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GENERALIZED JORDAN DERIVATIONS ON SEMIPRIME RINGS

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Abstract

The purpose of this note is to prove the following. Suppose $\mathfrak{R}$ is a semiprime unity ring having an idempotent element e ($e \neq 0$, $e \neq 1$) which satisfies mild conditions. It is shown that every additive generalized Jordan derivation on $\mathfrak{R}$ is a generalized derivation.

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1. Introduction

Let $\mathfrak{R}$ be a ring. Recall that an additive (linear) map δ from $\mathfrak{R}$ to itself is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathfrak{R}$; a Jordan derivation if $\delta(a^2) = \delta(a)a + a\delta(a)$ for each $a \in \mathfrak{R}$; and a Jordan triple derivation if $\delta(aba) = \delta(a)ba + a\delta(b)a + ab\delta(a)$ for all $a, b \in \mathfrak{R}$. More generally, if there is a derivation $\tau : \mathfrak{R} \to \mathfrak{R}$ such that $\delta(ab) = \delta(a)b + a\tau(b)$ for all $a, b \in \mathfrak{R}$, then δ is called a generalized derivation and τ is the relating derivation; if there is a Jordan derivation $\tau : \mathfrak{R} \to \mathfrak{R}$ such that $\delta(a^2) = \delta(a)a + a\tau(a)$ for all $a \in \mathfrak{R}$, then δ is called a generalized Jordan derivation and τ is the relating Jordan derivation. The structures of derivations, Jordan derivations, generalized derivations and generalized Jordan derivations have been systematically studied. It is obvious that every generalized derivation is a generalized Jordan derivation and every derivation is a Jordan derivation. But the converse is in general not true. Herstein [3] showed that every Jordan derivation from a 2-torsion free prime ring into itself is a derivation. Brešar [1] proved that Herstein's result is true for 2-torsion free semiprime rings. Jing and Lu, motivated by the concept of generalized derivation, initiate the concept of generalized Jordan derivation in [5]. Moreover, in [5] the authors conjecture that every generalized Jordan derivation on a 2-torsion free semiprime ring is a generalized derivation.

In the present paper we characterize generalized Jordan derivation on a semiprime ring $\mathfrak{R}$. We prove that if there is a nontrivial idempotent element in $\mathfrak{R}$ which satisfies mild conditions, then every generalized Jordan derivation is a generalized derivation.

In the ring $\mathfrak{R}$, let e be an idempotent element so that $e \neq 0$, $e \neq 1$. As in [4], the two-sided Peirce decomposition of $\mathfrak{R}$ relative to the idempotent e takes the form $\mathfrak{R} = e\mathfrak{R}e \oplus e\mathfrak{R}(1-e) \oplus (1-e)\mathfrak{R}e \oplus (1-e)\mathfrak{R}(1-e)$. We will formally set $e_1 = e$ and $e_2 = 1-e$. So letting $\mathfrak{R}_{mn} = e_m\mathfrak{R}e_n$, $m, n = 1, 2$, we may write $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$. Moreover, an element of the subring $\mathfrak{R}_{mn}$ will be denoted by a_{mn}.

2. Results and proofs

In this section we discuss the generalized Jordan derivations on rings. The following theorem is our main result.

Theorem 2.1. *Let $\mathfrak{R}$ be a 2-torsion free semiprime unity ring containing a nontrivial idempotent e_1. Consider $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$ the Peirce decomposition relative to the idempotent e_1 satisfying the following conditions:*

(♠) *if $x_{11} \cdot \mathfrak{R}_{12} = 0$ then $x_{11} = 0$;*
if $x_{21} \cdot \mathfrak{R}_{12} = 0$ then $x_{21} = 0$.

Then every generalized Jordan derivation from $\mathfrak{R}$ into itself is a generalized derivation.

Henceforth, let $\mathfrak{R}$ be a 2-torsion free semiprime unity ring containing a nontrivial idempotent e_1. Consider $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$ the Peirce decomposition relative to the idempotent e_1 satisfying the following conditions:

(♠) if $x_{11} \cdot \mathfrak{R}_{12} = 0$ then $x_{11} = 0$;
if $x_{21} \cdot \mathfrak{R}_{12} = 0$ then $x_{21} = 0$.

Let $\delta : \mathfrak{R} \to \mathfrak{R}$ be a generalized Jordan derivation and $\tau : \mathfrak{R} \to \mathfrak{R}$ the relating Jordan derivation such that $\delta(a^2) = \delta(a)a + a\tau(a)$ for all $a \in \mathfrak{R}$. We shall complete the proof of the above theorem by proving several lemmas.

Lemma 2.2. *For all $a, b, c \in \mathfrak{R}$, the following statements hold:*

(i) $\delta(ab + ba) = \delta(a)b + a\tau(b) + \delta(b)a + b\tau(a)$;
(ii) $\delta(aba) = \delta(a)ba + a\tau(b)a + ab\tau(a)$;
(iii) $\delta(abc + cba) = \delta(a)bc + a\tau(b)c + ab\tau(c) + \delta(c)ba + c\tau(b)a + cb\tau(a)$.

Proof. See [5, Lemma 2.1]. □

Lemma 2.3. *$\tau(e_1) = [e_1, s]$ for some $s \in \mathfrak{R}$, where $[x, y] = xy - yx$ for $x, y \in \mathfrak{R}$.*

Proof. Write $\tau(e_1) = s_{11} + s_{12} + s_{21} + s_{22}$. Since $\tau(e_1) = \tau(e_1)e_1 + e_1\tau(e_1)$, we have $s_{11} + s_{12} + s_{21} + s_{22} = 2s_{11} + s_{12} + s_{21}$, which implies that $s_{11} = s_{22} = 0$ and $\tau(e_1) = s_{12} + s_{21}$. Let $s = s_{12} - s_{21}$. It is obvious that $\tau(e_1) = [e_1, s]$. □

Observe that $d_s : \mathfrak{R} \to \mathfrak{R}$ so that $d_s(a) = [a, s]$ is a derivation and thus a Jordan derivation. Define Δ by $\Delta(a) = \delta(a) - d_s(a)$ for each $a \in \mathfrak{R}$. Clearly, Δ is also a generalized Jordan derivation from $\mathfrak{R}$ into itself, and $\Xi : \mathfrak{R} \to \mathfrak{R}$, defined by $\Xi(a) = \tau(a) - d_s(a)$ for each $a \in \mathfrak{R}$, is the relating Jordan derivation. Note that

$$\Xi(e_1) = \Xi(e_2) = 0. \tag{†}$$

Lemma 2.4. *$\Xi(a_{ij}) \subset \mathfrak{R}_{ij}$ for any $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2)$.*

Proof. *Case 1.* For $i = j = 1$, $a_{11} = e_1 a_{11} e_1$, we have from Lemma 2.2(ii) that

$$\Xi(a_{11}) = \Xi(e_1 a_{11} e_1) = \Xi(e_1) a_{11} e_1 + e_1 \Xi(a_{11}) e_1 + e_1 a_{11} \Xi(e_1) = e_1 \Xi(a_{11}) e_1.$$

By (†) we get $\Xi(a_{11}) \in \mathfrak{R}_{11}$.

Case 2. For $i = j = 2$ write $\Xi(a_{22}) = b_{11} + b_{12} + b_{21} + b_{22}$ we have from Lemma 2.2 item (i)

$$\begin{aligned} 0 = \Xi(e_1 a_{22} + a_{22} e_1) &= \Xi(e_1) a_{22} + e_1 \Xi(a_{22}) + \Xi(a_{22}) e_1 + a_{22} \Xi(e_1) \\ &= e_1 \Xi(a_{22}) + \Xi(a_{22}) e_1 = 2b_{11} + b_{12} + b_{21}, \end{aligned}$$

by (†) we have $\Xi(a_{22}) \in \mathfrak{R}_{22}$.

Case 3. For $i = 1$ and $j = 2$, write $\Xi(a_{12}) = b_{11} + b_{12} + b_{21} + b_{22}$. We have from Lemma 2.2(i), (ii) and the fact that Ξ is a derivation because Ξ is defined on a 2-torsion free semiprime ring [1] that

$$\Xi(a_{12}) = \Xi(e_1 a_{12} + a_{12} e_1) = e_1 \Xi(a_{12})$$

and

$$0 = \Xi(e_1 a_{12} e_1) = e_1 \Xi(a_{12}) e_1.$$

Hence, $\Xi(a_{12}) \in \mathfrak{R}_{12}$ by (†).

Case 4. Finally, for $i = 2$ and $j = 1$, write $\Xi(a_{21}) = b_{11} + b_{12} + b_{21} + b_{22}$. We have from Lemma 2.2(i), (ii) that

$$\Xi(a_{21}) = \Xi(e_1 a_{21} + a_{21} e_1) = \Xi(a_{21}) e_1$$

and

$$0 = \Xi(e_1 a_{21} e_1) = e_1 \Xi(a_{21}) e_1.$$

Thus, $\Xi(a_{21}) \in \mathfrak{R}_{21}$ by (†). □

Lemma 2.5. *$\Delta(a_{ij}) \subset \mathfrak{R}_{ij} + \mathfrak{R}_{jj}$ for $i \neq j$.*

Proof. Firstly, we prove that $\Delta(e_1) \in \mathfrak{R}_{11} + \mathfrak{R}_{21}$. Let $\Delta(e_1) = a_{11} + a_{12} + a_{21} + a_{22}$. Since, by (†), $\Delta(e_1) = \Delta(e_1) e_1 + e_1 \Xi(e_1) = \Delta(e_1) e_1$, we see that $a_{11} + a_{12} + a_{21} + a_{22} = a_{11} + a_{21}$, which implies that $a_{12} = a_{22} = 0$ and $\Delta(e_1) = a_{11} + a_{21} \in \mathfrak{R}_{11} + \mathfrak{R}_{21}$.

Case 1. For $i = 1$ and $j = 2$, let $a_{12} \in \Re_{12}$ and $\Delta(a_{12}) = b_{11} + b_{12} + b_{21} + b_{22}$. Then

$$\begin{aligned} b_{11} + b_{12} + b_{21} + b_{22} &= \Delta(a_{12}) \\ &= \Delta(e_1 a_{12} + a_{12} e_1) \\ &= \Delta(e_1)a_{12} + e_1\Xi(a_{12}) + \Delta(a_{12})e_1 + a_{12}\Xi(e_1) \\ &= \Delta(e_1)a_{12} + \Xi(a_{12}) + b_{11} + b_{21}. \end{aligned}$$

Hence, $b_{12} + b_{22} = \Delta(e_1)a_{12} + \Xi(a_{12}) \in \Re_{12} + \Re_{22}$ by (†). On the other hand,

$$\begin{aligned} b_{11} + b_{12} + b_{21} + b_{22} &= \Delta(a_{12}) = \Delta(a_{12}e_2 + e_2 a_{12}) \\ &= \Delta(a_{12})e_2 + a_{12}\Xi(e_2) + \Delta(e_2)a_{12} + e_2\Xi(a_{12}) \\ &= \Delta(a_{12})e_2 + \Delta(e_2)a_{12} \\ &= b_{12} + b_{22} + \Delta(e_2)a_{12}. \end{aligned}$$

Thus, by (†), we get $b_{11} + b_{12} + b_{21} + b_{22} = \Delta(e_1)a_{12} + \Xi(a_{12}) + \Delta(e_2)a_{12}$, which implies that $\Delta(a_{12}) \in \Re_{12} + \Re_{22}$.

Case 2. For $i = 2$ and $j = 1$, let $a_{21} \in \Re_{21}$ and $\Delta(a_{21}) = b_{11} + b_{12} + b_{21} + b_{22}$. Then

$$\begin{aligned} b_{11} + b_{12} + b_{21} + b_{22} &= \Delta(a_{21}) \\ &= \Delta(a_{21}e_1 + e_1 a_{21}) \\ &= \Delta(a_{21})e_1 + a_{21}\Xi(e_1) + \Delta(e_1)a_{21} + e_1\Xi(a_{21}) \\ &= b_{11} + b_{21}. \end{aligned}$$

Therefore, by (†), $\Delta(a_{21}) \in \Re_{11} + \Re_{21}$. □

LEMMA 2.6. *$\Delta(a_{ii}) \subset \Re_{ii} + \Re_{ji}$, with $i \neq j$.*

PROOF. *Case 1.* For $i = 1$, by Lemma 2.2(ii) we have

$$\begin{aligned} \Delta(a_{11}) &= \Delta(e_1 a_{11} e_1) \\ &= \Delta(e_1)a_{11}e_1 + e_1\Xi(a_{11})e_1 + e_1 a_{11}\Xi(e_1) \\ &= \Delta(e_1)a_{11} + \Xi(a_{11}). \end{aligned}$$

Therefore, by (†), $\Delta(a_{11}) \in \Re_{11} + \Re_{21}$.

Case 2. The proof is similar to Case 1. □

LEMMA 2.7. (1) *$\Delta(a_{11}b_{12}) = \Delta(a_{11})b_{12} + a_{11}\Xi(b_{12})$ holds for all $a_{11} \in \Re_{11}$ and $b_{12} \in \Re_{12}$.*

(2) *$\Delta(a_{12}b_{22}) = \Delta(a_{12})b_{22} + a_{12}\Xi(b_{22})$ holds for all $a_{12} \in \Re_{12}$ and $b_{22} \in \Re_{22}$.*

(3) *$\Delta(a_{21}b_{12}) = \Delta(a_{21})b_{12} + a_{21}\Xi(b_{12})$ holds for all $a_{21} \in \Re_{21}$ and $b_{12} \in \Re_{12}$.*

(4) *$\Delta(a_{22}b_{22}) = \Delta(a_{22})b_{22} + a_{22}\Xi(b_{22})$ holds for all $a_{22}, b_{22} \in \Re_{22}$.*

PROOF. For any $a_{11} \in \mathfrak{R}_{11}$ and $b_{12} \in \mathfrak{R}_{12}$, it follows from Lemmas 2.2 and 2.5 that

$$\begin{aligned}\Delta(a_{11}b_{12}) &= \Delta(a_{11}b_{12} + b_{12}a_{11})\\ &= \Delta(a_{11})b_{12} + a_{11}\Xi(b_{12}) + \Delta(b_{12})a_{11} + b_{12}\Xi(a_{11})\\ &= \Delta(a_{11})b_{12} + a_{11}\Xi(b_{12}).\end{aligned}$$

Similarly, (2) is true for all $a_{12} \in \mathfrak{R}_{12}$ and $b_{22} \in \mathfrak{R}_{22}$.

Now for any $a_{21} \in \mathfrak{R}_{21}$ and $b_{12} \in \mathfrak{R}_{12}$, it follows from Lemmas 2.2, 2.4, 2.5 and (†) that

$$\begin{aligned}\Delta(a_{21}b_{12}) &= \Delta(a_{21}b_{12}e_2 + e_2b_{12}a_{21})\\ &= \Delta(a_{21})b_{12}e_2 + a_{21}\Xi(b_{12})e_2 + a_{21}b_{12}\Xi(e_2)\\ &\quad + \Delta(e_2)(b_{12}a_{21}) + e_2\Xi(b_{12})a_{21} + e_2b_{12}\Xi(a_{21})\\ &= \Delta(a_{21})b_{12} + a_{21}\Xi(b_{12}).\end{aligned}$$

Finally, for any $a_{22} \in \mathfrak{R}_{22}$, by Lemma 2.2(ii) and (†), we have

$$\begin{aligned}\Delta(a_{22}) &= \Delta(e_2a_{22}e_2)\\ &= \Delta(e_2)a_{22}e_2 + e_2\Xi(a_{22})e_2 + e_2a_{22}\Xi(e_2)\\ &= \Delta(e_2)a_{22} + \Xi(a_{22}),\end{aligned}$$

and hence $\Delta(a_{22}b_{22}) = \Delta(e_2)a_{22}b_{22} + \Xi(a_{22}b_{22})$ holds for all $a_{22}, b_{22} \in \mathfrak{R}_{22}$. Since

$$\begin{aligned}\Delta(a_{22})b_{22} + a_{22}\Xi(b_{22}) &= \Delta(e_2)a_{22}b_{22} + \Xi(a_{22})b_{22} + a_{22}\Xi(b_{22})\\ &= \Delta(e_2)a_{22}b_{22} + \Xi(a_{22}b_{22}),\end{aligned}$$

we get that $\Delta(a_{22}b_{22}) = \Delta(a_{22})b_{22} + a_{22}\Xi(b_{22})$. □

LEMMA 2.8. *$\Delta(ab) = \Delta(a)b + a\Xi(b)$ for all $a, b \in \mathfrak{R}$, that is, Δ is a generalized derivation.*

PROOF. First, for any $a, b \in \mathfrak{R}$ and $x_{12} \in \mathfrak{R}_{12}$, by Lemmas 2.2–2.7, we have

$$\begin{aligned}\Delta(abx_{12}) &= \Delta(a_{11}b_{11}x_{12} + a_{12}b_{21}x_{12} + a_{22}b_{21}x_{12} + a_{21}b_{11}x_{12})\\ &= \Delta(a_{11}b_{11})x_{12} + a_{11}b_{11}\Xi(x_{12}) + \Delta(a_{12}b_{21})x_{12} + a_{12}b_{21}\Xi(x_{12})\\ &\quad + \Delta(a_{22}b_{21})x_{12} + a_{22}b_{21}\Xi(x_{12}) + \Delta(a_{21}b_{11})x_{12} + a_{21}b_{11}\Xi(x_{12})\\ &= \Delta(a_{11}b_{11} + a_{12}b_{21} + a_{22}b_{21} + a_{21}b_{11})x_{12}\\ &\quad + (a_{11}b_{11} + a_{12}b_{21} + a_{22}b_{21} + a_{21}b_{11})\Xi(x_{12})\\ &= \Delta(ab)x_{12} + ab\Xi(x_{12}).\end{aligned}$$

Second, for any $x_{12} \in \mathfrak{R}_{12}$, by Lemmas 2.2–2.7, we get

$$\begin{aligned}\Delta(abx_{12}) &= \Delta(a_{11}b_{11}x_{12} + a_{12}b_{21}x_{12} + a_{22}b_{21}x_{12} + a_{21}b_{11}x_{12})\\ &= \Delta(a_{11})b_{11}x_{12} + a_{11}\Xi(b_{11}x_{12}) + \Delta(a_{12})b_{21}x_{12} + a_{12}\Xi(b_{21}x_{12})\\ &\quad + \Delta(a_{22})b_{21}x_{12} + a_{22}\Xi(b_{21}x_{12}) + \Delta(a_{21})b_{11}x_{12} + a_{21}\Xi(b_{11}x_{12})\\ &= \Delta(a)bx_{12} + a\Xi(b)x_{12} + ab\Xi(x_{12}).\end{aligned}$$

So $(\Delta(ab) - \Delta(a)b - a\Xi(b))x_{12} = 0$ for any $x_{12} \in \mathfrak{R}_{12}$. Hence $e_1(\Delta(ab) - \Delta(a)b - a\Xi(b))e_1 = 0 = e_2(\Delta(ab) - \Delta(a)b - a\Xi(b))e_1$ by condition (♠).

Third, for any $x_{22} \in \mathfrak{R}_{22}$, we compute

$$\begin{aligned}
\Delta(abx_{22}) &= \Delta(a_{11}b_{12}x_{22}) + \Delta(a_{12}b_{22}x_{22}) + \Delta(a_{21}b_{12}x_{22}) + \Delta(a_{22}b_{22}x_{22}) \\
&= \Delta(a_{11}b_{12})x_{22} + a_{11}b_{12}\Xi(x_{22}) + \Delta(a_{12}b_{22})x_{22} + a_{12}b_{22}\Xi(x_{22}) \\
&\quad + \Delta(a_{21}b_{12})x_{22} + a_{21}b_{12}\Xi(x_{22}) + \Delta(a_{22}b_{22})x_{22} + a_{22}b_{22}\Xi(x_{22}) \\
&= \Delta(ab)x_{22} + a_{11}b_{12}\Xi(x_{22}) + a_{12}b_{22}\Xi(x_{22}) + a_{21}b_{12}\Xi(x_{22}) \\
&\quad + a_{22}b_{22}\Xi(x_{22}).
\end{aligned}$$

Fourth, on the other hand,

$$\begin{aligned}
\Delta(abx_{22}) &= \Delta(a_{11}b_{12}x_{22}) + \Delta(a_{12}b_{22}x_{22}) + \Delta(a_{21}b_{12}x_{22}) + \Delta(a_{22}b_{22}x_{22}) \\
&= \Delta(a_{11})b_{12}x_{22} + a_{11}\Xi(b_{12}x_{22}) + \Delta(a_{12})b_{22}x_{22} + a_{12}\Xi(b_{22}x_{22}) \\
&\quad + \Delta(a_{21})b_{12}x_{22} + a_{21}\Xi(b_{12}x_{22}) + \Delta(a_{22})b_{22}x_{22} + a_{22}\Xi(b_{22}x_{22}) \\
&= \Delta(a)bx_{22} + a_{11}\Xi(b_{12}x_{22}) + a_{12}\Xi(b_{22}x_{22}) + a_{21}\Xi(b_{12}x_{22}) \\
&\quad + a_{22}\Xi(b_{22}x_{22}) \\
&= \Delta(a)bx_{22} + a_{11}\Xi(b_{12})x_{22} + a_{11}b_{12}\Xi(x_{22}) + a_{12}\Xi(b_{22})x_{22} \\
&\quad + a_{12}b_{22}\Xi(x_{22}) + a_{21}\Xi(b_{12})x_{22} + a_{21}b_{12}\Xi(x_{22}) + a_{22}\Xi(b_{22})x_{22} \\
&\quad + a_{22}b_{22}\Xi(x_{22}) \\
&= \Delta(a)bx_{22} + a\Xi(b)x_{22} + a_{11}b_{12}\Xi(x_{22}) + a_{12}b_{22}\Xi(x_{22}) \\
&\quad + a_{21}b_{12}\Xi(x_{22}) + a_{22}b_{22}\Xi(x_{22}).
\end{aligned}$$

Thus, comparing the above two equations, we obtain $(\Delta(ab) - \Delta(a)b - a\Xi(b))x_{22} = 0$ for any $x_{22} \in \mathfrak{R}_{22}$, and then $e_1(\Delta(ab) - \Delta(a)b - a\Xi(b))e_2 = 0 = e_2(\Delta(ab) - \Delta(a)b - a\Xi(b))e_2$. Therefore $\Delta(ab) = \Delta(a)b + a\Xi(b)$. □

Proof of Theorem 2.1. From the above lemmas, we have proved that $\Delta : \mathfrak{R} \to \mathfrak{R}$ is a generalized derivation. Since $\Delta(a) = \delta(a) - d_s(a)$ for each $a \in \mathfrak{R}$, by a simple calculation, we see that δ is also a generalized derivation. The proof is complete. □

Corollary 2.9. *Let $M_2(\mathbb{C})$ denote the algebra of all 2×2 complex matrices. Suppose that $\delta : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is a linear mapping such that $\delta(E^2) = \delta(E)E + E\tau(E)$ holds for all idempotent E in $M_2(\mathbb{C})$, where $\tau : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is a linear mapping satisfying $\tau(E) = \tau(E)E + E\tau(E)$ for any idempotent E in $M_2(\mathbb{C})$. Then δ is a generalized derivation.*

Proof. Let $M_2(\mathbb{C}) = E_1M_2(\mathbb{C})E_1 \oplus E_1M_2(\mathbb{C})E_2 \oplus E_2M_2(\mathbb{C})E_1 \oplus E_2M_2(\mathbb{C})E_2$ be the Peirce decomposition relative to the idempotent $E_1 = \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$. Clearly $M_2(\mathbb{C})$ is semiprime and satisfies (♠). By [5, Theorem 4.1] we have that τ is a derivation and $\delta(A^2) = \delta(A)A + A\tau(A)$ for any $A \in M_2(\mathbb{C})$. Therefore, by Theorem 2.1, δ is a generalized derivation. □

Definition 2.10. Let $U(\mathfrak{R})$ be the group of units of $\mathfrak{R}$. An ideal I of a ring $\mathfrak{R}$ is unit-prime if, for any $a, b \in \mathfrak{R}$, $aU(\mathfrak{R})b \subseteq I$ implies $a \in I$ or $b \in I$, and unit-semiprime if, for any $a \in \mathfrak{R}$, $aU(\mathfrak{R})a \subseteq I$ implies $a \in I$. A ring $\mathfrak{R}$ is unit-(semi)prime if (0) is a unit-(semi)prime ideal of $\mathfrak{R}$.

Theorem 2.11. *Matrix rings over unit-semiprime rings are unit-semiprime.*

Proof. See [2, Theorem 11]. □

The purpose of the following example is to show the existence of a ring that satisfies the hypotheses of the main theorem of this paper.

Example 2.12. Let M_2 be a 2×2 matrix ring over a unit-semiprime ring. Suppose that $\delta : M_2 \to M_2$ is a generalized Jordan derivation and $\tau : M_2 \to M_2$ is the related Jordan derivation. Then δ is a generalized derivation.

Proof. First observe that M_2 is a unit-semiprime ring by Theorem 2.11. Consider $E = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ nontrivial idempotent in M_2 and

$$M_2 = (M_2)_{11} \oplus (M_2)_{12} \oplus (M_2)_{21} \oplus (M_2)_{22}$$

the Peirce decomposition relative to E. Suppose $X_{11}(M_2)_{12} = 0$, where $X_{11} = \left[\begin{smallmatrix} x_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right] \in (M_2)_{11}$. As $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in (M_2)_{12}$ it follows that $X_{11} = 0$. Similarly, we show that if $X_{21}(M_2)_{12} = 0$ then $X_{21} = 0$. Therefore M_2 satisfies (♠). It is worth noting that with a fixed non-trivial idempotent satisfying (♠), we can demonstrate Theorem 2.1. Hence $\delta : M_2 \to M_2$ is a generalized derivation. □

In [5], the authors introduced the concept of generalized Jordan triple derivation. Let $\mathfrak{R}$ be a ring and $\delta : \mathfrak{R} \to \mathfrak{R}$ an additive map. If there is a Jordan triple derivation $\tau : \mathfrak{R} \to \mathfrak{R}$ such that $\delta(aba) = \delta(a)ba + a\tau(b)a + ab\tau(a)$ for every $a, b \in \mathfrak{R}$, then δ is called a generalized Jordan triple derivation, and τ is the relating Jordan triple derivation. Recall that τ is a Jordan triple derivation if $\tau(aba) = \tau(a)ba + a\tau(b)a + ab\tau(a)$ for any $a, b \in \mathfrak{R}$.

The authors conjecture that every generalized Jordan triple derivation on 2-torsion free semiprime ring is a generalized derivation. In our case we have the following corollary.

Corollary 2.13. *Let $\mathfrak{R}$ be a 2-torsion free semiprime unity ring satisfying (♠) and δ be a generalized Jordan triple derivation from $\mathfrak{R}$ into itself. If there exist an idempotent e so that $e \neq 0$, $e \neq 1$ in $\mathfrak{R}$, then δ is a generalized derivation.*

Proof. Let $\delta : \mathfrak{R} \to \mathfrak{R}$ be a generalized Jordan triple derivation and $\tau : \mathfrak{R} \to \mathfrak{R}$ the relating Jordan triple derivation. Note that $\tau(e_1 + e_2) = 0$, so τ is in fact a Jordan derivation. Now it is easy to check that a generalized Jordan triple derivation on $\mathfrak{R}$ is a generalized Jordan derivation. Therefore, by Theorem 2.1, δ is a generalized derivation. □

The open question that remains is whether the Jing and Lu conjectures hold if $\mathfrak{R}$ does not contain a nontrivial idempotent.

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