RT-MAE 2015-07

LINKING STRUCTURAL SIGNATURE AND SIGNATURE POINT PROCESS

by

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Palavras-Chaves: Signature point processes, system signatures; coherent systems.

AMS Classification: 60G55, 60G44.

- Novembro de 2015 -

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Abstract. In this paper we relate, under certain conditions, structural signature generated by independent and identically distributed coherent system's component lifetimes and signature point process, generated by dependent coherent system's component lifetimes without simultaneous failures.

Keywords: Signature point processes, system signatures; coherent systems.

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1.Introduction. As in Barlow and Proschan (1981) a complex engineering system is completely characterized by its structure function ϕ which relates its lifetime T and its components lifetimes T_i, $1 \leq i \leq n$, defined in a complete probability space $(\Omega, \Im, P)$ as, by example, the series parallel decomposition:

$$T = \phi(\mathbf{T}) = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i,$$

where $K_j, 1 \leq j \leq k$ are minimal cut sets, that is, a minimal set of components whose joint failure causes the system fail.

The performance of a coherent system can be measured from this structural relationship and the joint distribution function of its component lifetimes. The structure functions offer a way of indexing the class of coherent system but such representations make the system lifetime distribution function analytically very complicated. An alternative representation for the coherent system distribution function is through the system signature, as in Samaniego (1985), that, while narrower in scope than the structure function, is substantially more useful:

Definition 1.1 Let T be the lifetime of a coherent system of order n, with component lifetimes $T_1, ..., T_n$, independent and identically distributed (i.i.d.) positive random variables with continuous distribution F. Then the signature vector s is defined as

$$s = (P(T = T_{(1)}), P(T = T_{(2)}) ..., P(T = T_{(n)}))$$

where $\{T_{(i)}, 1 \leq i \leq n\}$ are the order statistics of $\{T_i, 1 \leq i \leq n\}$.

A detailed treatment of the theory and applications of system signature may be found in Samaniego (2007). This reference gives detailed justification for the i.i.d. assumption used in the system signature definition. By the way there are some applications in which this assumption is appropriate, and in such case, the use of system signature for comparisons among systems is wholly appropriate; such applications range from batteries in lighting, to wafers or chips in a digital computer to the subsystem of spark plugs in an automobile engine.

An important feature of system signature is the fact that, in the context of i.i.d. continuous component lifetimes, they are distribution free measures of system quality, depending solely on the design characteristics of the system and independent of the system components behavior.

Samaniego(2007), Kochar, et al. (1999), Shaked and Suarez-Llorens (2003) and Navarro and Rychlik(2007) extended the signature concept to the case where the component lifetimes $T_1, ..., T_n$ are exchangeable (i.e. the joint continuous distribution function,

$F(t_1, ..., t_n)$, of $(T_1, ..., T_n)$ is the same for any permutation of $t_1, ..., t_n)$, an interesting and practical situation in reliability theory.

Even with the widely used concept of signature and its development, researchers does not have considered the general case where the component lifetimes are stochastically dependent on time.

In a presentation at the 26th European Conference on Operational Research, Bueno (2013) define the signature point process introducing the above concept under more general conditions of dependence. Observing the ordered component lifetimes as they appear on time, the signature point process vector is defined as

$$(1_{\{T=T_{(1)}\}}, 1_{\{T=T_{(2)}\}}, ..., 1_{\{T=T_{(n)}\}}).$$

Also, we can recover the component lifetimes if we consider the complete information level, where we observe the components and its ordering on time. In this approach we can talk about reliability component importance for system reliability, redundancy of components, redundancy allocation, ...

In this paper, the natural tool to be used is the point process martingale theory. In Section 2 we give the mathematical details of the signature point process, and natural representation of the distribution function coherent system with dependent components lifetimes without simultaneous failures, as a function of the ordered component lifetimes. Associated with this point process we give its compensator process. In Section 3 we relate, under certain conditions, the structural signature generated by independent and identically distributed coherent system's component lifetimes and signature point process, generated by dependent coherent system's component lifetimes without simultaneous failures.

2.1 Signatures.

2.1 Structural signature.

Under the signature definition assumption, Samaniego considers the order statistics of independent and identically distributed components lifetimes of a coherent system of order n with continuous distribution. Clearly $\{T = T_{(i)}\}$ $1 \leq i \leq n$ is a (P-a.s.) partition of the probability space and we have the system distribution function representation

$$P(T \leq t) = \sum_{i=1}^{n} P(T = T_{(i)})P(T_{(i)} \leq t | T = T_{(i)}) =$$

$$\sum_{i=1}^{n} P(T = T_{(i)})P(T_{(i)} \leq t).$$

Interesting results extending the above decomposition are Theorem 2.1.1 which relates signature with exchangeable component lifetimes, and Theorem 2.1.2, which relates signatures of systems with different number of components.

Theorem 2.1.1 If $(T_1, T_2, ..., T_n)$ is an exchangeable random vector, $T = \phi(T_1, ..., T_n)$ is the lifetime of a coherent system and $\mathbf{s} = (s_1, s_2, ..., s_n)$ is the signature vector of a system with the same structure function as that of the system with lifetime T, but with independent and identically distributed components having a common absolutely continuous distribution, then

$$P(T > t) = \Sigma_{i=1}^{n} s_i P(T_(i) > t).$$

Theorem 2.1.2 If $(T_1, T_2, ..., T_n)$ is an exchangeable random vector and T is the lifetime of a coherent system with components $T_1, ..., T_k$, $k \leq n$, then

$$P(T > t) = \Sigma_{i=1}^{n} s_i^* P(T_(i) > t)$$

for all t where $\mathbf{s}^*$ is the signature of order n of the system with the same structure as that the lifetime T but with independent and identically distributed component lifetimes having a common absolutely continuous distribution.

2.2 The Signature Point Process.

In our general setup, we consider the vector $(T_1, ..., T_n)$ of n component lifetimes which are finite and positive random variables defined in a complete probability space $(\Omega, \Im, P)$, with $P(T_i \neq T_j) = 1$, for all $i \neq j, i, j$ in $C = \{1, ..., n\}$, the index set of components. The lifetimes can be dependent but simultaneous failures are ruled out.

In what follows, to simplify the notation, we assume that relations such as $\subset, =, \leq, <, \neq$ between random variables and measurable sets, respectively, always hold with probability one, which means that the term P-a.s., is suppressed.

The evolution of components in time define a point process given through the failure times. We denote by $T_{(1)} < T_{(2)} < ... < T_{(n)}$ the ordered lifetimes $T_1, T_2, ..., T_n$, as they appear on time and, as a convention we set $T_{(n+1)} = T_{(n+2)} = ... = \infty$ Therefore the sequence $(T_n)_{n \geq 1}$ defines a point process.

The mathematical description of our observations, is given by a family of sub σ-algebras of $\Im$, denoted by $(\Im_t)_{t \geq 0}$, where

$$\Im_t = \sigma\{1_{\{T_{(i)} > s\}}, 1 \leq i \leq n, 0 < s \leq t\},$$

satisfies the Dellacherie conditions of right continuity and completeness .

Intuitively, at each time t the observer knows if the event $\{T_{(i)} \leq t\}$ have either occurred or not and if it had, he knows exactly the value $T_{(i)}$.

We observe that, in the original concept of signature, Samaniego does not use any information on time represented by the trivial σ- algebra $\Im_t = \{\Omega, \emptyset\}, \quad \forall t.$

In the following, we consider the informationi $\Im_t$. The behavior of the stochastic process $P(T > t|\Im_t)$, as the information flows continuously in time is given by Theorem 2.2.1.

Theorem 2.2.1 Let $T_1, T_2, ..., T_n$ be the component lifetimes of a coherent system with lifetime T. Then,

$$P(T \leq t|\Im_t) = \sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)} \leq t\}}.$$

Proof From the total probability rule we have $P(T \leq t|\Im_t) =$

$$\sum_{k=1}^{n} P(\{T \leq t\} \cap \{T = T_{(k)}|\Im_t) = \sum_{k=1}^{n} E[1_{\{T=T_{(k)}\}} 1_{\{T_{(k)} \leq t\}}|\Im_t].$$

As T and $T_{(k)}$ are $\Im_t$-stopping time and it is well known that the event $\{T = T_{(k)}\} \in \Im_{T_{(k)}}$ where

$$\Im_{T_{(k)}} = \{A \in \Im_\infty : A \cap \{T_{(k)} \leq t\} \in \Im_t, \forall t \geq 0\},$$

we conclude that $\{T = T_{(k)}\} \cap \{T_{(k)} \leq t\}$ is $\Im_t$-measurable.

Therefore $P(T \leq t|\Im_t) =$

$$\sum_{k=1}^{n} E[1_{\{T=T_{(k)}\}} 1_{\{T_{(k)} \leq t\}}|\Im_t] = \sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)} \leq t\}}.$$

The above decomposition allows us to define the signature point process.

Definition 2.2.2 The vector $(1_{\{T=T_{(k)}\}}, 1 \leq k, j \leq n)$ is defined as the marked point signature process of the system ϕ.

Remark 2.2.3 As the collection of events $\{\{T = T_{(i)}\}, 1 \leq i \leq n\}$ is a partition of Ω, we have $\sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} = 1$ and

$$P(T > t|\Im_t) = 1 - P(T \leq t|\Im_t) =$$

$$\sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} - \sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)} \leq t\}} =$$

$$\sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} [1 - 1_{\{T_{(k)} \leq t\}}] = \sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)} > t\}}.$$

Remark 2.2.4 Using Remark 2.2.3 we can calculate the system reliability as

$$P(T > t) = E[P(T > t|\Im_t)] =$$

$$E[\sum_{k=1}^{n} 1_{\{T=T_{(k)}\}} 1_{\{T_{(k)} > t\}}] = \sum_{k=1}^{n} P(\{T = T_{(k)}\} \cap \{T_{(k)} > t\}).$$

If the component lifetimes are independent, continuous and identically distributed we have,

$$P(T > t) = \sum_{k=1}^{n} P(T = T_{(k)}) P(T_{(k)} > t)$$

recovering the classical result as in Samaniego (1985).

Remark 2.2.5

The marked point $N_t((i)) = 1_{\{T_{(i)} \leq t\}}$ is an $\Im_t$-sub-martingale, that is, $T_{(i)}$ is $\Im_t$-measurable and $E[N_t((i))|\Im_s] \geq N_s((i))$ for all $0 \leq s \leq t$. From Doob-Meyer decomposition there exists a unique $\Im_t$-predictable process, $(A_t((i))_{t \geq 0}$, called the $\Im_t$-compensator of $N_t((i))$, with $A_0((i)) = 0$ and such that $M_t((i)) = N_t((i)) - A_t((i))$ is a zero mean uniformly integrable $\Im_t$-martingale.

The compensator process $(A_t((i)))_{t \geq 0}$ generalizes the classical hazard function notion, on the basis of all observations available up to, but not including, the present.

As $N_t((i))$ can only count on the time interval $(T_{(i-1)}, T_{(i)}]$, the corresponding compensator differential $dA_t((i))$ must vanish outside this interval. We assume that $T_i, 1 \leq i \leq n$ are totally inaccessible $\Im_t$-stopping time.

Remark 2.2.6 An extended and positive random variable τ is an $\Im_t$-stopping time if, and only if, $\{\tau \leq t\} \in \Im_t$, for all $t \geq 0$; an $\Im_t$-stopping time τ is called predictable if an increasing sequence $(\tau_n)_{n\geq 0}$ of $\Im_t$-stopping time, $\tau_n < \tau$, exists such that $\lim_{n\to\infty} \tau_n = \tau$; an $\Im_t$-stopping time τ is totally inaccessible if $P(\tau = \sigma < \infty) = 0$ for all predictable $\Im_t$-stopping time σ. For a mathematical basis of stochastic processes applied to reliability theory see the books of Aven and Jensen(2009) and Bremaud (1981).

Remark 2.2.7

Under the totally inaccessible assumption, $A_t((i))$ is continuous. In certain sense, an absolutely continuous lifetime is totally inaccessible. In the case where $(P(T_{(k)} \leq t|\Im_t))_{t\geq 0}$ is absolutely continuous (with respect to Lebesgue measure), $P(T_{(k)} \leq t|\Im_t)$ is $\Im_t$-predictable, of finite variation and $A_t((k)) = -\ln(1 - P(T_{(k)} \leq t \wedge T_{(k)}|\Im_t)$. See Arjas and Yashin (1988).

Such situation generalizes the case of independent components: If the component lifelength $T_{(k)}$ has a continuous compensator which is deterministic (except that it is stopped when $T_{(k)}$ occurs), then the lifelengths of the other components have no causal effect on $T_{(k)}$. However, other components may well dependent casually on $T_{(k)}$, so that the components need not be statistically independent.

The $\Im_t$-compensator of $1_{\{T\leq t\}}$, where T is the system lifetime is set in the following Theorem:

Theorem 2.2.8

Let $T_1, T_2, ..., T_n$, be the components lifetimes of a coherent system with lifetime T. Then, the $\Im_t$-submartingale $P(T \leq t|\Im_t)$, has the $\Im_t$-compensator

$$A_t = \sum_{k=1}^{n} \int_0^t 1_{\{T=T_{(k)}\}} dA_s((k)).$$

Proof

We consider the process

$1_{\{T=T_{(k)}\}}(w,s) = 1_{\{T=T_{(k)}\}}(w).$

It is left continuous and $\Im_t$-predictable. Therefore

$\int_0^t 1_{\{T=T_{(k)}\}}(s)dM_s((k))$

is an $\Im_t$-martingale.

As a finite sum of $\Im_t$-martingales is an $\Im_t$-martingale, we have

$$\sum_{k=1}^{n}\int_0^t 1_{\{T=T_{(k)}\}}dM_s((k)) =$$

$$\sum_{k=1}^{n}\int_0^t 1_{\{T=T_{(k)}\}}d1_{\{T_{(k)}\leq s\}} - \sum_{k=1}^{n}\int_0^t 1_{\{T=T_{(k)}\}}dA_s((k)).$$

is an $\Im_t$-martingale. As the compensator is unique we finish the proof.

Remark 2.2.9 It is well know that the total hazard A_T is a standard exponential random variable. Follows, from Theorem 2.2.7 that, in the set $\{T = T_{(k)}\}$ $A_{T_{(k)}} = A_{T_{(k)}}((k))$ is a standard exponential random variable. Also, it can e proved, see Norros (1986), that $A_{T_{(1)}}((1)), A_{T_{(2)}}((2)), ..., A_{T_{(n)}}((n))$ are independent and identically distributed standard exponential random variables.

3. Linking structural signature and signature point process

To relates the structural signature generated by independent and identically distributed coherent system's components lifetimes and signature point process, generated by dependent coherent system's components lifetimes without simultaneous failures we are going to use the record values theory.

Chandler (1952) introduced a suitable stochastic model to study the theory of record values. After the introduction a broad spectrum of researchers worked in this field. Glick (1978) provides a survey of the first 25 years and Nagaraja (1988) discussed the posterior contributions. In the present section we use the standard value process just as it was introduced.

We consider to observe a sequence, $(T_n)_{n\geq 1}$, of continuous random variables, independent and identically distributed with distribution function F. A random variable T_j will be called a record if it exceeds in value all preceding observation, that is, $T_j > T_i, \forall i < j$. Let $(S_n)_{n\geq 0}$ be the sequence of record times with $P(S_0 = 1) = 1$ and

$$S_n = \min\{j : j > S_{n-1}, T_j > T_{S_{n-1}}, \quad \forall n \geq 1\}.$$

The corresponding record values $(R_n)_{n\geq 0}$ is defined as $R_n = T_{S_n}, n \geq 0$.

Increasing transformations of the $T_n's$ will not affect the values of S_n, but the sequence $(R_n)_{n\geq 0}$ which has distributions dependents on the specific common distribution function of $T_n's$. We intend to calculate theR_n distribution function.

We consider a sequence, $(E_n)_{n\geq 1}$, of independent and identically distributed random variables with standard exponential distribution and its corresponding sequence of records, denoted by $(E_n^R)_{n\geq 0}$. From the exponential distribution lack of memory, the differences between successive records, $E_n^R - E_{n-1}^R$ will again be independent and identically distributed standard exponential random variables, for all $n \geq 1$. Follows that

$$E_n^R = (E_n^R - E_{n-1}^R) + (E_{n-1}^R - E_{n-2}^R) + \ldots + (E_1^R - E_0^R) + E_0^R$$

which has a gamma distribution with parameters $n+1$and 1, with

$$P(E_n^R > t) = \sum_{k=0}^{n} \frac{e^{-t}t^k}{k!}.$$

Now, considering to observe the sequence $(T_n)_{n\geq 1}$, of continuous random variables independent and identically distributed with distribution function F and the increasing transformation $E_n^R = -\ln(1 - F(R_n))$, of continuous random variables independent and identically with standard exponential distribution we have

$$P(R_n > t) = P(F^{-1}(1 - e^{-E_n^R}) > t) = P(1 - e^{-E_n^R} > F(t)) =$$
$$P(E_n^R > -\ln(1 - F(t))) = \sum_{k=0}^{n} \frac{e^{-\ln(1-F(t))}(-\ln(1 - F(t)))^k}{k!}.$$

However this expression is the reliability function of $T_{(k)}$, the k-th event in a non homogeneous poisson process with the mean value function $\bigwedge(t) = -\ln(1 - P(T_{(k)} \leq t|\Im_t)$, which, by Remark 2.2.7, is the compensator $A_t((k)) = -\ln(1 - P(T_{(k)} \leq t \wedge T_{(k)}|\Im_t)$, which is a determinist increasing function in the set $\{T_{(k)} < t\}$.

This shows that, in a NHPP with mean value function $\Lambda(t) = -\ln \overline{F}(t)$, R_k and $T_{(k)}$ are equal in distribution. In other words, an NHPP is essentially a record counting process in the sense that its mean value function $\Lambda(t)$ is continuous and tends to ∞ as $t \to \infty$, and the sequence of occurrence times of NHPP can be considered as the record values of a sequence of independent and identically distributed random variables each having distribution function F.

Consider a complex system under a maintenance program, so that when the system fails it is repaired and continued in operation. Assume that the components are subject to failures according to a well-known NHPP with deterministic compensator. Follows that the occurrence times are stochastically dependent and generates the signature point process. However,they can be interpreted as the ordered record values of random variables independent and identically distributed which generates the signatures, linking the two concepts.

4. Acknowledgements

This work was partially supported by FAPESP, Proc. No.2015/02249-1.

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