_i = ix_i\}$, $i = 0, \frac{1}{2}, 1$, satisfying the following multiplicative relations:

$$\mathfrak{J}_0\mathfrak{J}_0 \subseteq \mathfrak{J}_0; \quad \mathfrak{J}_1\mathfrak{J}_1 \subseteq \mathfrak{J}_1; \quad \mathfrak{J}_1\mathfrak{J}_0 = 0; \quad (\mathfrak{J}_1 \oplus \mathfrak{J}_0)\mathfrak{J}_{\frac{1}{2}} \subseteq \mathfrak{J}_{\frac{1}{2}};$$
$$\mathfrak{J}_{\frac{1}{2}}\mathfrak{J}_{\frac{1}{2}} \subseteq \mathfrak{J}_1 \oplus \mathfrak{J}_0.$$

In studying preservers on algebras or rings, one usually assumes additivity in advance. Recently, however, a growing number of papers began investigating preservers that are not necessarily additive, characterizing the interrelation between the multiplicative and additive structures of a ring or algebra in an interesting topic. The first result about the additivity of maps on rings was given by Martindale III [13]. He established a condition on a ring $\mathfrak{R}$, such that every multiplicative isomorphism on

$\mathfrak{R}$ is additive. Besides, over the years, several works have been published considering different types of maps on non-associative rings or algebras among them we can mention [1–12]. Ferreira and Ferreira [3] also considered this question in the context of $n-$multiplicative maps on alternative rings satisfying Martindale's conditions. Ferreira and Nascimento proved the additivity of multiplicative derivations [12]. Motivated by all these results mentioned above, the present paper considers a similar Ferreira and Ferreira's problems [3] in the context of Jordan rings. We investigate the problem of when a n-multiplicative isomorphism and a n-multiplicative derivation are additive for the class of Jordan rings.

2 n-Multiplicative Isomorphism

It will be convenient for us to change notation at this point. Henceforth, the ring $\mathfrak{J}$ will be 2-torsion free and the nonassociative monomial m of degree n will be an expression of the form:

$$m(x_1, x_2, \ldots, x_{n-1}, x_n) := x_1(x_2(\ldots(x_{n-1}x_n)\ldots)).$$

When the first i variables in the non-associative monomial m assume equal values, we will denote by:

$$m(\underbrace{z, z, \ldots, z}_{i}, x_{i+1}, \ldots, x_n) := \xi_z(x_{i+1}, \ldots, x_n).$$

The main technique which we will use is the following argument which will be termed a "standard argument". Suppose, $x, y, s \in \mathfrak{J}$ are such that $\varphi(s) = \varphi(x) + \varphi(y)$. Multiplying this equality by $\varphi(t_i)$, $(i = 1, 2, \ldots, n-1)$, we get:

$$\begin{aligned}\varphi(t_1)(\varphi(t_2)(\ldots(\varphi(t_{n-1})\varphi(s))\ldots)) &= \varphi(t_1)(\varphi(t_2)(\ldots(\varphi(t_{n-1})\varphi(x))\ldots)) \\ &\quad + \varphi(t_1)(\varphi(t_2)(\ldots(\varphi(t_{n-1})\varphi(y))\ldots));\end{aligned}$$

then

$$\begin{aligned}m(\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_{n-1}), \varphi(s)) &= m(\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_{n-1}), \varphi(x)) \\ &\quad + m(\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_{n-1}), \varphi(y)).\end{aligned}$$

It follows that:

$$\varphi\big(m(t_1, t_2, \ldots, t_{n-1}, s)\big) = \varphi\big(m(t_1, t_2, \ldots, t_{n-1}, x)\big) + \varphi\big(m(t_1, t_2, \ldots, t_{n-1}, y)\big).$$

Moreover, if

$$\begin{aligned}&\varphi\big(m(t_1, t_2, \ldots, t_{n-1}, x)\big) + \varphi\big(m(t_1, t_2, \ldots, t_{n-1}, y)\big) \\ &\quad = \varphi\big(m(t_1, t_2, \ldots, t_{n-1}, x) + m(t_1, t_2, \ldots, t_{n-1}, y)\big),\end{aligned}$$

then by injectivity of φ, we have that:

$$m(t_1, t_2, \ldots, t_{n-1}, s) = m(t_1, t_2, \ldots, t_{n-1}, x) + m(t_1, t_2, \ldots, t_{n-1}, y).$$

The main result of this section reads as follows.

Theorem 2.1 *Let $\mathfrak{J}$ and $\mathfrak{J}'$ be Jordan rings and e a non-trivial idempotent in $\mathfrak{J}$. Let $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_{\frac{1}{2}} \oplus J_0$ be the Peirce decomposition of $\mathfrak{J}$ with respect to e. If $\mathfrak{J}$ satisfies the following conditions:*

(i) *Let $a_i \in \mathfrak{J}_i (i = 1, 0)$. If $t_{\frac{1}{2}} a_i = 0$ for all $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$, then $a_i = 0$;*
(ii) *Let $a_0 \in \mathfrak{J}_0$. If $t_0 a_0 = 0$ for all $t_0 \in \mathfrak{J}_0$, then $a_0 = 0$;*
(iii) *Let $a_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$. If $t_0 a_{\frac{1}{2}} = 0$ for all $t_0 \in \mathfrak{J}_0$, then $a_{\frac{1}{2}} = 0$.*

Then every n-multiplicative isomorphism from $\mathfrak{J}$ onto $\mathfrak{J}'$ is additive.

The proof is organized in a series of Lemmas.

Lemma 2.2 $\varphi(0) = 0$.

Proof Since φ is surjective, there exists $x \in \mathfrak{J}$, such that $\varphi(x) = 0$. Therefore, $\varphi(\xi_0(x)) = \xi_{\varphi(0)}(\varphi(x)) = \xi_{\varphi(0)}(0) = 0$. □

Lemma 2.3 *Let $a_i \in \mathfrak{J}_i$, $i = 1, \frac{1}{2}, 0$. Then,* $\varphi(a_1 + a_{\frac{1}{2}} + a_0) = \varphi(a_1) + \varphi(a_{\frac{1}{2}}) + \varphi(a_0)$.

Proof Since φ is surjective, we can find an element $s = s_1 + s_{\frac{1}{2}} + s_0 \in \mathfrak{J}$, such that:

$$\varphi(s) = \varphi(a_1) + \varphi\left(a_{\frac{1}{2}}\right) + \varphi(a_0). \tag{2.1}$$

For e, applying a standard argument to (2.1), we get:

$$\begin{aligned}\varphi\big(\xi_{2e}(s)\big) &= \varphi\big(\xi_{2e}(a_1)\big) + \varphi\big(\xi_{2e}\big(a_{\frac{1}{2}}\big)\big) + \varphi\big(\xi_{2e}(a_0)\big) \\ &= \varphi(2^{n-1}a_1) + \varphi\big(a_{\frac{1}{2}}\big) + \varphi(0) = \varphi(2^{n-1}a_1) + \varphi\big(a_{\frac{1}{2}}\big).\end{aligned}$$

Since $\varphi(\xi_{2e}(s)) = \varphi(2^{n-1}s_1 + s_{\frac{1}{2}})$, we have:

$$\varphi\left(2^{n-1}s_1 + s_{\frac{1}{2}}\right) = \varphi(2^{n-1}a_1) + \varphi\left(a_{\frac{1}{2}}\right). \tag{2.2}$$

Now, for $t_0 \in \mathfrak{J}_0$ applying the standard argument to (2.2), we have:

$$\begin{aligned}\varphi\big(\xi_{2e}(t_0, 2^{n-1}s_1 + s_{\frac{1}{2}})\big) &= \varphi\big(\xi_{2e}(t_0, a_1)\big) + \varphi\big(\xi_{2e}\big(t_0, a_{\frac{1}{2}}\big)\big) \\ &= \varphi\big(t_0 a_{\frac{1}{2}}\big).\end{aligned}$$

Hence:

$$\varphi\left(t_0 s_{\frac{1}{2}}\right) = \varphi\left(t_0 a_{\frac{1}{2}}\right).$$

Therefore, $t_0 s_{\frac{1}{2}} = t_0 a_{\frac{1}{2}}$ for every $t_0 \in \mathfrak{J}_0$. It follows from the item (iii) of Theorem 2.1 that $s_{\frac{1}{2}} = a_{\frac{1}{2}}$. With a similar argument, we can show that $s_1 = a_1$.

Now, it only remains show that $s_0 = a_0$. For this, let $t_0 \in \mathfrak{J}_0$; applying the standard argument to (2.1), we get:

$$\begin{aligned}
&\varphi\big(\xi_{t_0}(s)\big)\\
&\quad= \varphi\big(\xi_{t_0}(a_1)\big) + \varphi\big(\xi_{t_0}\big(a_{\frac{1}{2}}\big)\big) + \varphi\big(\xi_{t_0}(a_0)\big)\\
&\quad= \varphi(0) + \varphi\big(\xi_{t_0}\big(a_{\frac{1}{2}}\big)\big) + \varphi\big(\xi_{t_0}(a_0)\big)\\
&\quad= \varphi\big(\xi_{t_0}\big(a_{\frac{1}{2}}\big)\big) + \varphi\big(\xi_{t_0}(a_0)\big).
\end{aligned} \tag{2.3}$$

For $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$, applying the standard argument to (2.3), we have:

$$\begin{aligned}
&\varphi\big(\xi_{2e}\big(t_{\frac{1}{2}}, \xi_{t_0}(s)\big)\big)\\
&\quad= \varphi\big(\xi_{2e}\big(t_{\frac{1}{2}}, \xi_{t_0}\big(a_{\frac{1}{2}}\big)\big)\big) + \varphi\big(\xi_{2e}\big(t_{\frac{1}{2}}, \xi_{t_0}(a_0)\big)\\
&\quad= \varphi\big(2^{n-2}\big(t_{\frac{1}{2}}\xi_{t_0}\big(a_{\frac{1}{2}}\big)\big)_1\big) + \varphi\big(t_{\frac{1}{2}}\xi_{t_0}(a_0)\big),
\end{aligned} \tag{2.4}$$

where $(t_{\frac{1}{2}}\xi_{t_0}(a_{\frac{1}{2}}))_1 \in \mathfrak{J}_1$ and $t_{\frac{1}{2}}\xi_{t_0}(a_0) \in \mathfrak{J}_{\frac{1}{2}}$. Now, for $t_0^{'} \in \mathfrak{J}_0$, applying the standard argument to (2.4), we have that:

$$\begin{aligned}
\varphi\big(\xi_{2e}\big(t_0', \xi_e\big(t_{\frac{1}{2}}, \xi_{t_0}(s)\big)\big)\big) &= \varphi\big(\xi_{2e}\big(t_0', 2^{n-2}\big(t_{\frac{1}{2}}\xi_{t_0}\big(a_{\frac{1}{2}}\big)\big)_1\big)\big) + \varphi\big(\xi_{2e}\big(t_0', t_{\frac{1}{2}}\xi_{t_0}(a_0)\big)\big)\\
&= \varphi(0) + \varphi\big(\xi_{2e}\big(t_0', t_{\frac{1}{2}}\xi_{t_0}(a_0)\big)\big)\\
&= \varphi\big(\xi_{2e}\big(t_0', t_{\frac{1}{2}}\xi_{t_0}(a_0)\big)\big) = \varphi\big(t_0'\big(t_{\frac{1}{2}}\xi_{t_0}(a_0)\big)\big).
\end{aligned}$$

Since $\xi_{2e}(t_0', \xi_e(t_{\frac{1}{2}}, \xi_{t_0}(s))) = t_0'(t_{\frac{1}{2}}\xi_{t_0}(s_0))$, we have:

$$t_0'\left(t_{\frac{1}{2}}\xi_{t_0}(s_0)\right) = t_0'\left(t_{\frac{1}{2}}\xi_{t_0}(a_0)\right).$$

It follows from the items (i), (ii), and (iii) of Theorem 2.1 that $s_0 = a_0$. Thus, $s = a_1 + a_{\frac{1}{2}} + a_0$. ☐

Lemma 2.4 *Let $a_{\frac{1}{2}}, b_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$ and $a_0 \in \mathfrak{J}_0$. Then, $\varphi\left(a_{\frac{1}{2}}a_0 + b_{\frac{1}{2}}\right) = \varphi\left(a_{\frac{1}{2}}a_0\right) + \varphi\left(b_{\frac{1}{2}}\right)$.*

Proof We note that:

$$\xi_{2e}\left(2e + a_{\frac{1}{2}}, a_0 + b_{\frac{1}{2}}\right) = \xi_{2e}\left(2e, a_{\frac{1}{2}}a_0 + b_{\frac{1}{2}}\right) + \xi_{2e}\left(b_{\frac{1}{2}}, a_{\frac{1}{2}}\right).$$

Using Lemma 2.3, we have: $\varphi(\xi_{2e}\left(2e + a_{\frac{1}{2}}, a_0 + b_{\frac{1}{2}})\right) = \varphi(\xi_{2e}\left(2e, a_{\frac{1}{2}}a_0 + b_{\frac{1}{2}})\right) + \varphi(\xi_{2e}\left(b_{\frac{1}{2}}, a_{\frac{1}{2}})\right)$, because $\xi_{2e}(2e, a_{\frac{1}{2}}a_0 + b_{\frac{1}{2}}) \in \mathfrak{J}_{\frac{1}{2}}$ and $\xi_{2e}(b_{\frac{1}{2}}, a_{\frac{1}{2}}) \in \mathfrak{J}_1$. Consequently, by Lemmas 2.2 and 2.3, we have:

$$\begin{aligned}
&\varphi\left(\xi_{2e}\left(2e, a_{\frac{1}{2}}a_0 + b_{\frac{1}{2}}\right)\right) + \varphi\left(\xi_{2e}\left(a_{\frac{1}{2}}, b_{\frac{1}{2}}\right)\right)\\
&= \varphi\left(\xi_{2e}\left(2e + a_{\frac{1}{2}}, a_0 + b_{\frac{1}{2}}\right)\right)\\
&= \xi_{\varphi(2e)}\left(\varphi\left(2e + a_{\frac{1}{2}}\right), \varphi\left(a_0 + b_{\frac{1}{2}}\right)\right)\\
&= \xi_{\varphi(2e)}\left(\varphi(2e) + \varphi\left(a_{\frac{1}{2}}\right), \varphi(a_0) + \varphi\left(b_{\frac{1}{2}}\right)\right)\\
&= \xi_{\varphi(2e)}(\varphi(2e) + \varphi(a_0)) + \xi_{\varphi(2e)}(\varphi(2e), \varphi\left(b_{\frac{1}{2}})\right)\\
&\quad + \xi_{\varphi(2e)}\left(\varphi(a_{\frac{1}{2}}), \varphi(a_0)\right)\\
&\quad + \xi_{\varphi(2e)}\left(\varphi\left(a_{\frac{1}{2}}\right), \varphi\left(b_{\frac{1}{2}}\right)\right)\\
&= \varphi(0) + \varphi\left(\xi_{2e}(2e, b_{\frac{1}{2}})\right)\\
&\quad + \varphi\left(\xi_{2e}\left(a_{\frac{1}{2}}, a_0\right)\right) + \varphi\left(\xi_{2e}\left(a_{\frac{1}{2}}, b_{\frac{1}{2}}\right)\right).
\end{aligned}$$

Thus, $\varphi(\xi_{2e}(2e, a_{\frac{1}{2}}a_0 + b_{\frac{1}{2}})) = \varphi(\xi_{2e}(2e, b_{\frac{1}{2}})) + \varphi(\xi_{2e}(a_{\frac{1}{2}}, a_0))$, that is $\varphi(a_{\frac{1}{2}}a_0 + b_{\frac{1}{2}}) = \varphi(a_{\frac{1}{2}}a_0) + \varphi(b_{\frac{1}{2}})$. □

Lemma 2.5 *Let $a_{\frac{1}{2}}, b_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$. Then, $\varphi(a_{\frac{1}{2}} + b_{\frac{1}{2}}) = \varphi(a_{\frac{1}{2}}) + \varphi(b_{\frac{1}{2}})$.*

Proof Let $s = s_1 + s_{\frac{1}{2}} + s_0 \in \mathfrak{J}$, such that:

$$\varphi(s) = \varphi\left(a_{\frac{1}{2}}\right) + \varphi\left(b_{\frac{1}{2}}\right). \tag{2.5}$$

For $t_0 \in \mathfrak{J}_0$, applying the standard argument to (2.5) and using Lemma 2.4, we have:

$$\varphi(\xi_{t_0}(s, t_0)) = \varphi\left(\xi_{t_0}\left(a_{\frac{1}{2}}, t_0\right)\right) + \varphi\left(\xi_{t_0}\left(b_{\frac{1}{2}}, t_0\right)\right) = \varphi\left(\xi_{t_0}\left(a_{\frac{1}{2}}, t_0\right) + \xi_{t_0}\left(b_{\frac{1}{2}}, t_0\right)\right).$$

Hence, $\xi_{t_0}(s, t_0) = \xi_{t_0}(a_{\frac{1}{2}}, t_0) + \xi_{t_0}(b_{\frac{1}{2}}, t_0)$. By (ii) and (iii) of Theorem 2.1, we obtain that $s_0 = 0$ and $s_{\frac{1}{2}} = a_{\frac{1}{2}} + b_{\frac{1}{2}}$. Now, for $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$ and $e \in \mathfrak{J}_1$, applying the standard argument to (2.5) again, we have:

$$\begin{aligned}
&\varphi\left(\xi_{2e}\left(s_1, t_{\frac{1}{2}}\right) + \xi_{2e}\left(s_{\frac{1}{2}}, t_{\frac{1}{2}}\right)\right)\\
&= \varphi\left(\xi_{2e}\left(s, t_{\frac{1}{2}}\right)\right) = \varphi\left(\xi_{2e}\left(a_{\frac{1}{2}}, t_{\frac{1}{2}}\right)\right) + \varphi\left(\xi_{2e}\left(b_{\frac{1}{2}}, t_{\frac{1}{2}}\right)\right).
\end{aligned} \tag{2.6}$$

For u_0, applying the standard argument to (2.6), we get that:

$$\begin{aligned}\varphi\left(\xi_{u_0}\left(\xi_{2e}\left(s_1, t_{\frac{1}{2}}\right)\right)\right) &= \varphi\left(\xi_{u_0}\left(\xi_{2e}\left(a_{\frac{1}{2}}, t_{\frac{1}{2}}\right)\right)\right) + \varphi\left(\xi_{u_0}\left(\xi_{2e}\left(b_{\frac{1}{2}}, t_{\frac{1}{2}}\right)\right)\right) \\ &= \varphi(0) + \varphi(0) = 0.\end{aligned}$$

Hence, $\xi_{u_0}(\xi_{2e}(s_1, t_{\frac{1}{2}})) = 0$ for every $u_0 \in \mathfrak{J}_0$ and $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$, it follows from the items (iii) and (i) of Theorem 2.1 that $s_1 = 0$. Thus, $s = s_{\frac{1}{2}} = a_{\frac{1}{2}} + b_{\frac{1}{2}}$. □

Lemma 2.6 *Let $a_1, b_1 \in \mathfrak{J}_1$. Then, $\varphi(a_1 + b_1) = \varphi(a_1) + \varphi(b_1)$.*

Proof Let $s = s_1 + s_{\frac{1}{2}} + s_0 \in \mathfrak{J}$, such that:

$$\varphi(s) = \varphi(a_1) + \varphi(b_1). \tag{2.7}$$

For $t_0 \in \mathfrak{J}_0$, applying the standard argument to (2.7), we have:

$$\begin{aligned}\varphi\left(\xi_{t_0}\left(s_{\frac{1}{2}}, t_0\right) + \xi_{t_0}(s_0, t_0)\right) &= \varphi(\xi_{t_0}(s, t_0)) = \varphi(\xi_{t_0}(a_1, t_0)) + \varphi(\xi_{t_0}(b_1, t_0)) \\ &= \varphi(0) + \varphi(0) = 0.\end{aligned}$$

Thus, $\xi_{t_0}(s_{\frac{1}{2}}, t_0) + \xi_{t_0}(s_0, t_0) = 0$ for every $t_0 \in J_0$. Since $\xi_{t_0}(s_{\frac{1}{2}}, t_0) \in \mathfrak{J}_{\frac{1}{2}}$ and $\xi_{t_0}(s_0, t_0) \in \mathfrak{J}_0$, we get that $\xi_{t_0}(s_{\frac{1}{2}}, t_0) = 0$ and $\xi_{t_0}(s_0, t_0) = 0$. By (ii) and (iii) of Theorem 2.1, we obtain that $s_{\frac{1}{2}} = 0$ and $s_0 = 0$. Now, for $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$ and $e \in \mathfrak{J}_1$, applying the standard argument to (2.7) again and using Lemma 2.5, we have:

$$\begin{aligned}\varphi\left(\xi_{2e}\left(s_1, t_{\frac{1}{2}}\right)\right) &= \varphi\left(\xi_{2e}\left(s, t_{\frac{1}{2}}\right)\right) \\ &= \varphi\left(\xi_{2e}\left(a_1, t_{\frac{1}{2}}\right)\right) + \varphi\left(\xi_{2e}\left(b_1, t_{\frac{1}{2}}\right)\right) \\ &= \varphi\left(\xi_{2e}\left(a_1, t_{\frac{1}{2}}\right) + \xi_{2e}\left(b_1, t_{\frac{1}{2}}\right)\right).\end{aligned} \tag{2.8}$$

Hence, $\xi_{2e}(s_1, t_{\frac{1}{2}}) = \xi_{2e}(a_1, t_{\frac{1}{2}}) + \xi_{2e}(b_1, t_{\frac{1}{2}})$ for every $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$, it follows from the item (i) of Theorem 2.1 that $s_1 = a_1 + b_1$. Thus, $s = s_1 = a_1 + b_1$. □

Lemma 2.7 *Let $a_0, b_0 \in \mathfrak{J}_0$. Then, $\varphi(a_0 + b_0) = \varphi(a_0) + \varphi(b_0)$.*

Proof Let $s = s_1 + s_{\frac{1}{2}} + s_0 \in \mathfrak{J}$, such that:

$$\varphi(s) = \varphi(a_0) + \varphi(b_0). \tag{2.9}$$

For $e \in \mathfrak{J}_1$, applying the standard argument to (2.9), we have:

$$\begin{aligned}\varphi\left(\xi_{2e}\left(s_{\frac{1}{2}}, 2e\right) + \xi_{2e}(s_1, 2e)\right) &= \varphi(\xi_{2e}(s, 2e)) = \varphi(\xi_{2e}(a_0, 2e)) + \varphi(\xi_{2e}(b_0, 2e)) \\ &= \varphi(0) + \varphi(0) = 0.\end{aligned}$$

Therefore, $s_{\frac{1}{2}} + 2^{n-1}s_1 = 0$. Hence, $s_{\frac{1}{2}} = 0$ and $s_1 = 0$. Now, for $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$ and $e \in \mathfrak{J}_1$, applying the standard argument to (2.9) again and using Lemma 2.5, we have:

$$\begin{aligned}\varphi\left(\xi_{2e}\left(s_0, t_{\frac{1}{2}}\right)\right) &= \varphi\left(\xi_{2e}\left(s, t_{\frac{1}{2}}\right)\right)\\ &= \varphi\left(\xi_{2e}\left(a_0, t_{\frac{1}{2}}\right)\right) + \varphi\left(\xi_{2e}\left(b_0, t_{\frac{1}{2}}\right)\right)\\ &= \varphi\left(\xi_{2e}\left(a_0, t_{\frac{1}{2}}\right) + \xi_{2e}\left(b_0, t_{\frac{1}{2}}\right)\right). \end{aligned} \tag{2.10}$$

Hence, $\xi_{2e}(s_0, t_{\frac{1}{2}}) = \xi_{2e}(a_0, t_{\frac{1}{2}}) + \xi_{2e}(b_0, t_{\frac{1}{2}})$ for every $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$; it follows from the item (i) of Theorem 2.1 that $s_0 = a_0 + b_0$. Thus, $s = s_0 = a_0 + b_0$. □

Now, we are ready to prove our first result.

Proof of Theorem 2.1. Let $a = a_1 + a_{\frac{1}{2}} + a_0$, $b = b_1 + b_{\frac{1}{2}} + b_0$. By Lemmas 2.3, 2.5, 2.6, and 2.7, we have:

$$\begin{aligned}\varphi(a+b) &= \varphi\left((a_1+b_1) + \left(a_{\frac{1}{2}} + b_{\frac{1}{2}}\right) + (a_0+b_0)\right)\\ &= \varphi(a_1+b_1) + \varphi\left(a_{\frac{1}{2}} + b_{\frac{1}{2}}\right) + \varphi(a_0+b_0)\\ &= \varphi(a_1) + \varphi(b_1) + \varphi\left(a_{\frac{1}{2}}\right) + \varphi\left(b_{\frac{1}{2}}\right) + \varphi(a_0) + \varphi(b_0)\\ &= \varphi\left(a_1 + a_{\frac{1}{2}} + a_0\right) + \varphi\left(b_1 + b_{\frac{1}{2}} + b_0\right)\\ &= \varphi(a) + \varphi(b).\end{aligned}$$

That is, φ is additive on $\mathfrak{J}$. □

3 n-Multiplicative Derivation

We now investigate the problem of when a n-multiplicative derivations is additive for the class of Jordan rings.

For this purpose, we will assume that the Jordan ring $\mathfrak{J}$ is $\{2, (n-1), (2^{n-1}-1)\}$-torsion free for $n \geq 2$ where n is degree of the nonassociative monomial $m = m(x_1, \ldots, x_n)$.

Let d be a n-multiplicative derivation of Jordan ring $\mathfrak{J}$. If we put $d(e) = a_1 + a_{\frac{1}{2}} + a_0$, then $d(m(e, e, \ldots, e)) = \sum_{i=1}^{n} m(e, \ldots, d(e), \ldots, e) = na_1 + a_{\frac{1}{2}}$. Since $d(m(e, e, \ldots, e)) = d(e)$, then $(n-1)a_1 - a_0 = 0$. Thus, $a_1 = a_0 = 0$ and $d(e) = a_{\frac{1}{2}}$. By [14, p. 77], we have:

$$\mathcal{D}_{y,z}(x) = [L_y, L_z] + [L_y, R_z] + [R_y, R_z],$$

is a derivation for all $y, z \in \mathfrak{J}$. In particular, if $y = a_{\frac{1}{2}}$ and $z = 4e$, then $\mathcal{D}_{y,z}(e) = 3d(e)$. Indeed:

$$\begin{aligned}\mathcal{D}_{a_{\frac{1}{2}},4e}(e) &= \left(\left[L_{a_{\frac{1}{2}}}, L_{4e}\right] + \left[L_{a_{\frac{1}{2}}}, R_{4e}\right] + \left[R_{a_{\frac{1}{2}}}, R_{4e}\right]\right)(e) \\ &= L_{a_{\frac{1}{2}}} L_{4e}(e) - L_{4e}L_{a_{\frac{1}{2}}}(e) + L_{a_{\frac{1}{2}}} R_{4e}(e) - R_{4e}L_{a_{\frac{1}{2}}}(e) \\ &= R_{a_{\frac{1}{2}}} R_{4e}(e) - R_{4e}R_{a_{\frac{1}{2}}}(e) = a_{\frac{1}{2}}(4ee) - 4e\left(a_{\frac{1}{2}}e\right) + a_{\frac{1}{2}}(e4e) \\ &\quad - \left(a_{\frac{1}{2}}e\right)4e + (e4e)a_{\frac{1}{2}} - \left(ea_{\frac{1}{2}}\right)4e = 2a_{\frac{1}{2}} - a_{\frac{1}{2}} + 2a_{\frac{1}{2}} \\ &\quad - a_{\frac{1}{2}} + 2a_{\frac{1}{2}} - a_{\frac{1}{2}} = 3a_{\frac{1}{2}},\end{aligned}$$

so $(\mathcal{D}_{y,z} - 3d)(e) = 0$. Let $\mathfrak{D} = \mathcal{D}_{y,z} - 3d$, note that $\mathfrak{D}$ is additive if and only if d is additive, since $\mathcal{D}_{y,z}$ is additive. Furthermore, observe that $\mathfrak{D}$ is a multiplicative derivation, such that $\mathfrak{D}(e) = 0$.

The next is the main result of this section. Its proof shares the same outline as that of Theorem 2.1, but it needs different technique.

Theorem 3.1 *Let $\mathfrak{J}$ be a Jordan ring with a non-trivial idempotent e. Let $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_{\frac{1}{2}} \oplus J_0$ be the Peirce decomposition of $\mathfrak{J}$ with respect to e. If $\mathfrak{J}$ satisfies the conditions of Theorem* 2.1, then every n-multiplicative derivation d from $\mathfrak{J}$ is additive.

The proof will be organized in a series of auxiliary lemmas.

Lemma 3.2 $\mathfrak{D}(0) = 0$.

Proof Note that $\mathfrak{D}(0) = \mathfrak{D}(\xi_0(0)) = 0$. □

Lemma 3.3 $\mathfrak{D}(\mathfrak{J}_i) \subseteq \mathfrak{J}_i$ *for* $i = 1, \frac{1}{2}, 0$.

Proof Let $a_1 \in \mathfrak{J}_1$ as $\mathfrak{D}(e) = 0$, we have $\mathfrak{D}(a_1) = \mathfrak{D}(\xi_e(a_1)) = \xi_e(\mathfrak{D}(a_1))$. If we express $\mathfrak{D}(a_1) = \mathfrak{D}(a_1)_1 + \mathfrak{D}(a_1)_{\frac{1}{2}} + \mathfrak{D}(a_1)_0$, it follows that:

$$\mathfrak{D}(a_1) = \mathfrak{D}(a_1)_1 + \mathfrak{D}(a_1)_{\frac{1}{2}} + \mathfrak{D}(a_1)_0 = \xi_e(\mathfrak{D}(a_1)) = \mathfrak{D}(a_1) + \frac{1}{2^{n-1}}\mathfrak{D}(a_1)_{\frac{1}{2}}.$$

Thus, $\mathfrak{D}(a_1) = 0 = \mathfrak{D}(a_1)_{\frac{1}{2}}$ and $\mathfrak{D}(a_1) = \mathfrak{D}(a_1)_1$.

Let $a_0 \in \mathfrak{J}_0$ as $\mathfrak{D}(e) = 0$, we have $0 = \mathfrak{D}(\xi_e(a_0)) = \xi_e(\mathfrak{D}(a_0)) = \mathfrak{D}(a_0)_1 + \frac{1}{2^{n-1}}\mathfrak{D}(a_0)_{\frac{1}{2}}$. If we express $\mathfrak{D}(a_0) = \mathfrak{D}(a_0)_1 + \mathfrak{D}(a_0)_{\frac{1}{2}} + \mathfrak{D}(a_0)_0$, it follows that $\mathfrak{D}(a_0)_1 = 0 = \mathfrak{D}(a_0)_{\frac{1}{2}}$. Thus, $\mathfrak{D}(a_0) = \mathfrak{D}(a_0)_0$.

Let $a_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$, we have:

$$\begin{aligned}\mathfrak{D}\left(a_{\frac{1}{2}}\right) = \mathfrak{D}\left(\xi_{2e}\left(a_{\frac{1}{2}}\right)\right) &= \xi_{\mathfrak{D}(2e)}\left(2e, \ldots, 2e, a_{\frac{1}{2}}\right) + \xi_{2e}(\mathfrak{D}(2e), 2e, \ldots, 2e, a_{\frac{1}{2}}) \\ &+ \cdots + \xi_{2e}(2e, \ldots, \mathfrak{D}(2e), a_{\frac{1}{2}}) + \xi_{2e}\left(2e, \ldots, 2e, \mathfrak{D}\left(a_{\frac{1}{2}}\right)\right).\end{aligned}$$

If we express $\mathfrak{D}(a_{\frac{1}{2}}) = \mathfrak{D}(a_{\frac{1}{2}})_1 + \mathfrak{D}(a_{\frac{1}{2}})_{\frac{1}{2}} + \mathfrak{D}(a_{\frac{1}{2}})_0$, it follows that $(2^{n-1}-1)\mathfrak{D}(a_{\frac{1}{2}})_1 = 0 = \mathfrak{D}(a_{\frac{1}{2}})_0$ and $0 = x_{\frac{1}{2}} = \xi_{\mathfrak{D}(2e)}(2e, \ldots, 2e, a_{\frac{1}{2}}) + \xi_{2e}(\mathfrak{D}(2e), 2e, \ldots, 2e, a_{\frac{1}{2}}) + \cdots + \xi_{2e}(2e, \ldots, \mathfrak{D}(2e), a_{\frac{1}{2}}) \in \mathfrak{J}_{\frac{1}{2}}$ once that $\xi_{2e}(2e, \ldots, 2e, \mathfrak{D}(a_{\frac{1}{2}})) = 2^{n-1}\mathfrak{D}(a_{\frac{1}{2}})_1 + \mathfrak{D}(a_{\frac{1}{2}})_{\frac{1}{2}}$. Thus, $\mathfrak{D}(a_{\frac{1}{2}}) = \mathfrak{D}(a_{\frac{1}{2}})_{\frac{1}{2}}$.

Lemma 3.4 *Let $a_1 \in \mathfrak{J}_1$ and $a_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$. Then, $\mathfrak{D}(2^{n-1}a_{\frac{1}{2}}) = 2^{n-1}\mathfrak{D}(a_{\frac{1}{2}})$, $\mathfrak{D}(2e) = 0$ and $\mathfrak{D}(2^{n-1}a_1) = 2^{n-1}\mathfrak{D}(a_1)$.*

Proof Let $a_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$ as $\mathfrak{D}(e) = 0$ and $\mathfrak{D}(2^{n-1}a_{\frac{1}{2}}) \in \mathfrak{J}_{\frac{1}{2}}$; then:

$$\mathfrak{D}(a_{\frac{1}{2}}) = \mathfrak{D}\left(\xi_e\left(2^{n-1}a_{\frac{1}{2}}\right)\right) = \xi_e\left(\mathfrak{D}(2^{n-1}a_{\frac{1}{2}})\right) = \frac{1}{2^{n-1}}\mathfrak{D}(2^{n-1}a_{\frac{1}{2}}).$$

Thus, $\mathfrak{D}(2^{n-1}a_{\frac{1}{2}}) = 2^{n-1}\mathfrak{D}(a_{\frac{1}{2}})$.

Now note that $(n-1)\mathfrak{D}(2e)a_{\frac{1}{2}} = \mathfrak{D}(\xi_{2e}(a_{\frac{1}{2}})) - \xi_{2e}(\mathfrak{D}(a_{\frac{1}{2}})) = \mathfrak{D}(a_{\frac{1}{2}}) - \mathfrak{D}(a_{\frac{1}{2}}) = 0$. By item (i) of Theorem 3.1, we have $\mathfrak{D}(2e) = 0$.

Let $a_1 \in \mathfrak{J}_1$ as $\mathfrak{D}(2e) = 0$ and $\mathfrak{D}(a_1) \in \mathfrak{J}_1$; then:

$$\mathfrak{D}(2^{n-1}a_1) = \mathfrak{D}(\xi_{2e}(a_1)) = \xi_{2e}(\mathfrak{D}(a_1)) = 2^{n-1}\mathfrak{D}(a_1).$$

□

Lemma 3.5 *Let $a_1 \in \mathfrak{J}_1$, $a_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$ and $a_0 \in \mathfrak{J}_0$. Then, $\mathfrak{D}(a_1 + a_{\frac{1}{2}} + a_0) = \mathfrak{D}(a_1) + \mathfrak{D}(a_{\frac{1}{2}}) + \mathfrak{D}(a_0)$.*

Proof Consider $\mathfrak{D}(a_1 + a_{\frac{1}{2}} + a_0) = d_1 + d_{\frac{1}{2}} + d_0$; by Lemma 3.4, we get:

$$\begin{aligned}\mathfrak{D}\left(2^{n-1}a_1 + a_{\frac{1}{2}}\right) &= \mathfrak{D}\left(\xi_{2e}\left(a_1 + a_{\frac{1}{2}} + a_0\right)\right)\\ &= \xi_{2e}\left(\mathfrak{D}\left(a_1 + a_{\frac{1}{2}} + a_0\right)\right)\\ &= \xi_{2e}\left(d_1 + d_{\frac{1}{2}} + d_0\right) = 2^{n-1}d_1 + d_{\frac{1}{2}}.\end{aligned} \tag{3.1}$$

Let $t_0 \in \mathfrak{J}_0$; then:

$$\begin{aligned}t_0 d_{\frac{1}{2}} &= \xi_{2e}\left(t_0, \mathfrak{D}\left(2^{n-1}a_1 + a_{\frac{1}{2}}\right)\right)\\ &= \mathfrak{D}\left(\xi_{2e}\left(t_0, 2^{n-1}a_1 + a_{\frac{1}{2}}\right)\right)\\ &\quad -\xi_{2e}\left(\mathfrak{D}(t_0), 2^{n-1}a_1 + a_{\frac{1}{2}}\right)\\ &= \mathfrak{D}\left(t_0 a_{\frac{1}{2}}\right) - \mathfrak{D}(t_0)a_{\frac{1}{2}} = t_0\mathfrak{D}\left(a_{\frac{1}{2}}\right).\end{aligned} \tag{3.2}$$

Thus, by item (iii) of Theorem 3.1, we have $\mathfrak{D}(a_{\frac{1}{2}}) = d_{\frac{1}{2}}$. Let $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$ by (3.1), we have:

$$\begin{aligned}
&\xi_{2e}\left(t_{\frac{1}{2}}, 2^{n-1}d_1\right) + \xi_{2e}\left(t_{\frac{1}{2}}, d_{\frac{1}{2}}\right) \\
&= \xi_{2e}\left(t_{\frac{1}{2}}, 2^{n-1}d_1 + d_{\frac{1}{2}}\right) \\
&= \xi_{2e}\left(t_{\frac{1}{2}}, \mathfrak{D}\left(2^{n-1}a_1 + a_{\frac{1}{2}}\right)\right) \\
&= \mathfrak{D}\left(\xi_{2e}\left(t_{\frac{1}{2}}, 2^{n-1}a_1 + a_{\frac{1}{2}}\right)\right) \\
&\quad -\xi_{2e}\left(\mathfrak{D}\left(t_{\frac{1}{2}}\right), 2^{n-1}a_1 + a_{\frac{1}{2}}\right) \\
&= \mathfrak{D}\left(\xi_{2e}\left(t_{\frac{1}{2}}, 2^{n-1}a_1\right) + \xi_{2e}\left(t_{\frac{1}{2}}, a_{\frac{1}{2}}\right)\right) \\
&\quad -\xi_{2e}\left(\mathfrak{D}(t_{\frac{1}{2}}), 2^{n-1}a_1\right) - \xi_{2e}\left(\mathfrak{D}\left(t_{\frac{1}{2}}\right), a_{\frac{1}{2}}\right).
\end{aligned} \tag{3.3}$$

As $\xi_{2e}(t_{\frac{1}{2}}, d_{\frac{1}{2}}), \xi_{2e}(\mathfrak{D}(t_{\frac{1}{2}}), a_{\frac{1}{2}}) \in \mathfrak{J}_1 \oplus \mathfrak{J}_0$ follows that for $u_0 \in \mathfrak{J}_0$ by (3.3), we get:

$$\begin{aligned}
u_0\xi_{2e}\left(t_{\frac{1}{2}}, 2^{n-1}d_1\right) &= \xi_{2e}\left(u_0, \xi_{2e}\left(t_{\frac{1}{2}}, 2^{n-1}d_1\right)\right) \\
&= \xi_{2e}\left(u_0, \xi_{2e}\left(t_{\frac{1}{2}}, 2^{n-1}d_1\right) + \xi_{2e}\left(t_{\frac{1}{2}}, d_{\frac{1}{2}}\right)\right) \\
&= \xi_{2e}\left(u_0, \mathfrak{D}\left(\xi_{2e}\left(t_{\frac{1}{2}}, 2^{n-1}a_1\right) + \xi_{2e}\left(t_{\frac{1}{2}}, a_{\frac{1}{2}}\right)\right)\right) \\
&\quad -\xi_{2e}\left(u_0, \xi_{2e}\left(\mathfrak{D}\left(t_{\frac{1}{2}}\right), 2^{n-1}a_1\right)\right) = u_0\xi_{2e}\left(t_{\frac{1}{2}}, \mathfrak{D}(2^{n-1}a_1)\right).
\end{aligned}$$

Therefore by item (iii) of Theorem 3.1 and Lemma 3.4, we have $d_1 = \mathfrak{D}(a_1)$.
Finally, we show that $d_0 = \mathfrak{D}(a_0)$. Let $e \in \mathfrak{J}_1, t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$ and $h_0, t_0 \in \mathfrak{J}_0$. We have:

$$\begin{aligned}
\xi_{h_0}\left(h_0, 2e, t_{\frac{1}{2}}, t_0, \mathfrak{D}\left(a_1 + a_{\frac{1}{2}} + a_0\right)\right) &= \xi_{h_0}\left(h_0, 2e, t_{\frac{1}{2}}, t_0, d_1 + d_{\frac{1}{2}} + d_0\right) \\
&= \xi_{h_0}(h_0, t_{\frac{1}{2}}, t_0, d_0).
\end{aligned}$$

On the other hand, using the identity of n-multiplicative derivation, we get:

$$\xi_{h_0}\left(h_0, 2e, t_{\frac{1}{2}}, t_0, \mathfrak{D}\left(a_1 + a_{\frac{1}{2}} + a_0\right)\right) = \xi_{h_0}\left(h_0, t_{\frac{1}{2}}, t_0, \mathfrak{D}(a_0)\right).$$

Thus, $\xi_{h_0}(h_0, t_{\frac{1}{2}}, t_0, d_0) = \xi_{h_0}(h_0, t_{\frac{1}{2}}, t_0, \mathfrak{D}(a_0))$. Now, by items (iii), (i) and (ii) of Theorem 3.1, we have $d_0 = \mathfrak{D}(a_0)$.

Therefore, $\mathfrak{D}(a_1 + a_{\frac{1}{2}} + a_0) = \mathfrak{D}(a_1) + \mathfrak{D}(a_{\frac{1}{2}}) + \mathfrak{D}(a_0)$. □

Lemma 3.6 *Let $a_{\frac{1}{2}}, b_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$. Then, $\mathfrak{D}(a_{\frac{1}{2}} + b_{\frac{1}{2}}) = \mathfrak{D}(a_{\frac{1}{2}}) + \mathfrak{D}(b_{\frac{1}{2}})$.*

Proof Note that $\xi_{2e}((2e+a_{\frac{1}{2}}),(2e+b_{\frac{1}{2}})) = \xi_{2e}(a_{\frac{1}{2}}+b_{\frac{1}{2}}+2^2e+a_{\frac{1}{2}}b_{\frac{1}{2}})$. Since $a_{\frac{1}{2}}+b_{\frac{1}{2}} = \xi_{2e}(a_{\frac{1}{2}}+b_{\frac{1}{2}}) \in \mathfrak{J}_{\frac{1}{2}}$ and $\xi_{2e}(2^2e+a_{\frac{1}{2}}b_{\frac{1}{2}}) \in \mathfrak{J}_1$ by Lemmas 3.4 and 3.5, we get:

$$\begin{aligned}
&\mathfrak{D}\left(\xi_{2e}\left(a_{\frac{1}{2}}+b_{\frac{1}{2}}\right)\right)+\mathfrak{D}\left(\xi_{2e}\left(2^2e+a_{\frac{1}{2}}b_{\frac{1}{2}}\right)\right)\\
&=\xi_{2e}\left(\mathfrak{D}\left(a_{\frac{1}{2}}+b_{\frac{1}{2}}\right)\right)+\xi_{2e}\left(\mathfrak{D}\left(2^2e+a_{\frac{1}{2}}b_{\frac{1}{2}}\right)\right)\\
&=\xi_{2e}\left(\mathfrak{D}\left(a_{\frac{1}{2}}+b_{\frac{1}{2}}\right)+\mathfrak{D}\left(2^2e+a_{\frac{1}{2}}b_{\frac{1}{2}}\right)\right)\\
&=\xi_{2e}\left(\mathfrak{D}\left(a_{\frac{1}{2}}+b_{\frac{1}{2}}+2^2e+a_{\frac{1}{2}}b_{\frac{1}{2}}\right)\right)\\
&=\mathfrak{D}\left(\xi_{2e}\left(a_{\frac{1}{2}}+sb_{\frac{1}{2}}+2^2e+a_{\frac{1}{2}}b_{\frac{1}{2}}\right)\right)\\
&=\mathfrak{D}\left(\xi_{2e}\left(\left(2e+a_{\frac{1}{2}}\right),\left(2e+b_{\frac{1}{2}}\right)\right)\right)\\
&=\xi_{2e}\left(\mathfrak{D}\left(2e+a_{\frac{1}{2}}\right),\left(2e+b_{\frac{1}{2}}\right)\right)\right)\\
&\quad+\xi_{2e}\left(\left(2e+a_{\frac{1}{2}}\right),\mathfrak{D}\left(2e+b_{\frac{1}{2}}\right)\right)\right)\\
&=\xi_{2e}\left(\mathfrak{D}(2e)+\mathfrak{D}\left(a_{\frac{1}{2}}\right),\left(2e+b_{\frac{1}{2}}\right)\right)\\
&\quad+\xi_{2e}\left(\left(2e+a_{\frac{1}{2}}\right),\mathfrak{D}(2e)+\mathfrak{D}\left(b_{\frac{1}{2}}\right)\right)\\
&=\xi_{2e}\left(\mathfrak{D}\left(a_{\frac{1}{2}}\right),2e\right)+\xi_{2e}\left(\mathfrak{D}\left(a_{\frac{1}{2}}\right),b_{\frac{1}{2}}\right)\\
&\quad+\xi_{2e}\left(2e,\mathfrak{D}\left(b_{\frac{1}{2}}\right)\right)+\xi_{2e}\left(a_{\frac{1}{2}},\mathfrak{D}\left(b_{\frac{1}{2}}\right)\right)\\
&=\mathfrak{D}\left(\xi_{2e}\left(a_{\frac{1}{2}},b_{\frac{1}{2}}\right)\right)+\xi_{2e}\left(\mathfrak{D}\left(a_{\frac{1}{2}}\right),2e\right)\\
&\quad+\xi_{2e}\left(2e,\mathfrak{D}\left(b_{\frac{1}{2}}\right)\right).
\end{aligned}$$

Observe that $\mathfrak{D}(\xi_{2e}(2^2e+a_{\frac{1}{2}}b_{\frac{1}{2}})), \mathfrak{D}(\xi_{2e}(a_{\frac{1}{2}},b_{\frac{1}{2}})) \in \mathfrak{J}_1$ and

$$\mathfrak{D}\left(\xi_{2e}\left(a_{\frac{1}{2}}+b_{\frac{1}{2}}\right)\right),\xi_{2e}\left(\mathfrak{D}\left(a_{\frac{1}{2}}\right),2e\right)+\xi_{2e}\left(2e,\mathfrak{D}\left(b_{\frac{1}{2}}\right)\right)\in\mathfrak{J}_{\frac{1}{2}};$$

it follows that:

$$\begin{aligned}
\mathfrak{D}\left(a_{\frac{1}{2}}+b_{\frac{1}{2}}\right)&=\mathfrak{D}\left(\xi_{2e}\left(a_{\frac{1}{2}}+b_{\frac{1}{2}}\right)\right)=\xi_{2e}\left(\mathfrak{D}(a_{\frac{1}{2}}),2e\right)+\xi_{2e}\left(2e,\mathfrak{D}\left(b_{\frac{1}{2}}\right)\right)\\
&=\mathfrak{D}\left(a_{\frac{1}{2}}\right)+\mathfrak{D}\left(b_{\frac{1}{2}}\right).
\end{aligned}$$

□

Lemma 3.7 *Let* $a_i, b_i \in \mathfrak{J}_i$, $(i=1,0)$. *Then,* $\mathfrak{D}(a_i+b_i)=\mathfrak{D}(a_i)+\mathfrak{D}(b_i)$.

Proof Let $a_i, b_i \in \mathfrak{J}_i$ $(i = 1, 0)$ and $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$; by Lemma 3.6, we have:

$$\begin{aligned}\mathfrak{D}(a_i + b_i)t_{\frac{1}{2}} = \xi_{2e}\left(\mathfrak{D}(a_i + b_i), t_{\frac{1}{2}}\right) &= \mathfrak{D}\left(\xi_{2e}\left((a_i + b_i), t_{\frac{1}{2}}\right)\right)\\ -\xi_{2e}\left((a_i + b_i), \mathfrak{D}\left(t_{\frac{1}{2}}\right)\right) &= \mathfrak{D}\left(\xi_{2e}\left(a_i, t_{\frac{1}{2}}\right)\right) + \mathfrak{D}\left(\xi_{2e}\left(b_i, t_{\frac{1}{2}}\right)\right)\\ -\xi_{2e}\left(a_i, \mathfrak{D}\left(t_{\frac{1}{2}}\right)\right) &- \xi_{2e}\left(b_i, \mathfrak{D}\left(t_{\frac{1}{2}}\right)\right)\\ &= \xi_{2e}\left(\mathfrak{D}(a_i), t_{\frac{1}{2}}\right) + \xi_{2e}\left(\mathfrak{D}(b_i), t_{\frac{1}{2}}\right)\\ &= (\mathfrak{D}(a_i) + \mathfrak{D}(b_i))t_{\frac{1}{2}}.\end{aligned}$$

Therefore, by item (i) of Theorem 3.1, we get $\mathfrak{D}(a_i + b_i) = \mathfrak{D}(a_i) + \mathfrak{D}(b_i)$. □

Now, we are in a position to show that $\mathfrak{D}$ preserves addition.

Proof of Theorem 3.1. Let $a = a_1 + a_{\frac{1}{2}} + a_0$, $b = b_1 + b_{\frac{1}{2}} + b_0$. By Lemmas 3.5–3.7, we have:

$$\begin{aligned}\mathfrak{D}(a + b) &= \mathfrak{D}((a_1 + b_1) + \left(a_{\frac{1}{2}} + b_{\frac{1}{2}}\right) + (a_0 + b_0)) = \mathfrak{D}(a_1 + b_1) + \mathfrak{D}\left(a_{\frac{1}{2}} + b_{\frac{1}{2}}\right)\\ &+\mathfrak{D}(a_0 + b_0) = \mathfrak{D}(a_1) + \mathfrak{D}(b_1) + \mathfrak{D}\left(a_{\frac{1}{2}}\right) + \mathfrak{D}\left(b_{\frac{1}{2}}\right) + \mathfrak{D}(a_0) + \mathfrak{D}(b_0)\\ &= \mathfrak{D}\left(a_1 + a_{\frac{1}{2}} + a_0\right) + \mathfrak{D}(b_1 + b_{\frac{1}{2}} + b_0) = \mathfrak{D}(a) + \mathfrak{D}(b).\end{aligned}$$

That is, $\mathfrak{D}$ is additive on $\mathfrak{J}$. □

The following two examples show that there are non-trivial noncommutative Jordan algebra and Jordan algebra, respectively, that satisfy the conditions of the Theorem 2.1.

Example 3.8 Let $\mathfrak{F}$ be a field of characteristic different from 2, $\mathfrak{J}$ a four-dimensional algebra over $\mathfrak{F}$, and a basis $\{e_{11}, e_{10}, e_{01}, e_{00}\}$ with the multiplication table given by: $e_{ij}e_{kl} = \delta_{jk}e_{il}$ $(i, j, k, l = 1, 2)$, where δ_{jk} is the Kronecker delta. We can verify that $\mathfrak{J}$ is a Jordan algebra. In fact, $\mathfrak{J}$ is an associative algebra where e_{11} and e_{00} are orthogonal idempotents, such that $e = e_{11} + e_{00}$ is the unity element of $\mathfrak{J}$. Moreover, if $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_{\frac{1}{2}} \oplus \mathfrak{J}_0$ is the Peirce decomposition of $\mathfrak{J}$, relative to e_{11}, then we have $\mathfrak{J}_1 = \mathfrak{F}e_{11}, \mathfrak{J}_{\frac{1}{2}} = \mathfrak{F}e_{10} + \mathfrak{F}e_{01}, \mathfrak{J}_0 = \mathfrak{F}e_{00}$. From a direct calculation, we can verify that $\mathfrak{J}$ satisfies the conditions (i)–(iii) of Theorem 2.1.

Example 3.9 Let $\mathfrak{K}$ be the algebra obtained from the associative algebra $\mathfrak{J}$, in Example 3.8, on replacing the product xy by $x \cdot y = \frac{1}{2}(xy + yx)$. We can verify that $\mathfrak{K}$ is a Jordan algebra where e_{11} and e_{00} are orthogonal idempotents, such that $e = e_{11} + e_{00}$ is the unity element of $\mathfrak{K}$. Moreover, if $\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_{\frac{1}{2}} \oplus \mathfrak{K}_0$ is the Peirce decomposition of $\mathfrak{K}$, relative to e_{11}, then we have $\mathfrak{K}_i = \mathfrak{F}e_{ii}$ $(i = 1, 0)$ and $\mathfrak{K}_{\frac{1}{2}} = \mathfrak{F}e_{10} + \mathfrak{F}e_{01}$. From a direct calculation, we can verify that the algebra $\mathfrak{K}$ satisfies the conditions (i)–(iii) of the Theorem 2.1.

4 Corollaries

In this section, we will give some consequences of our main results.

Corollary 4.1 *Let* $\mathfrak{J}$ *satisfy the conditions of Theorem* 2.1. Then, any multiplicative map φ of $\mathfrak{J}$ onto an arbitrary Jordan ring $\mathfrak{J}'$ is additive.

In the case of unital Jordan rings, we have the following.

Corollary 4.2 *Let* $\mathfrak{J}$ *be a unital Jordan ring and e a non-trivial idempotent in* $\mathfrak{J}$*. Let* $\mathfrak{J}'$ *be a Jordan ring. Let* $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_{\frac{1}{2}} \oplus \mathfrak{J}_0$ *be the Peirce decomposition of* $\mathfrak{J}$ *with respect to e. If* $\mathfrak{J}$ *satisfies the following condition:*

(i) *Let* $a_i \in \mathfrak{J}_i (i = 1, 0)$*. If* $t_{\frac{1}{2}} a_i = 0$ *for all* $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$*, then* $a_i = 0$*;*

then every multiplicative map from $\mathfrak{J}$ *onto* $\mathfrak{J}'$ *is additive.*

Corollary 4.3 *Let* $\mathfrak{J}$ *be a unital Jordan ring and e a non-trivial idempotent in* $\mathfrak{J}$*. Let* $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_{\frac{1}{2}} \oplus \mathfrak{J}_0$ *be the Peirce decomposition of* $\mathfrak{J}$ *with respect to e. If* $\mathfrak{J}$ *satisfies the following condition:*

(i) *Let* $a_i \in \mathfrak{J}_i (i = 1, 0)$*. If* $t_{\frac{1}{2}} a_i = 0$ *for all* $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$*, then* $a_i = 0$*;*

then every multiplicative derivation from $\mathfrak{J}$ *is additive.*

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