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1 Introduction

Let $U(KG)$ denote the group of units of the group ring of a given group G over a field K. Also, we shall denote by $T = T(G)$ and $TU(KG)$ the set of elements of finite order in G and $U(KG)$ respectivelly.

In this note, we shall consider groups G that are either nilpotent or FC and determine conditions on G and K for $TU(KG)$ to be closed under multiplication, i.e. to be a subgroup of $U(KG)$. This question was first studied in [4] but the answer was incomplete because it depended on the fact that every idempotent of KT is central in KG, a condition not fully understood at that time. Using the results in [1], [2], we are able to give a complete answer to this question. In particular, we do not need a technical hypothesis assumed in [4, theorem 4.1] and we correct a gap in [1, theorem 5.2]. In what follows, if a ring R is such that its torsion units form a subgroup, we shall say, briefly, that R has the t.p.p. (*torsion product property*).

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2 Group rings in characteristic $p > 0$

We remark first that, if G is either a nilpotent or FC group, then T is locally finite and that if $G \neq T$, then G contains a central element of infinite order (see [5]).

Lemma 2.1 *Let G be a group such that $T = T(G)$ is locally finite, and assume that either G contains a central element of infinite order or K is not algebraic over its prime field $\mathcal{P}(K)$. If $TU(KG)$ is a subgroup then, for every finite subgroup $T_1 \subset T$ the quotient ring $KT_1/J(KT_1)$, where $J(KT_1)$ denotes the Jacobson radical of KT_1, is a direct sum of fields.*

Proof. Let x denote be a central element of infinite order in G. Denote by $KT_1[x]$ the smallest subring of KG containing KT and $\{x\}$ and let $\varphi : (KT_1)[x] \to (KT_1/J(KT_1))[x]$ the epimorphism induced by the natural map $KT_1 \to KT_1/J(KT_1)$. Since $J(KT_1)$ is nilpotent and x is central, it follows that $Ker(\varphi) = J(KT_1)[x]$ is a nilpotent ideal. Hence, φ induces, by restriction, epimorphisms of the respective unit groups and also of the respective sets of torsion elements. Then, it is easily seen that $(KT_1/J(KT_1))[x]$ also has the t.p.p.

Since $KT_1/J(KT_1)$ is semisimple artinian, we have that

$$\frac{KT_1}{J(KT_1)} \cong \oplus_{i=1}^{t} M_{n_i}(D_i),$$

where D_i is a division ring containig K, $1 \leq i \leq t$.

For each index i we have:

$(M_{n_i}(D_i))[x] \cong (M_{n_i}(D_i) \otimes_K K)[x]$

$$\cong (D_i \otimes_K M_{n_i}(K) \otimes K)[x] \cong D_i \otimes_K M_{n_i}(K[x]).$$

Then, also $M_{n_i}(K[x])$ has the t.p.p.. It follows from [4, prop. 2.2] that $n_i = 1$. Then:

$$\frac{KT_1}{J(KT_1)} \cong \oplus_{i=1}^{t} D_i.$$

Given any two elements $x, y \in T_1$ we have that $\bar{x}, \bar{y} \in \times_{i=1}^{t} TU(D_i)$ and [4, prop. 2.1] shows they are central. Hence, $KT_1/J(KT_1)$ is commmutative and the result follows.

A similar argument proves the statement in the case where K contains an element x which is transcendental over $\mathcal{P}(K)$ □

Lemma 2.2 *Let K and G be as in the previous lemma. If $TU(KG)$ is a subgroup, then the P set of p-elements in G is a normal subgroup of G and $T' \subset P$.*

Proof. Assume that α is a p-element. Then, for some integer $n \geq 1$ we have that $(\alpha - 1)^{p^n} = \alpha^{p^n} - 1 = 0$ i.e. $\alpha - 1$ is a nilpotent element. We set $T_1 =< supp(\alpha) >$. Then T_1 is finite and the image $\overline{\alpha}$ in $KT_1/J(KT_1)$ is also nilpotent. Then, lemma 2.1 shows that $\alpha \in 1 + J(KT_1)$

Hence, given two p-elements $\alpha, \beta \in G$, we have that $\alpha\beta \in 1 + J(KT_1)$, which is a p-group.

Given $x, y \in T$, lemma 2.1 shows that $K < x, y > /J(K < x, y >)$ $\cong \oplus_i D_i$, a direct sum of fields; hence, $(x, y) - 1 \in J(< K < x, y >)$. Thus, there exists an integer $n \geq 1$ such that $(x, y)^{p^n} = 1$. Consequently $T' \subset P$. □

Theorem 2.3 *Let G be a nilpotent or FC group and let K be a field with $char(K) = p > 0$. Then $TU(KG)$ is a subgroup if and only if one of the following conditions hold:*

(i) *G is abelian.*

(ii) *$G = T$ and K is algebraic over its prime field $\mathcal{P}(K)$.*

(iii) *The set P of p-elements in G is a subgroup, $T' \subset P$ and if T/P is non central in G/P then Ω, the algebraic closure of $\mathcal{P}(K)$ in K is finite and, for all $x \in G$ and all p'-element $a \in T$, we have that xax^{-1} is of the form $xax^{-1} = a^{p^r}y$, where $r \geq 0$ and $y \in P$. Furthermore, for every such an exponent r we have that $[\Omega : \mathcal{P}(K)] \mid r$.*

Proof. It is clear that either *(i)* or *(ii)* implies that KG has the t.p.p. Assume then that G is not abelian and that either $G \neq T$ or K is not algebraic over $\mathcal{P}(K)$. From lemmma 2.2 we see that P is a subgroup and that $T' \subset P$.

Since $\Delta(G : P)$ is a locally nilpotent ideal, it follows that $K(G/P)$ also has the t.p.p. Since T/P contains no p-elements, [7, lemma VI.3.12] shows that if there exists a non central idempotent $e \in K(T/P)$, then $U(K(G/P))$ contains a subgroup which is isomorphic to $GL(m, K)$ with $m > 1$. If K is not algebraic over $\mathcal{P}(K)$ this yields a contradiction. On the other hand, if $G \neq T$, then [4, theorem 4.1] shows directly that every idempotent of $K(T/P)$ is central in $K(G/P)$.

In both cases, [1] shows that *(iii)* holds.

To prove sufficiency, we observe that both *(i)* and *(ii)* imply readily that KG has the t.p.p. Thus, assume that *(iii)* holds. Then, [1] shows that every idempotent of $K(T/P)$ is central in $K(G/P)$ and, as in [4, theorem 4.4] we see that KG has the t.p.p. also in this case. □

3 Group rings in characteristic 0

Out first result holds in a slightly more general setting.

Lemma 3.1 *Let G be a group such that $T(G)$ is locally finite and let K be a field of characteristic 0. If $TU(KG)$ is a subgroup, then T is abelian.*

To prove our statemet, we can assume that T is finite. Then, we can write $KT \cong \oplus_{i=1}^{t} M_{n_i}(D_i)$. Since $M_2(\mathbf{Q})$ does not have the t.p.p. (see, for example [6, p. 20]), it follows immediately that $n_i = 1$, $1 \leq i \leq t$.

Thus, $KT \cong \oplus_{i=1}^{t} D_i$ contains no nilpotent elements, so [7, theorem VI.1.11] shows that T_1 is either abelian or a Hamiltonian group. Finally, if T_1 is Hamiltonian, it contains a subgroup of the form

$$Q = < a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} > .$$

Let p be any prime. It was shown in [3, theorem 2] that $\alpha = x + ya$ with $x, y \in \mathbf{Z}$, $p \nmid x, p \mid y$, is a unit in $\mathbf{Q}_{(p)}Q$, and therefore in $\mathbf{Q}G$ and that $(b, \alpha) = b(b^{-1})^{\alpha}$ is not an element of finite order. Hence, T_1 must be abelian. □

We can now correct [4, theorem 5.2], which should be stated as follows.

Theorem 3.2 *Let G be a nilpotent or FC group and let K be a field of characteristic 0. Then, $TU(KG)$ is a subgroup if and only if the following conditions hold:*

(i) *T is abelian.*

(ii) *For all $t \in T$ and all $x \in G$ we have that $xtx^{-1} = t^i$, where $i = i(x)$ locally and, for each non central element $t \in T$, K contains no root of unity of order $o(t)$.*

Proof. We know, from the lemma above, that T is abelian.

Also, every idempotent in KT is central in KG, since, as before, [7, lemma VI.3.12] shows that $U(KG)$ contains a copy of $GL(m, K)$, with $m \geq 1$. This yields a contradiction, because $M_2(\mathbf{Q})$ does not have the t.p.p.

Now, [2] shows that, for all $t \in T$ and all $x \in G$ we have that $xtx^{-1} = t^i$, where $i = i(x)$ locally and that for every non central element $t \in T$, K contains no root of unity of order $o(t)$.

To prove sufficiency, notice that we may suppose that G is finitely generated and, therefore, that T is finite. Thus $KT = \oplus_{i=1}^{t} K_i$, a direct sum of fields. Let $S = \{s_i\}_{i \in I}$ be a transversal of T in G. Then, we know from [7, lemma VI.3.22] that every unit $u \in KG$ can be written in the form $u = \sum_i f_i g_i$ wher $f_i \in K_i$, $g_i \in S$, $1 \leq i \leq t$.

Since [2] shows that the conditions in the statement of our theorem imply that every idempotent of KT is central in KG, we have that $g_i f_i = f_i' g_i$, for some $F_i' \in K$, $1 \leq i \leq t$. Hence:

$$u^m = \sum_{i=1}^{t} \overline{f_i} g_i^m,$$

where $\overline{f_i} \in K$, $1 \leq i \leq t$. Thus, $u \in TU(KG)$ if and only if there exists an integer m such that $g_i^m = 1$, $1 \leq i \leq t$, i.e. if and only if $u \in U(KT)$. Since T is abelian, it follows easily that KG has the t.p.p. □

References

[1] S. P. Coelho. *A Note on Central Idempotents in Group Rings*. **Proc. Edinburgh Math. Soc. 30** (1987), 69 - 72.

[2] S. P. Coelho and C. Polcino Milies. *A Note on Central Idempotents in Group Rings II*. **Proc. Edinburgh Math. Soc. 31** (1988), 211 - 215.

[3] M. M. Parmenter and C. Polcino Milies. *Group Rings whose Torsion Units form a Nilpotent or FC Group*. **Proc. Amer. Math. Soc.** 68 (1978), 247 - 248.

[4] C. Polcino Milies. *Group Rings whose Torsion Units form a Subgroup II*. **Comm. in Algebra**, 9.7 (1981), 699 - 712.

[5] D. J. S. Robinson. *A Course in the Theory of Groups*, Springer-Verlag, New York, 1982.

[6] J. J. Rotman, *The Theory of Groups: an introduction*. 2nd ed.. Allyn and Bacon. Boston. 1973.

[7] S. K. Sehgal. *Topics in Group Rings*. Marcel Dekker. New York 1978.

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