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Dan Henry's work on perturbation of the boundary problems

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Abstract
Perturbation of the boundary, or perturbation of the domain of definition of a boundary value problem have been investigated by many authors since the classical works of Rayleigh (Theory of sound, Dover, 1945) and Hadamard (Mémoire sur le problème d'analyse relatif à l'equilibre des plaques élastiques encastrées, Ouvres de J. Hadamard 2, ed. C.N.R.S. Paris, 1968). Many contributions to the theme and, in special to questions related to "generic properties" were made by Dan Henry, during his stay at the Instituto de Matemática e Estatística da USP. In characteristic fashion, his results were not published in separate articles but collected instead in the monograph (Henry in Perturbation of the boundary in boundary value problems of PDEs, London Mathematical Society lecture note series - 318, Cambridge University Press, 2005). We describe some of these results and, briefly, some ulterior developments by followers.

Keywords Boundary value problem · Generic property · Transversality theorem · Simple eigenvalue

1 Introduction

The mathematical work of Dan Henry, aside from some sporadic incursions, centered around two main themes: the "geometric theory" of semilinear parabolic equations and perturbation of the domains in boundary value problems of P.D.Es. These were, respectively, the subject of his monographs [5] and [6]. The first is, by far, his most well known opus. Since its first publication in 1981, it remains as the basic

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reference in the field, having been translated to many languages and received more than 10,000 citations accordingly to the Mathscinet index. The second monograph, though, in my opinion, of comparable depth and brilliance, did not have a comparable impact, for a series of reasons, including the narrower scope of its subject and the relative isolation in which the work has been done. I decided to restrict this review to the second monograph for a couple of reasons. Firstly, a personal one: it is the subject I am more at ease with, having it been the subject of my PhD thesis, under Dan's supervision. And, more importantly, it was entirely developed and written during his stay with us at the University of São Paulo, from the earlier eighties until his untimely death in 2002.

Perturbation of the boundary in boundary value problems have been investigated by many authors, since the classical works of Rayleigh [15] and Hadamard [4].

More recently, the theme was extensively investigated, mainly from the point of view of Optimization Theory, by many authors, who developed the theory known as "Shape Optimization" or "Shape Theory". An extensively introduction to the subject as well as a comprehensive bibliography can be found in the monographs: [3] and [7].

In his monograph [6], Henry, working independently, considered the problem from a somewhat different viewpoint. He developed a kind of "Differential Calculus" for domain perturbations. We start by describing briefly his approach and then state some of the results obtained.

2 Henry's framework

Let $\Omega \subset \mathbb{R}^n$ be a C^m domain, $m \geq 1$,

Consider a nonlinear differential operator F_Ω in $\mathcal{C}^m(\Omega)$ given by

$$F_\Omega(u)(x) = f(x, Lu(x)), \quad x \in \Omega$$

where f is a (generally nonlinear) function and

$$Lu(x) = \left(u(x), \frac{\partial u}{\partial y_1}(x), \dots, \frac{\partial u}{\partial y_n}(x), \frac{\partial^2 u}{\partial y_1^2}(x), \frac{\partial^2 u}{\partial y_1 \partial y_2}(x), \dots \right), \quad x \in \mathbb{R}^n.$$

The aim here is to study the behavior of solutions of the boundary value problem

$$\begin{cases} f(\cdot, Lu(\cdot)) = 0 & \text{in } \Omega, \\ b(,\cdot, Rv(\cdot), M\, N_\Omega(\cdot)) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

when the region Ω is changed. Here R, M are constant-coefficient differential operators and $N_\Omega(\cdot)$ is the outward unit normal on $\partial\Omega$.

2.1 Eulerian and Lagrangian formulations

One conceptual difficulty in the task is that the function spaces involved change as the region changes. In his work, Henry overcomes this problem by bringing it back to the reference region, using a change of coordinates as described in the sequel.

Let $\Omega \subset \mathbb{R}^n$ be a C^m, $m \geq 1$, domain. If $h : \Omega \to \mathbb{R}^n$ is a C^m diffeomorphism into its image $\Omega_h = h(\Omega)$ and $v \in C^m(h(\Omega))$, we may define the "pull-back map"

$$\begin{aligned} h^* : C^m(\Omega_h) &\to C^m(\Omega) \\ u &\mapsto u \circ h \end{aligned}$$

which is an isomorphism in various function spaces, with inverse $h^{*-1} = (h^{-1})^*$.

The formal differential operator $(F_{\Omega_h} v)(y) = f(y, Lv(y))$, $y \in \Omega_h$, acting on functions defined in the "perturbed" region $\Omega_h = h(\Omega)$:

$$F_{h(\Omega)} : D_{F_{h(\Omega)}} \subset C^m(h(\Omega)) \to C^0(h(\Omega)).$$

is the *Eulerian form* of the problem. Using the pull-back we may define the differential operator:

$$h^* F_{h(\Omega)} h^{*-1} : h^* D_{F_{h(\Omega)}} \subset C^m(\Omega) \to C^0(\Omega),$$

the *Lagrangian form*, acting now in the *fixed* region Ω.

The Eulerian form is more natural and, in general, more convenient for computations. The Lagrangian form is more convenient for formal proofs.

The advantage of the Lagrangian form is that it facilitates the use of analytical tools like the Implicit Function and Transversality theorems, However to use this tools we need:

- To prove the differentiability of the map

$$(u, h) \mapsto h^* F_{h(\Omega)} h^{*-1} u,$$

- To compute its derivatives.

The differentiability in h follows from the fact that the map is a rational function of the derivatives of h up to order m. Therefore, we need only to compute the derivative along a C^1 curve of diffeomorphisms $t \mapsto h(t, .)$.

2.2 Computation of the derivative

We need to compute the derivative of

$$h \mapsto h^* F_{h(\Omega)} h^{*-1} u$$

$$\frac{\partial}{\partial t} F_{\Omega(t)}(v)(y) = \frac{\partial}{\partial t} f(y, Lv(y))$$

with $y = (h(t, x))$ fixed in $\Omega(t) = h(t, \Omega)$.

To this end, it is convenient to introduce the *"anti-convective derivative"*

$$D_t = \frac{\partial}{\partial t} - U(t,x)\frac{\partial}{\partial x}, \qquad U(x,t) = \left(\frac{\partial h}{\partial x}\right)^{-1}\frac{\partial h}{\partial t}.$$

The anti convective derivative satisfies the following nice property for sufficiently regular h and v:

$$D_t(h^{\star}v)(x) = h^{\star}\left(\frac{\partial v}{\partial t}\right)(x), \quad \text{for } x \in \Omega.$$

Using this property, it is not difficult to prove the following formulas (with suitable regularity hypotheses), showing how to pass from one formulation to the other.

Theorem 1 *Let $F_Q(t)v$ be the map defined by*

$$y \mapsto f(t, y, Lv(y)), \quad y \in Q.$$

and $t \to h(t,.)$ a curve of embeddings defined in Ω, Then, at points in Ω

$$D_t(h^* F_{\Omega(t)}(t) h^{*-1})u = h^* \dot{F}_{\Omega(t)}(t) h^{*-1} u + h^* F'_{\Omega(t)}(t) h^{*-1} \cdot D_t u$$

where

$$\dot{F}_Q(t) = \frac{\partial f}{\partial t}(t, y, Lv) \quad \textit{and}$$
$$F'_Q(t)v \cdot \omega(y) = \frac{\partial f}{\partial \lambda}(t, y, Lv(y)) \cdot L\omega(y).$$

We also need to differentiate boundary conditions of the form

$$b(t, y, Lv, MN_{\Omega(t)}(y)) = 0 \text{ para } y \in \partial\Omega(t),$$

where L and M are linear differential operators with constant coefficients and $N_{\Omega(t)}$ is the unit exterior normal at the boundary $\partial\Omega(t)$. Let us define

$$\mathcal{B}_{h(\Omega)}(t)v(y) = b(t, y, Lv(y), MN_{h(\Omega)}, (y))$$

for y in a neighborhood of $\partial\Omega(t)$.

$$D_t(h^* \mathcal{B}_{h(\Omega)}(t) h^{*-1})(u) = h^* \dot{\mathcal{B}}_{h(\Omega)}(t) h^{*-1} u + h^* \mathcal{B}'_{h(\Omega)}(t) h^{*-1}(u) \cdot D_t u + \left(h^* \frac{\partial \mathcal{B}_{h(\Omega)}}{\partial N}(t) h^{*-1}\right) u \cdot D_t(h^* N_{\Omega(t)}),$$

Theorem 2
where $h = h(t,.)$, $\dot{\mathcal{B}}$, $\mathcal{B}'_{h(\Omega)}(t)$ are as before and

$$\frac{\partial \mathcal{B}_{h(\Omega)}}{\partial N}(t)(v) \cdot \eta(y) = \frac{\partial b}{\partial \mu}(t, y, Lv(y), MN_{\Omega}(y)) \cdot M\eta(y).$$

3 The transversality theorem

A basic tool in the proof of many "generic results" is the Transversality Theorem, originally proved by Thom for finite dimensional problems in [16] and extended by Abraham and Quine for Fredholm maps in infinite dimensional spaces (see [1] and [14]). A generalized version, for semi-Fredholm operators, better suited for variation of the boundary problems, was obtained by Henry in [6]. To state the result, we first need some preliminary definitions.

3.1 Generic properties

If X is a topological space, we say that a subset F of X is

- *rare* if its closure has empty interior.
- *meager* if it is contained in a countable union of rare subsets of X.
- *residual* if its complement in X is meager.

We say that X is a *Baire space* if any residual subset of X is dense.

Definition 1 A property $\mathcal{P}(x)$ depending on a parameter $x \in X$ is *generic* if it holds for a *residual* subset of X.

Let X be a Baire space and $I = [0, 1]$. For any closed or $\sigma - closed$ $F \subset X$ and any non negative integer m we say that the codimension of F is greater or equal to m (*codim* F≥ m) if the subset $\{\phi \in \mathcal{C}(I^m, X) \mid \phi(I^m) \cap F$ is non-empty $\}$ is meager in $\mathcal{C}(I^m, X)$. We say *codim* $F = k$ if k is the largest integer satisfying *codim* $F \geq k$.

Let now X and Y be Banach spaces A map $T \in \mathcal{L}(X, Y)$ is a *semi-Fredholm* map if the range of T is closed and at least one (or both, for Fredholm) of dim $\mathcal{N}(T)$, codim $\mathcal{R}(T)$ is finite; the *index* of T is then

$$ind(T) = dim\mathcal{N}(T) - codim\mathcal{R}(T).$$

Let $f : X \to Y$ be a $\mathcal{C}^k$ function between Banach spaces. We say that x is a *regular point* of f if the derivative $f'(x)$ is surjective. Otherwise, x is called a *critical* point of f. A point $y \in Y$ is a *critical value* if it is the image of some critical point of f.

Given an open, bounded, C^m region $\Omega_0 \subset \mathbb{R}^n$, consider the following open subset of $C^m(\Omega, \mathbb{R}^n)$

$$\text{Diff}^m(\Omega) = \left\{ h \in C^m(\Omega, \mathbb{R}^n) \mid h \text{ is injective and } \frac{1}{|deth'(x)|} \text{ is bounded in } \Omega \right\}.$$

and the collection of regions $\{h(\Omega_0) \mid h \in \text{Diff}^m(\Omega_0)\}$.

A neighborhood of Ω is given by

$$\{h(\Omega); \|h - i_\Omega\|_{C^m(\Omega,\mathbb{R}^n)} < \varepsilon, \varepsilon > 0 \text{ sufficiently small}\},$$

where $i_\Omega : \Omega \mapsto \mathbb{R}^n$ is the inclusion. When $\|h - i_\Omega\|_{C^m(\Omega,\mathbb{R}^n)}$ is small, h is a C^m embedding of Ω in $\mathbb{R}^n$, a C^m diffeomorphism to its image $h(\Omega)$.

Michelleti showed in [9] that this topology is metrizable, and the set of regions C^m-diffeomorphic to Ω may be considered a separable metric space which we denote by $\mathcal{M}_m(\Omega) = \mathcal{M}_m$.

A function F defined in the space $\mathcal{M}_m$ with values in a Banach space is C^m or analytic if $h \mapsto F(h(\Omega))$ is C^m or analytic as a map of Banach spaces (h near i_Ω in $C^m(\Omega, \mathbb{R}^n)$). In this sense, we may express problems of perturbation of the boundary of a boundary value problem as problems of differential calculus in Banach spaces.

3.2 Thom-Abraham transversality theorem

Theorem 3 Theorem (Thom-Abraham-Quine) *Suppose given Banach manifolds* X, Y, Z *of class* C^k, $k \geq 1$; *an open set* $A \subset X \times Y$; *a* C^k map $f : A \mapsto Z$ *and a point* $\xi \in Z$.

Assume for each (x,y) $\in f^{-1}(\xi)$ *that:*

(1) $\frac{\partial f}{\partial x}(x,y) : T_x X \mapsto T_\xi Z$ *is Fredholm.*
(2) $Df(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) : T_x X \times T_y Y \mapsto T_\xi Z$ *is surjective*
(3) $(x,y) \mapsto y : f^{-1}(\xi) \mapsto Y$ *is* σ*-proper,*

Then $Y_{crit} = \{y \mid \xi$ is a critical value of $f(\cdot, y) : A_y \mapsto Z$ *is a meager set in* Y.
In [6], Henry applied Abraham's Transversality Theorem to obtain a short proof of the generic simplicity of the eigenvalues of the Dirichlet problem for the Laplacian operator as follows.

$$\begin{cases} (\Delta + \lambda)u(x) = 0 & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases}$$

Theorem 4 *For a residual set in* $\mathrm{Diff}^2(\Omega)$, *the eigenvalues of the problem in* $h(\Omega)$ *are all simple.*

We give an sketch of the proof here. For simplicity, we suppose that our regions are of class C^3.

Consider the map:

$$F : H^2 \cap H_0^1(\Omega) \times \mathbb{R} \times Diff^3(\Omega) \longrightarrow L^2(\Omega),$$

defined by

$$F(u, \lambda, h) = h^\star (\Delta + \lambda) h^{\star^{-1}} u.$$

Then, the eigenvalues in the region $h(\Omega)$ are simple if and only if 0 is a regular value of the map

$$F_h := F(\cdot,\cdot,h) : H^2 \cap H_0^1(\Omega) \times \mathbb{R} \longrightarrow L^2(\Omega).$$

By the Transversality Theorem, it is enough to show that 0 is a regular value of F. Suppose this is false, and let (u_0, λ_0, h_0) be a critical point. (We may, by "transferring the origin", suppose that $h_0 = i_\Omega$). Then

$$\begin{aligned}(\dot{u},\dot{\lambda},\dot{h}) \mapsto DF(u_0,\lambda_0,i_\Omega)(\dot{u},\dot{\lambda},\dot{h})\\ = (\Delta+\lambda)\dot{u} + \dot{\lambda}u_0 + (\dot{h}\cdot\nabla)(\Delta+\lambda)u_0 - (\Delta+\lambda)\dot{h}\cdot\nabla u_0\end{aligned}$$

is not surjective. Thus, there exists $\psi \neq 0$ in $L^2(\Omega)$ which is orthogonal to the image of DF, that is

$$0 = \int_\Omega \psi(\Delta+\lambda)(\dot{u} - \dot{h}\cdot\nabla u_0) + \dot{\lambda}\psi u_0.$$

Taking $\dot{h} = 0$ and $\dot{u} = 0$ we see that $\int_\Omega \psi u_0 = 0$. Taking $\dot{h} = 0$ and varying $\dot{u}$:

$$0 = \int_\Omega (\Delta+\lambda)\dot{u}\psi \quad \forall \dot{u} \in L^2(\Omega).$$

It follows that ψ is a weak, therefore strong, solution of $(\Delta+\lambda)\psi = 0$ in Ω, $\psi = 0$ in $\partial\Omega$. Varying now $\dot{h}$ and using Green's identity:

$$\begin{aligned}0 &= \int_\Omega -((\Delta+\lambda)\dot{h}\cdot\nabla u_0)\cdot\psi\\ &= \int_\Omega \dot{h}\cdot\nabla u_0(\Delta+\lambda)\psi - ((\Delta+\lambda)\dot{h}\cdot\nabla u_0)\cdot\psi\\ &= \int_\Omega (\dot{h}\cdot\nabla u_0)\cdot\Delta\psi - \Delta(\dot{h}\cdot\nabla u_0)\cdot\psi\\ &= \int_{\partial\Omega} \dot{h}\cdot\nabla u_0 \frac{\partial\psi}{\partial N}\\ &= \int_{\partial\Omega} \dot{h}\cdot N\cdot\frac{\partial u_0}{\partial N}\cdot\frac{\partial\psi}{\partial N}.\end{aligned}$$

Therefore, we conclude that ψ satisfies $\Delta\psi + \lambda\psi = 0$ in Ω, $\psi = \frac{\partial\psi}{\partial N} = 0$ in $\partial\Omega$. By uniqueness in the Cauchy problem for second order elliptic equations, it follows that $\psi \equiv 0$, and we have reached the searched for contradiction. □

Remark 1 Here and afterward, one can extend the result to less regular (say, C^2) regions in the following way. The property of the first N eigenvalues being simple is open in the weaker topology by continuity. For the density part, we can choose a stronger topology and then take intersection.

3.3 Henry's transversality theorem

We highlight two important ingredients in the proof above:

- The perturbation function $\dot{h}$ was general enough to allow the conclusion that the integrand above should vanish.
- The conditions obtained were sufficient for the Uniqueness Theorem to apply.

In many boundary value problems these conditions are not present, and new tools are needed. Perhaps the simplest example is the following nonlinear problem.

Theorem 5 *For a residual set of C^2 regions $\Omega \subset \mathbb{R}^n$, the solutions of*

$$\begin{cases} \Delta u + f(x,u) = 0 & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases}$$

are simple, that is, for "most" C^2 regions $\Omega \subset \mathbb{R}^n$, the linearization

$$\Phi \to \Delta\Phi + \frac{\partial f}{\partial u}(\cdot, u)\phi \,:\, H^2(\Omega) \cap H^1(\Omega) \to L^2(\Omega)$$

is an isomorphism.

Remark 2 If we try to apply the Abraham's Transversality Theorem here, reasoning by contradiction, we conclude that, if the result is not true, there must exist a solution of the problem satisfying the additional condition $\frac{\partial u}{\partial N} = 0$ in $\partial\Omega$. If $f(x,0) \equiv 0$ on $\partial\Omega$, it would follow by uniqueness in the Cauchy Problem that $u \equiv 0$. Without this hypothesis, it is possible to find examples where u is a solution with $u = \frac{\partial u}{\partial N} = 0$ in $\partial\Omega$.

If we want to prove that this is an "unusual" property, we may add the condition $\frac{\partial u}{\partial N} = 0$ in the definition of the map $F(u, h)$ above. In the present case, this may be done simply by changing the domain, that is, we may consider the map

$$G \,:\, W_0^{2,p}(\Omega) \times Diff^3(\Omega) \longrightarrow L^p(\Omega), n < p < \infty,$$
$$F(u,h) = h^\star(\Delta + f)h^{\star^{-1}} u$$

But now

- $G(\cdot, h)$ is semi-Fredholm with index $-\infty$ (not Fredholm),
- There is little hope to prove that $Dg(u, h)$ is surjective at a critical point.

These considerations strongly suggest the following extension of the Transversality Theorem, which is one of the central results obtained in [6]

Theorem 6 *Suppose given positive numbers k and m; Banach manifolds X, Y, Z of class C^k; an open set $A \subset X \times Y$; a C^k map $f \,:\, A \mapsto Z$ and a point $\xi \in Z$.*

Assume, for each (x,y) $\in f^{-1}(\xi)$, *that*:

(1) $\frac{\partial f}{\partial x}(x,y) \,:\, T_x X \mapsto T_\xi Z$ *is semi-Fredholm with index* $< k$.

(2) *Either*

(α) $Df(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) : T_xX \times T_yY \mapsto T_\xi Z$ *is surjective or*

(β) $\dim\left\{\mathcal{R}(Df(x,y))/\mathcal{R}\left(\frac{\partial f}{\partial x}(x,y)\right)\right\} \geq m + dim\mathcal{N}\left(\frac{\partial f}{\partial x}(x,y)\right)$.

Further assume

(3) $(x,y) \mapsto y : f^{-1}(\xi) \mapsto Y$ *is σ-proper, that is* $f^{-1}(\xi) = \bigcup_{j=1}^{\infty} \mathcal{M}_j$ *is a countable union of sets* $\mathcal{M}_j$ *such that* $(x,y) \mapsto y : \mathcal{M}_j \mapsto Y$, *is a proper map for each* j.

Let$A_y = \{x|(x,y) \in A\}$ *and* $Y_{crit} = \{y \mid \xi$ *is a critical value of* $f(\cdot,y) : A_y \mapsto Z\}$.

Then Y_{crit} *is a meager set in* Y *and, if* (x,y) $\mapsto$ y : $f^{-1}(\xi) \mapsto Y$ *is proper*, Y_{crit} *is also closed. If ind* $\frac{\partial f}{\partial x} \leq -m < 0$ *on*$f^{-1}(\xi)$, *then* $2(\alpha)$ *implies* $2(\beta)$ *and*

$$Y_{crit} = \{y \mid \xi \in f(A_y, y)\}$$

has codimension $\geq$ m in Y. [*Note* Y_{crit} *is meager iff codim* $Y_{crit} \geq 1$].

Proof of Theorem 5 We verify condition (2)(β). that is , we show that

$$\dim\left\{\mathcal{R}(Df(x,y))/\mathcal{R}\left(\frac{\partial f}{\partial x}(x,y)\right)\right\} = \infty.$$

Suppose not. Then, there exist $f_1, \dots, f_m \in L^p(\Omega)$, such that $f_j = L(\dot{u}_j - \dot{h}_j \cdot \nabla u)$, for some $\dot{u}_j \in W_0^{2,p}, \dot{h}_j \in C^3(\Omega)$ and such that, for every $(\dot{u}, \dot{h})$, there exists $\dot{v} \in W_0^{2,p}$ and $c_1, \dots, c_m \in \mathbb{R}$ so

$$\sum_1^m c_j f_j + L\dot{v} = L(\dot{u} - \dot{h} \cdot \nabla u), \text{ or}$$

$$L\left(\dot{u} - \dot{v} - \sum_1^m c_j \dot{u}_j - \left(\dot{h} - \sum_1^m c_j \dot{h}_j\right) \cdot \nabla u\right) = 0.$$

Let $\phi_1, \dots, \phi_k$ be a basis for the kernel of L. Then, for some constants $b_1, \dots, b_k$

$$\dot{u} - \dot{v} - \sum_1^m c_j \dot{u}_j - \left(\dot{h} - \sum_1^m c_j \dot{h}_j\right) \cdot \nabla u = \sum_1^k b_j \phi_j.$$

Computing the normal derivative

$$\left(\dot{h} - \sum_1^m c_j \dot{h}_j\right) \cdot N \frac{\partial^2 u}{\partial N^2} = \sum_1^k b_j \frac{\partial \phi_j}{\partial N} \quad \text{on } \partial\Omega.$$

Since $\frac{\partial^2 u}{\partial N^2}_{|\partial\Omega} = \Delta u_{|\partial\Omega} = -f(x,0)$, we obtain

$$\dot{h} \cdot f(x,0) = \sum_1^m c_j \dot{h}_j \cdot Nf(x,0) + \sum_1^k b_j \frac{\partial \phi_j}{\partial N}.$$

Thus, the map

$$\Gamma : \dot{h} \mapsto \dot{h} \cdot Nf(\cdot, 0)$$

must be of finite rank, which is not possible, since $f(\cdot, 0)_{|\partial\Omega} \not\equiv 0$.

The conclusion is that there is an open dense set of $h \in Diff^3(\Omega)$ such that either $f(\cdot, 0)_{|h(\partial\Omega)} \equiv 0$, or there are no solutions in $W^{2,p}(\Omega)$. Applying Abraham's Transversality Theorem in this set gives the desired result. □

4 Eigenvalues of the Dirichlet Laplacian on symmetric regions

In many problems, the application of Henry's Transversality Theorem leads, as above (by contradiction). to the conclusion that a certain map Γ defined in $\text{Diff}^m(\Omega)$ must be of finite rank. There, the map Γ was unusually simple, and a contradiction immediately ensues. In other cases, the map Γ is a complicated gadget and it is far from immediate to obtain a contradiction. Many examples were considered by Henry in his monograph [6], where a completely new method, the *method of rapidly oscillating functions* was developed to overcome this difficulty. Even with this powerful tool, however, some questions remain open or partially answered. We consider now only some examples more familiar to this author.

The difficulties mentioned appear even in the comparatively simple case of the eigenvalues of the Dirichlet problem of the Laplacian, considered in Sect. 3, if we restrict attention to *symmetric* regions.

To be more precise, we start with some basic definitions

Definition 2

- If G is a subgroup of the orthogonal group $O(n)$, we say that that a region $\Omega \subset \mathbb{R}^n$ is *G-symmetric* (or *G-invariant*) if $g\Omega = \Omega$, for all $g \in G$.
- A map $h : \Omega \to \mathbb{R}^n$ is *G-equivariant* if $h \circ g = g \circ h$ for all $g \in G$.

If we restrict attention to the set of G-symmetric regions, the problem becomes much more difficult, since the set of "perturbations" is also restricted. In fact, except in a very exceptional case, multiple eigenvalues must exist in these regions since whenever u is an eigenfunction of the Laplacian in a G-symmetric region Ω, then $u \circ g$ is also an eigenfunction. It is then convenient to change the concept of simplicity of eigenvalues, as follows.

Definition 3 We say that an eigenvalue λ is G-simple if the associated eigenspace is generated by the orbit $\{u \circ g, \; g \in G\}$ of any eigenfunction u.

The simplest situation that can occur then, in the presence of symmetry, is the G-simplicity of the eigenvalues. It is shown in [10] that this is indeed the"generic situation" at least for compact commutative semigroups of $O(n)$.

To avoid algebraic complications we consider here only the simplest nontrivial case: $G = \mathbb{Z}_3$, the group generated by a rotation of angle $\theta = 2\pi/3$.

The *complex* space $L^2(\Omega, \mathbb{C})$ can be decomposed as an orthogonal sum

$$L^2(\Omega, \mathbb{C}) = M_0 \oplus M_1 \oplus M_{-1}, \qquad M_k = \{u \; : \; u \circ g = e^{ik\theta} u\}.$$

These subspaces are nontrivial and invariant for the Laplacian.

If u is a (complex) eigenfunction in M_1, then $\overline{u}$ is an independent eigenfunction in M_{-1} and $Re(u)$ and $Im(u)$ are linearly independent real eigenfunctions. Therefore, the eigenvalues associated to eigenfunctions in M_1 must have multiplicity at least 2 both in the real and complex sense. The same is true in M_{-1} The eigenvalues in M_0 may be simple. We want to show that this "simplest possible situation" is generic. This is the main result in [10] (for finite commutative groups).

Theorem 7 *For a residual set of* C^3, *connected G-symmetric regions, the eigenvalues of the Dirichlet Laplacian have multiplicity at most* 2 (*that is, the associated eigenspaces have dimension at most* 2).

Sketch of the proof

We first consider the problem in the invariant subspace M_1. Consider the map:

$$F \; : \; H^2 \cap H^1_0(\Omega, \mathbb{C}) \cap M_1 \times \mathbb{R} \times E \mapsto M_1$$

defined by

$$F(u, \lambda, h) = h^{\star}(\Delta + \lambda) {h^{\star}}^{-1} u,$$

where E is the set of C^3 equivariant embeddings of $\Omega \to \mathbb{R}^n$.

As before the eigenvalues in the region $h(\Omega)$ are simple if and only if 0 is a regular value of the map

$$F_h := F(\cdot, \cdot, h) \; : \; H^2 \cap H^1_0(\Omega, \mathbb{C}) \cap M_1 \times \mathbb{R} \mapsto M_1.$$

By (Abraham's) Transversality Theorem, it is enough to show that 0 is a regular value of F. Suppose this is false, and let $(u_0, \lambda_0, i_\Omega)$ be a critical point.

Then

$$(\dot{u}, \dot{\lambda}, \dot{h}) \mapsto DF(u_0, \lambda_0, i_\Omega)(\dot{u}, \dot{\lambda}, \dot{h}) = \\ = (\Delta + \lambda)\dot{u} + \dot{\lambda} u_0 + (\dot{h} \cdot \nabla)(\Delta + \lambda) u_0 - (\Delta + \lambda)\dot{h} \cdot \nabla u_0$$

is not surjective. We want to argue, as before, that there must exist a (complex-valued) function $\psi \neq 0$ in M^1 , which is orthogonal to the image of DF. However, the image of DF *is not a complex subspace of* $L^2(\Omega, \mathbb{C})$, just a *real* subspace, therefore this orthogonality property is obtained in the real space structure of $L^2(\Omega, \mathbb{C})$, that is

$$0 = Re \int_\Omega \overline{\psi}\big((\Delta + \lambda)(\dot{u} - \dot{h} \cdot \nabla u_0) + \dot{\lambda} u_0\big).$$

Proceeding as before, we then obtain that ψ must be another eigenfunction satisfying $\int_\Omega \psi\overline{u_0} = 0$ and $Re\left(\frac{\partial u_0}{\partial N} \cdot \frac{\overline{\partial\psi}}{\partial N}\right) \equiv 0.$

However, this is not enough to obtain a contradiction via the Uniqueness Theorem.

We then apply Henry's Transversality Theorem to show that the set of regions where this occur must have empty interior. To this end, we consider the map

$$G(u,\psi,\lambda,h) = \left(h^\star(\Delta+\lambda)h^{\star^{-1}}u, h^\star(\Delta+\lambda)h^{\star^{-1}}\psi, \int_\Omega \overline{\psi}u, h^\star B h^{\star^{-1}}(u,\psi)\right)$$

where $B(v,w) = Re\left(\frac{\partial v}{\partial N} \cdot \frac{\overline{\partial w}}{\partial N}\right)$, defined in appropriate subsets of $M_1 \times M_1 \times \mathbb{R} \times E$.

The crucial condition to be verified is (2)(β). If we suppose it is not verified then, after a long computation, we conclude that the map

$$\sigma \overset{\Gamma}{\mapsto} Re\left[u_N \frac{\partial}{\partial N}\mathcal{B}_{\Delta+\lambda}\left(\sigma\overline{\psi}_N\right) + \psi_N \frac{\partial}{\partial N}\mathcal{B}_{\Delta+\lambda}\left(\sigma\overline{u}_N\right)\right]_{|\partial\Omega}$$

with $\sigma = \dot{h} \cdot N$, must be of finite rank, where $\mathcal{B}_{\Delta+\lambda}$ is given by

$$v = \mathcal{B}_{\Delta+\lambda} g$$

when

$$\begin{cases} (\Delta+\lambda)v \;\in\; \mathcal{N}(\Delta+\lambda) \\ v_{|\partial\Omega} = g \quad , \quad v \perp \mathcal{N}(\Delta+\lambda) \end{cases}$$

The operator Γ above is a pseudo-differential operator, which can only be of finite rank if its symbol is identically zero. However, the theory of pseudo-differential operators ordinarily computes only the principal symbol, and in our problems the (apparent) principal symbol usually vanishes. This is the main technical difficulty in this problem. To overcome it, at least in some cases, Henry developed in [6] the "method of oscillating functions" which consists in applying the operator Γ to functions of the form $\rho = \gamma e^{i\omega\theta}$, where γ and θ are smooth even functions in $\partial\Omega$ and ω is a real number and using asymptotic expansion in ω.

In the present case the method gives (if $n \geq 3$)

$$\Gamma(\gamma\cos\omega\theta) = \omega^{-1}\cos\omega\theta Re(\partial_\theta u_N \partial_\theta \overline{\psi}_N) + O(\omega^{-2}).$$

It follows that, if the rank is finite, then

$$Re\left(\frac{\partial u_N}{\partial\tau} \cdot \frac{\partial\psi_N}{\partial\tau}\right) \equiv 0$$

in a open set U of $\partial\Omega$, for any vector τ tangent to $\partial\Omega \cap U$.

With some more effort, we are now able to apply Uniqueness arguments to show that this implies $\psi \equiv 0$, finally obtaining the searched for contradiction. Therefore, the eigenvalues of the Laplacian restricted to the subspace M_1 are generically simple (in the complex sense).

Now, if λ is an eigenvalue in M_{-1} associated to an eigenfunction ψ, then it is also an eigenvalue in M_1, with associated eigenfunction $\overline{\psi}$.

Finally if λ_1 is an eigenvalue both in M_0 and M_1, we can compute the derivatives along a curve of (equivariant) diffeomorphisms and use the Transversality Theorem and the method of "rapidly oscillating function" again to show they can be separated. □

4.1 Some later results

Similar results were obtained for the *Neumann* Laplacian in [8]. The extension, though expected, is far from trivial.

Higher order problems can also be considered. For example the following eigenvalue problem for the Biharmonic operator

$$\begin{cases} (\Delta^2 + \lambda)u = 0 & \text{in } \Omega \\ u = \frac{\partial u}{\partial N} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

It was shown in [11], that the eigenvalues of this problem in $\mathbb{Z}_2$ symmetric regions are generically simple. For this proof an extension of the method of rapidly oscillating functions, obtained in [12], was used.

The general framework developed by Henry can also be used for time dependent problems. For instance, consider the following family of semilinear parabolic problem with nonlinear Neumann boundary conditions:

$$\begin{cases} u_t(x,t) &= \Delta u(x,t) - au(x,t) + f(u(x,t)), \quad x \in \Omega_\epsilon \quad \text{and} \quad t > 0, \\ \dfrac{\partial u}{\partial N}(x,t) &= g(u(x,t)), \quad x \in \partial\Omega_\epsilon \quad \text{and} \quad t > 0, \end{cases} \tag{3}$$

where a is a positive constant $f, g : \mathbb{R} \to \mathbb{R}$ are real functions and Ω_ϵ is the image of the unit square in R^2. under the family of diffeomorphisms

$$h_\epsilon(x_1, x_2) = (x_1\,,\, x_2 + x_2\,\epsilon\, sen(x_1/\epsilon^\alpha))$$

where $0 < \alpha < 1$ and $\epsilon > 0$ is small.

Using Henry's approach, one can pull the problem (3) back to the unit square, making it possible to compare solutions for different values of the parameter ϵ. This was done in [2], where the continuity of the global attractors at $\epsilon = 0$ was proved.

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