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ABOUT SINGULARITY OF TWISTED SUMS

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Abstract

In this talk we study some aspects of the structure of twisted sums. Although a twisted sums of Köthe spaces is not necessarily a Köthe space, those which are obtained by the complex interpolation method are equipped in a natural way with an L_∞ - module structure. In this case we study disjoint versions of basic notions of the theory of twisted sums. We also consider some properties in the direction of local theory.

1 Introduction

Recall that a twisted sum of two Banach spaces Y, Z is a quasi-Banach space X which has a closed subspace isomorphic to Y such that the quotient X/Y is isomorphic to Z. Equivalently, X is a twisted sum of Y, Z if there exists a short exact sequence

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0.$$

According to Kalton and Peck [5], twisted sums can be identified with homogeneous maps $\Omega : X \to Y$ satisfying

$$\|\Omega(x_1 + x_2) - \Omega x_1 - \Omega x_2\| \leq C(\|x_1\| + \|x_2\|),$$

which are called quasi-linear maps, and induce an equivalent quasi-norm on X (seen algebraically as $Y \times X$) by

$$\|(y, x)\|_\Omega = \|y - \Omega z\| + \|x\|.$$

This space is usually denoted $Y \oplus_\Omega X$. When Y and X are, for example, Banach spaces of non-trivial type, the quasi-norm above is equivalent to a norm; therefore, the twisted sum obtained is a Banach space. The quasi-linear map is said to be trivial when $Y \oplus_\Omega X$ is isomorphic to the direct sum $Y \oplus X$.

We are mainly interested in the ambient of Köthe functions spaces over a σ-finite measure space (Σ, μ) endowed with their L_∞-module structure. A Köthe function space K is a linear subspace of $L_0(\Sigma, \mu)$, the vector space of all measurable functions, endowed with a quasi-norm such that whenever $|f| \leq g$ and $g \in K$ then $f \in K$ and $\|f\| \leq \|g\|$ and so that for every finite measure subset $A \subset \Sigma$ the characteristic function 1_A belongs to X. A particular case of which is that of Banach spaces with a 1-unconditional basis with their associated ℓ_∞-module structure.

Definition 1.1. *An L_∞-centralizer (resp. an ℓ_∞-centralizer) on a Köthe function (resp. sequence) space $\mathcal{K}$ is a homogeneous map $\Omega : \mathcal{K} \to L_0$ such that there is a constant C satisfying that, for every $f \in L_\infty$ (resp. ℓ_∞) and for every $x \in \mathcal{K}$, the difference $\Omega(fx) - f\Omega(x)$ belongs to $\mathcal{K}$ and*

$$\|\Omega(fx) - f\Omega(x)\|_\mathcal{K} \leq C\|f\|_\infty \|x\|_\mathcal{K}.$$

Observe that a centralizer Ω on $\mathcal{K}$ does not take values in $\mathcal{K}$, but in L_0, and still it induces an exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{\jmath} d_\Omega\mathcal{K} \xrightarrow{Q} \mathcal{K} \longrightarrow 0$$

as follows: $d_\Omega\mathcal{K} = \{(w,x) : w \in L_0, x \in \mathcal{K} : w - \Omega x \in \mathcal{K}\}$ endowed with the quasi-norm

$$\|(w,x)\|_{d_\Omega\mathcal{K}} = \|x\|_\mathcal{K} + \|w - \Omega x\|_\mathcal{K}$$

and with obvious inclusion $\jmath(x) = (x,0)$ and quotient map $Q(w,x) = x$. The reason is that a centralizer "is" quasi-linear, in the sense that for all $x, y \in \mathcal{K}$ one has $\Omega(x+y)-\Omega(x)-\Omega(y) \in \mathcal{K}$ and $\|\Omega(x+y)-\Omega(x)-\Omega(y)\| \leq C(\|x\|+\|y\|)$ for some $C > 0$ and all $x, y \in \mathcal{K}$. Centralizers arise naturally by complex interpolation [1] as can be seen in [4].

In this talk we study the *disjointly supported* versions of the basic (trivial, locally trivial, singular and supersingular) notions in the theory of centralizers and present several examples.

2 Main Results

An operator between Banach spaces is said to be *strictly singular* if no restriction to an infinite dimensional closed subspace is an isomorphism. Analogously, a quasi-linear map (in particular, a centralizer) is said to be *singular* if its restriction to every infinite dimensional closed subspace is never trivial. An exact sequence induced by a singular quasi-linear map is called a *singular sequence.* A quasi-linear map is singular if and only if the associated exact sequence has strictly singular quotient map. Singular ℓ_∞-centralizers exist and the most natural example is the Kalton-Peck map $\mathcal{K}_p : \ell_p \to \ell_p$, $0 < p < +\infty$, defined by $\mathcal{K}_p(x) = x \log \frac{|x|}{\|x\|_p}$.

In [3] where the authors introduced the notion of disjointly singular centralizer on Köthe function spaces, and proved that disjoint singularity coincides with singularity on Banach spaces with unconditional basis and presented a technique to produce disjointly singular centralizers via complex interpolation. An important fact to consider is that the fundamental Kalton-Peck map [5] is disjointly singular on L_p [3, Proposition 5.4], but it is not singular [6]. In fact, as the last stroke one could wish to foster the study of disjoint singularity is the argument of Cabello [2] that no centralizer on L_p can be singular that we extend here by showing that no centralizer can be singular. It is thus obvious that while singularity is an important notion in the domain of Köthe sequence spaces, disjoint singularity is the core notion in Köthe function spaces.

Theorem 2.1. *No singular L_∞-centralizers exist on (admissible) superreflexive Köthe funcion spaces. More precisely, every L_∞-centralizer on an admissible superreflexive Köthe function space is trivial on some copy of ℓ_2.*

The results are part of the work *On disjointly singular centralizers*, https://arxiv.org/pdf/1905.08241.pdf.

References

[1] BERGH, J. AND LÖFSTRÖM, J. - *Interpolation spaces. An introduction.*,Springer-Verlag, 1976.

[2] CABELLO, F. - There is no strictly singular centralizer on L_p. *Proc. Amer. Math. Soc.*, **142**, 949–955, 2014.

[3] CASTILLO, J.M.F., FERENCZI, V. AND GONZALEZ, M. - Singular exact sequences generated by complex interpolation. *Trans. Amer. Math. Soc.*, **369**, 4671–4708, 2017.

[4] KALTON, N.J. - Differentials of complex interpolation processes for Köthe function spaces. *Trans. Amer. Math. Soc.*, **333**, 479–529, 1992.

[5] KALTON, N.J. AND PECK, N. T. -Twisted sums of sequence spaces and the three space problem. *Trans. Amer. Math. Soc.*, **255** , 1–30, 1979.

[6] SUÁREZ DE LA FUENTE, J. - The Kalton-Peck centralizer on $L_p[0,1]$ is not strictly singular. *Proc. Amer. Math. Soc.*, **141** , 3447 – 3451, 2013.