RT-MAE-8002

WALSH SPECTRAL ANALYSIS

by

PEDRO A. MORETTIN

Palavras Chaves: Walsh Functions
(key words) Dyadic-stationary Processes
Walsh Spectrum

Classificação AMS: 62M15
(AMS Classification)

- Março de 1980 -

WALSH SPECTRAL ANALYSIS

Pedro A. Morettin
University of São Paulo, Brazil

ABSTRACT

In this paper we make a review of the recent work on the stochastic approach of Walsh spectral analysis. We are mainly interested in the properties of the finite Walsh transform, which in turn allow us to derive properties of the estimators of the Walsh spectrum. We further discuss linear dyadic filters and Walsh characteristic functions. A rather complete list of references is given.

1 - INTRODUCTION

The orthogonal system of sine and cosine functions plays an important role in the analysis of time series. In communications engineering, especially in the theory of linear, time-invariant networks, Fourier Analysis is perhaps the most important mathematical tool.

In the last decade, with the recent advances of semiconductors, another system of orthogonal functions have become important, the Walsh functions, first introduced by J.L.Walsh in 1923. The Walsh functions (WF) are defined as products of the Rademacher functions. Walsh (1923), Paley (1932), Fine (1949) and Mogenthaler (1957) have developed a theory of Walsh-Fourier series and most of the results parallel those of the classical trigonometric series theory. The original definition of Walsh was a recursive one and the functions were ordered by the number of sign changes in the interval [0,1). See Harmuth (1972) and Pichler (1967) for further details.

The generalized WF were considered by Fine (1950). A class of generalized WF of order α was introduced by Chrestenson (1955) and Selfridge (1955) combined these generalizations to developed a theory of Walsh transform.

Walsh spectral analysis has been used for several purposes recently, mainly due to computational advantages. Its uses include description of biological and medical systems, monitoring of EEG and ECG signals, speech processing, word recognition, image coding and transmission, filtering, multiplexing, etc. See for example, the 1972 and 1973 Proceedings on Applications of Walsh Functions, Beauchamp (1975) and Harmuth (1977).

In this paper we will be interested in the stochastic approach to Walsh spectral analysis and we review the recent work in this area.

2 - WALSH FUNCTIONS

The Walsh functions $\{W(n,x),\ n=0,1,2,...,\ 0\le x<1\}$ are defined as follows:

(i) $W(0,x) = 1,\ 0 \le x < 1$,

(ii) if n has the dyadic expansion $n = \sum_{i=0}^{\infty} x_i 2^i$, with $x_i = 0$ or $x_i = 1$, and $x_i = 0$ for $i > m_r$, then

$$W(n,x) = \prod_{i=1}^{r} \{r_{m_i}(x)\}, \tag{2.1}$$

where $m_1,\ldots,m_r$ correspond to the coefficients $x_{m_i} = 1$ and where $\{r_k(x)\}$ are the Rademacker functions.

The system of WF is orthonormal on $[0,1)$, that is,

$$\int_0^1 W(n,x)W(m,x)\,dx = \delta_{nm} \tag{2.2}$$

and forms a complet set. Fine (1949) showed the WF may be identified with the full set of characters of the dyadic group, D. This is the set of all sequences $\bar{x} = \{x_n\}$, where $x_n = 0$ or $x_n = 1$, $n = 1,2,3,\ldots$, with the group operation defined by $\bar{z} = \bar{x}\dot{+}\bar{y}$, if $\bar{x},\bar{y}\in D$, where $z_n = x_n + y_n \pmod 2$. There is a topology for D,based on the system of neighborhoods of $\bar{0} = (0,0,\ldots)$, with which D becomes a locally compact Abelian group. To each $\bar{x}\in D$ we assign a real number $x = d(\bar{x}) = \sum_{i=1}^{\infty} x_i 2^{-i}$ in the interval $[0,1)$. Note that this map does not have a single-valued inverse on the dyadic rationals. Two important properties are:

(i) for each y and almost all x,

$$W(n,x\dot{+}y) = W(n,x)W(n,y), \tag{2.3}$$

the exceptional x being those for which $x\dot{+}y$ is a dyadic rational;

(ii) for every fixed y and f integrable,

$$\int_0^1 f(y\dot{+}x)\,dx = \int_0^1 f(x)\,dx. \tag{2.4}$$

In order to generalize the WF, let x and y be two non-negative real numbers with dyadic expansions

$$x = \sum_{k=-\infty}^{\infty} x_k 2^k, \qquad x_k = 0 \text{ or } 1, \tag{2.5}$$

$$y = \sum_{k=-\infty}^{\infty} y_k 2^k, \qquad y_k = 0 \text{ or } 1. \tag{2.6}$$

Define the dyadic addition by

$$x\dot{+}y = \sum_{k=-\infty}^{\infty} (x_k \dot{+} y_k)\, 2^k, \tag{2.7}$$

where $0\dot{+}0 = 1\dot{+}1 = 0$, $1\dot{+}0 = 0\dot{+}1 = 1$. Then the generalized WF are defined by

$$W(y,x) = W([y],x)\,W([x],y), \tag{2.8}$$

$x,y \in \mathbb{R}_+$. Analogues of (2.3) and (2.4) above hold.

The original definition of Walsh was such that the functions are ordered by the number of sign changes in the unit interval. Denoting these functions by $\{wal(n,x)\}$ we see that

$$wal(n,x) = \begin{cases} W(n/2\dot{+}n,x), & n=0,2,4,\ldots, \\ W((n-1)/2\dot{+}n,x), & n=1,3,5,\ldots, \end{cases} \tag{2.9}$$

If f is any function of period 1, Lebesgue integrable on $[0,1)$, then it can be expanded in a Walsh-Fourier series

$$f(x) \sim \sum_{n=0}^{\infty} a_n W(n,x), \tag{2.10}$$

with coefficients

$$a_n = \int_0^1 f(x)W(n,x)\,dx, \qquad n=0,1,2,\ldots . \tag{2.11}$$

If $f^{(N)}(x) = \sum_{k=0}^{N-1} a_k W(k,x)$ and if $D_N(x)$ is the Dirichlet kernel

$$D_N(x) = \sum_{k=0}^{N-1} W(k,x), \tag{2.12}$$

then (Fine (1949)),

$$f^{(N)}(t) = \int_0^1 f(x)\,D_N(t \dot{+} x)\,dx. \tag{2.13}$$

If $N = 2^m$, m a non-negative integer, then

$$D_N(x) = \begin{cases} N, & 0 \le x \le N^{-1} \\ 0, & \text{otherwise} \end{cases} \tag{2.14}$$

and (2.13) reduces to $2^m \int_0^{N^{-1}} f(x)\,dx$.

The Walsh transform of a function $f \in L_2(0,\infty)$ is defined by

$$\hat{f}(y) = \int_0^{\infty} f(x)W(y,x)\,dx. \tag{2.15}$$

For properties of $\hat{f}$, see Weiser (1964) and Crittenden (1970).

3 - STATIONARY AND DYADIC-STATIONARY PROCESSES

When we analyse stationary processes with continuous time we are essentially considering a locally compact Abelian group, namely the additive group $\mathbb{R}$ of the real numbers. The characters of this group are the exponentials e^{iyx}, y and x real, hence the importance of the orthogonal system of sine and cosine functions. For processes with discrete time the associated group is the additive group $\mathbb{Z}$ of the integers.

Let us consider a real, stationary (in the wide sense) process $X = \{X(t), t \in \mathbb{Z}\}$ with zero mean and covariance function $R(\tau) = E\{X(t)X(t+\tau)\}$, $t, \tau \in \mathbb{Z}$. Then the Fourier spectrum of X is defined as

$$g(x) = (2\pi)^{-1} \sum_{\tau=-\infty}^{\infty} R(\tau) e^{-ix\tau}, \tag{3.1}$$

$-\infty < x < +\infty$, assuming that

$$\sum_{\tau=-\infty}^{\infty} |R(\tau)| < \infty, \tag{3.2}$$

that is, well separated values of X are non-correlated. We know that $g(\cdot)$ is bounded, uniformly continuous, of period 2π and

$$R(\tau) = \int_{-\pi}^{\pi} e^{i\tau x} g(x)\,dx. \tag{3.3}$$

Here x stands for frequency in radians and both $R(\cdot)$ and $g(\cdot)$ are even functions.

A Fourier spectral analysis is well established and the appropriate references are Koopmans (1974), Brillinger (1975) and Jenkins & Watts (1968).

We call a process $X = \{X_n, n=0,1,2,\ldots\}$ *dyadic-stationary* if its mean value is constant (which we assume for simplicity to be zero) and its covariance function

$$B(n,m) = \text{Cov}\{X_n, X_m\} = E\{X_n X_m\}$$

depends only on $n \dot{+} m$, hence we can write

$$B(\tau) = E\{X_n X_{n \dot{+} \tau}\}, \tag{3.4}$$

$n, \tau \in \mathbb{P} = \{0,1,2,\ldots\}$.

EXAMPLE 3.1 - a) The simplest example is the purely random process or white noise, that is, a sequence of independent and identically distributed random variables $\{\varepsilon_n, n \in \mathbb{P}\}$, with $E(\varepsilon_n) = 0$ and $\text{Var}\{\varepsilon_n\} = \sigma^2$, for all $n \in \mathbb{P}$. It follows that the covariance function of this process is given by

$$B(\tau) = E\{\varepsilon_n \cdot \varepsilon_{n \dot{+} \tau}\} = \begin{cases} 0, & \tau \neq 0 \\ \sigma^2, & \tau = 0 \end{cases} . \tag{3.5}$$

b) Let $X = \{X_n, n \in \mathbb{P}\}$ be a superposition of N periodic "ocillations"

$$X_n = \sum_{k=1}^{N} \xi_k W(n, x_k), \tag{3.6}$$

with ξ_i, $i=1,\ldots,N$ being random variables such that $E(\xi_i) = 0$, $E(\xi_i^2) = b_i$, $E(\xi_i \xi_j) = 0$, $i \neq j$, $i,j = 1,\ldots,N$.

Then $E(X_n) = 0$ and

$$B(\tau) = \sum_{k=1}^{N} b_k W(\tau, x_k). \tag{3.7}$$

In particular, $B(0) = E(X_n^2) = \sum_{k=1}^{N} b_k$, which shows that the average power of the composite oscillation is equal to the sum of the average powers of the periodic components.

4 - THE WALSH SPECTRUM

Let $X = \{X_n, n \in \mathbb{P}\}$ be a dyadic-stationary process, with zero mean and dyadic covariance function $B(\tau)$, $\tau \in \mathbb{P}$.

The *Walsh spectrum* of X is defined as

$$f(x) = \sum_{\tau=0}^{\infty} B(\tau) W(\tau, x), \tag{4.1}$$

$0 \leq x < \infty$, provided we assume that

$$\sum_{\tau=0}^{\infty} |B(\tau)| < \infty. \tag{4.2}$$

Then it can be shown (Morettin (1972)) that $f(\cdot)$ has period one, is non-negative and uniformly W-continuous. See Morgenthaler (1957) for the definition of W-continuity. It follows that it suffices to consider $f(x)$ for $0 \leq x \leq 1$.

If (4.2) is not warranted we still can define the Walsh spectrum as follows. Assume that X is W-continuous in mean square. Then, from the theory of homogeneous random processes on locally compact Abelian groups, we have that X admits the spectral representation

$$X_n = \int_0^1 W(n,x)\,dZ(x), \tag{4.3}$$

$n \in \mathbb{P}$, where $\{Z(x),\ 0 \leq x \leq 1\}$ is a real process with orthogonal increments and such that

$$E\{[dZ(x)]^2\} = dF(x); \tag{4.4}$$

here, $F(\cdot)$ is a unique distribution function on $[0,1]$. For details, see Yaglom (1961) and Morettin (1980a). The covariance function $B(\cdot)$ also has a spectral representation

$$B(\tau) = \int_0^1 W(\tau,x)\,dF(x), \tag{4.5}$$

and we call $F(\cdot)$ the dyadic spectral distribution function of X. Nagai (1976a) gives alternate proofs of (4.3) and (4.5). If $F(\cdot)$ is absolutely continuous then $dF(x) = f(x)dx$ and $f(\cdot)$ is the dyadic spectral density of X.

EXAMPLE 4.1 - (a) If $\{\varepsilon_n, n \geq 0\}$ is white noise, then from (3.5) it follows that

$$f(x) = \sigma^2 W(0,x) = \sigma^2. \tag{4.6}$$

Also, the white noise has spectral representation

$$\varepsilon_n = \int_0^1 W_n(x)\,dU(x)\,, \tag{4.7}$$

where $\{U(x),\ 0\leq x\leq 1\}$, is a process with orthogonal increments such that $E\{[dU(x)]^2\} = \sigma^2 dx$, that is, the dyadic spectral distribution function of $\{\varepsilon_n\}$ is $F(x) = \sigma^2 x,\ 0 \leq x \leq 1$.

(b) Let X be the process given by (3.6). Then (4.2) is not satisfied and we cannot define $f(\cdot)$ through (4.1). If we assume that the discontinuities of $F(\cdot)$ in the representation (4.5) do not coincide with the discontinuities of $W(\tau,x)$ (that is, x is not dyadic rational), then $dF(x_k) = b_k$, $k = 1,\ldots,N$ and equals zero otherwise.

5 - LINEAR DYADIC FILTERS

Let F be an operation whose domain consists of series $\{X_n, n\in\mathbb{P}\}$ and whose range consists of series $\{Y_n, n\in\mathbb{P}\}$. Assume that the series are real-valued and write $Y_n = F[X_n] = F[X](n)$.

We say that F is a *linear dyadic filter* (LDF) if:

(i) F is *linear*: $F[\alpha_1 X_1+\alpha_2 X_2](n) = \alpha_1 F[X_1](n) + \alpha_2 F[X_2](n)$, α_1, α_2 real constants;

(ii) F is *dyadic-invariant*: $F[T^h X](n) = F[X](n\dot{+}h)$, where T^h is the dyadic translation operator: $T^h[X_n] = X_{n\dot{+}h}$.

An important class of LDF is given by

$$Y_n = \sum_{h=0}^{\infty} a(n\dot{+}h)X_n\,, \tag{5.1}$$

where $a(n)$ is the *impulse response function* of the LDF.

For LDF whose domain includes the series $e_n = W(n,x)$, $n \in \mathbb{P}$, there exists a function $A(\cdot)$ such that

$$F[e](n) = A(x)W(n,x), \tag{5.2}$$

and $A(\cdot)$ is the *transfer function* of F. For the LDF (5.1) it is given by

$$A(x) = \sum_{h=0}^{\infty} a(h)W(h,x). \tag{5.3}$$

A LDF of the form (5.1) is *summable* if

$$\sum_{h=0}^{\infty} |a(h)| < \infty. \tag{5.4}$$

Define the dyadic convolution of two summable filters $\{a_1(\cdot)\}$ and $\{a_2(\cdot)\}$ by

$$(a_1 * a_2)(h) = \int_0^{\infty} a_2(h \dot{+} v)\, a_1(v)\, dv. \tag{5.5}$$

The following results are easily proved (Morettin, (1973b)):

(a) If (5.1) holds and $g(\cdot)$ and $f(\cdot)$ are Walsh spectrum of $\{Y_n, n \in \mathbb{P}\}$ and $\{X_n, n \in \mathbb{P}\}$, respectively, then

$$g(x) = [A(x)]^2 f(x), \tag{5.6}$$

where $A(\cdot)$ is given by (5.3);

(b) If $\{a_1(\cdot)\}$ and $\{a_2(\cdot)\}$ are two summable filters, with transfer functions $A_1(\cdot)$ and $A_2(\cdot)$, respectively, then

$\{(a_1*a_2)(\cdot)\}$ is a summable filter with transfer function $A_2(\cdot)A_1(\cdot)$.

Problems related to the identification of LDF and quadratic dyadic systems are considered further in Morettin (1973a). A theory for LDF similar to that developed by Wiener was considered by Weiser (1964) and Pichler (1970a,1970b).

EXAMPLE 5.1 - Let $X = \{X_n, n\in\mathbb{P}\}$ be a process such that

$$X_n = \sum_{k=0}^{\infty} a_k \varepsilon_{n\dotplus k}, \tag{5.7}$$

where $\{\varepsilon_n, n\in\mathbb{P}\}$ is white noise and a_k, $k=0,1,2,\ldots$ are real numbers such that $\sum_{k=0}^{\infty} a_k^2 < \infty$. Then X is dyadic-stationary, $E(X_n)=0$ and its covariance function is easily seen to be

$$B(\tau) = \sigma^2 \sum_{k=0}^{\infty} a_k a_{k\dotplus\tau}. \tag{5.8}$$

We call X given by (5.7) a *linear dyadic process*. If $a_k = 0$, for $k > m$, we call X a *dyadic moving average process of order* m (DMA(m) process).

It follows from (5.6) and (4.6) that the spectrum of X is given by

$$g(x) = \sigma^2[A(x)]^2, \tag{5.9}$$

where

$$A(x) = \sum_{k=0}^{\infty} a_k W(k,x). \tag{5.10}$$

6 - THE FINITE WALSH TRANSFORM

Let $\{X_n, n=0,1,2,\ldots,N-1\}$ be observed values of a time series $X=\{X_n, n\in T\}$, where $T=\mathbb{Z}$ if X is a stationary series and $T=\mathbb{P}$ if X is a dyadic-stationary series. The statistic

$$d^{(N)}(x) = \sum_{n=0}^{N-1} X_n W(n,x), \qquad (6.1)$$

$0 \le x \le 1$ is called the *finite Walsh transform* of $\{X_n, n=0,1,\ldots,N-1\}$. Throughout this paper we assume that $N=2^m$, m a non-negative integer.

If $c_k(n_1,\ldots,n_k)$ denotes the k-th order joint cumulant function of X (see Brillinger (1975), chapter 2, for details on cumulants), that is

$$c_k(n_1,\ldots,n_k) = \text{cum}\{X_{n_1},\ldots,X_{n_k}\}, \qquad (6.2)$$

$n_1,\ldots,n_k \in T$, $k=1,2,3,\ldots$, then we assume that

$$\sum_{n_1} \cdots \sum_{n_k} |c_k(n_1,\ldots,n_{k-1})| < \infty. \qquad (6.3)$$

This is, again, a form of asymptotic independence for well separated values of X. Due to the stationarity of X (usual or dyadic) the cumulant (6.2) is really a function of k-1 arguments.

If X is stationary, the central limit theorem that follows is valid (Morettin (1973b)), THEOREM 6.1 - If $E(X_0^2) < \infty$, (6.3) is satisfied and

$$\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-k} \sum_{k=0}^{N-1} W(n+(n\dot{+}k),x)R(k) = A(x) \qquad (6.4)$$

exists for all non-negative real x, then $d^{(N)}(x)$ is asymptotically normal $N(0,NB(x))$, where $B(x) = E(X_0^2) + 2A(x)$.

This theorem holds true for an m-dependent stationary sequence, when $R(k) = 0$, for $|k| > m$. In particular, for m=0, $d^{(N)}(x)$ is asymptotically $N(0,N\int_{-\pi}^{\pi} g(\alpha)\,d\alpha)$, $g(\cdot)$ given by (3.1).

For an m-dependent stationary sequence the following result is a stronger version of Theorem 6.1, since we do not assume (6.3). Let h,k,r be integer such that, for each k, $n = hk + r$, $0 \le r < k$.

THEOREM 6.2 - Let $\{X_n, n \in \mathbb{Z}\}$ be an m-dependent stationary sequence, and assume that

$$A(x) = \lim_{k\to\infty} k^{-1} \sum_{j=1}^{k-m-|u|} \sum_{u=-m}^{m} W(j\dot{+}(j|u|),x)R(u) \qquad (6.5)$$

exists for all real x. Then $N^{-\frac{1}{2}}d^{(N)}(x)$ is asymptotically $N(0,A(x))$.

For the proof and details see Morettin (1974a).

For a dyadic-stationary series we have the central limit theorem that follows (Morettin (1974b)). Let $V_m = [0,T^{-1}] = [0,2^{-m}]$.

THEOREM 6.3 - If X is dyadic-stationary, (6.3) is valid and

$x_j \dot{+} x_k \notin V_m \pmod 1$, $1 \le j < k \le M$, then the random variables $d^{(N)}(x_1), \ldots, d^{(N)}(x_M)$ are asymptotically independent and with (asymptotic) distributions $N(0,Nf(x_1)),\ldots,N(0,Nf(x_M))$, $f(\cdot)$ being the Walsh spectrum of X.

Since, for every $n \le N-1$, $W(n,x)$ is constant in each sub-interval $(k/N,(k+1)/N)$ of $[0,1]$, the finite Walsh transform $d^{(N)}(\cdot)$ takes on only N different values, that is,

$$d^{(N)}(k) = d^{(N)}(x), \quad \frac{k}{N} < x \le \frac{k+1}{N}, \quad k=0,1,\ldots,N-1. \tag{6.6}$$

Nagai (1976b) proves that $\{d^{(N)}(k),\ k=0,1,\ldots,N-1\}$ are orthogonal if and only if $\{X_n, n\in T\}$ is dyadic-stationary, with

$$E\{d^{(N)}(j)d^{(N)}(k)\} = \begin{cases} \sigma^2(j), & j=k \\ 0, & j\ne k \end{cases}$$

and

$$\sigma^2(j) = N^2\int_{j/N}^{(j+1)/N} dF(x), \qquad j=0,1,\ldots,N-1. \tag{6.7}$$

The statistic (6.7) can be computed quickly with a Fast Walsh Transform (FWT) algorithm. See Shanks (1970) and Ulman (1969) for details.

7 - ESTIMATION OF THE WALSH SPECTRUM

In this section we deal with the problem of estimating the Walsh spectrum of a dyadic-stationary series, as defined by (4.1). Consider the observed values $\{X_n, n=0,1,\ldots,N-1\}$ and let $d^{(N)}(\cdot)$ be the finite Walsh transform of these values.

The Theorem 6.3 suggests that $f(x)$ can be estimated by

$$I^{(N)}(x) = N^{-1}[d^{(N)}(x)]^2, \qquad (7.1)$$

called the *Walsh periodogram*. According to (6.6) we have that $I^{(N)}(x) = I^{(N)}(k)$, $\frac{k}{N} \le x \le \frac{k+1}{N}$, $k=0,1,\ldots,N-1$. It follows that $E\{I^{(N)}(x)\} = N^{-1}E[d^{(N)}(x)]^2 \longrightarrow f(x)$, when $N \to \infty$, which shows that the estimate is asymptotically unbiased. Moreover, the following results hold.

THEOREM 7.1 - (a) If $\sum_\tau \tau|B(\tau)| < \infty$, then

$$E\{I^{(N)}(x)\} = f(x) + O(N^{-1}); \qquad (7.2)$$

(b)

$$\text{Cov}\{I^{(N)}(x), I^{(N)}(y)\} = 2N^{-2}[D_N(x \dot{+} y)]^2 f^2(x) + O(N^{-1}), \qquad (7.3)$$

where $D_N(\cdot)$ is the Dirichlet kernel.

The asymptotic distribution of the Walsh periodogram can be obtained from Theorem 6.3: $I^{(N)}(x)$ is an asymptotically $f(x)\chi^2(1)$ variable, where $\chi^2(r)$ denotes a chi-square variable with r degrees of freedom. Moreover, the variables $I^{(N)}(x_j)$, $i \le j \le M$, are asymptotically independent.

From (7.3) we obtain

$$\text{Var}\{I^{(N)}(x)\} = 2f^2(x) + O(N^{-1}), \qquad (7.4)$$

which shows that the Walsh periodogram is not consistent. In order to improve the stability of the periodogram, we consider the smoothed Walsh periodogram, obtained in the following way. Let the data $X_0,\ldots,X_{N-1}$ be divided into L stretches of length K each, that is, $LK = N$. Here, we require that K is a power of two. Then compute the Walsh periodogram for each section, at frequency x:

$$I^{(K)}(x,\ell) = K^{-1}\Big[\sum_{n=0}^{K-1} X_{n+\ell K} W(n+\ell K,x)\Big]^2, \tag{7.5}$$

for $\ell=0,1,\ldots,L-1$. These periodogram ordinates are asymptotically independent $f(x)\chi^2(1)$ variables, hence this leads us to estimate $f(x)$ by

$$f^{(N)}(x) = L^{-1} \sum_{\ell=0}^{L-1} I^{(K)}(x,\ell). \tag{7.6}$$

For L fixed and $N\to\infty$, it follows that $f^{(N)}(x)$ will be an asymptotically $f(x)\chi^2(L)/L$ variable.

For details on the proofs of the results above see Morettin (1976). Another class of estimates is considered by Morettin (1978b). If p_j are weights such that $\sum_{j=-m}^{m} p_j = 1$, and $s(N)$ is an integer such that $s(N)/N$ is near x, when $N\to\infty$, define the estimate

$$f^{(N)}(x) = \sum_{j=-m}^{m} p_j I\left(\frac{s(N)+j}{N}\right). \tag{7.7}$$

Then it is shown that $f^{(N)}(x)$ is asymptotically unbiased and that

$$\mathrm{Var}\{f^{(N)}(x)\} = 2f^2(x) \sum_{j=-m}^{m} p_j^2 + O(N^{-1}). \qquad (7.8)$$

Besides, the asymptotic distribution of $f^{(N)}(x)$ is a $f(x) \cdot \sum_{j=-m}^{m} p_j \chi_j^2(1)$ variable, the variables $\chi_j^2(1)$ being independent. It may be difficult to use this approximating distribution in practice. A standard procedures is to approximate the distribution of such a variate by a multiple $\theta\chi^2(\nu)$ of a chi-squared variable, whose mean and degrees of freedom are determined by equating first and second-order moments. It follows that $\nu = 1 / \sum_{j=-m}^{m} p_j^2$ and $\theta = 1/\nu$.

A consistent estimate (in quadratic mean sense) has been proposed by Tanigushi (1977). It is essentially the analogue of the estimate proposed by Brillinger and Rosenblatt (1967) for the Fourier spectrum. Let

$$\hat{f}(x) = N^{-1} \sum_{s=0}^{N-1} I^{(N)}(x \dotplus \tfrac{s}{N}) K_N(\tfrac{s}{N}) \Phi(\tfrac{s}{N}), \qquad (7.9)$$

where $\Phi(s) = \delta_{so}$, $K_N(x) = b_N^{-1}K(b_N^{-1}s)$ and $\int_0^1 K(s)\Phi(s)\,ds = 1$. Here the the sequence $\{b_N\}$ is chosen such that $b_N \to 0$ but $b_N^{-1}N \to \infty$, as $N \to \infty$. Under regularity conditions, $\hat{f}(x)$ is asymptotically unbiased and normally distributed.

8 - FURTHER COMMENTS

Some lines of generalizations are possible. A theory of homogeneous random fields on locally compact groups is considered by Yaglom (1961). Tanigushi (1977) develops a statistical spectral estimation theory that unifies both the ordinary and the dyadic case. Properties of the finite Fourier transform for homogeneous random processes defined on locally compact Abelian groups are derived in Morettin (1980a). The case of the circle and of the sphere are studied by Jones (1963) and Roy (1972, 1976) and some generalizations of their results for compact groups are considered by Morettin (1978b).

The problem of computing the fast Fourier transform on finite Abelian groups is discussed by Cairns (1971). Problems related to ergodicity are considered by Blum & Eisenberg (1972) and Jajte (1967). Generalized random processes on locally compact groups are discussed by Ponomarenko (1974) and for the questions of prediction and interpolation of homogeneous processes on groups see Weron (1972).

A concept that may prove useful in the analysis of time series through WF is the following. Let X be a random variable defined on a probability space (Ω, A, P) with values in $(\mathbb{R}_+, B_+, P_X)$, where B_+ is the σ-algebra of Borel sets of the non-negative real numbers of $\mathbb{R}_+$ and P_X is the induced probability measure.

We define the *Walsh characteristic function* (WCF) of X (or of the probability measure P_X or of the corresponding distribution function F of X) to be

$$\psi(t) = E\{W(t,X)\} = \int_\Omega W(t,X(\omega))\,dP(\omega) =$$

$$= \int_{\mathbb{R}_+} W(t,x)\,dP_X(x) = \int_0^\infty W(t,x)\,dF(x), \qquad (8.1)$$

where $t,x \in \mathbb{R}_+$. It follows immediatelly that $\psi(0) = 1$, $|\psi(t)| \leq 1$ and ψ is uniformly W-continuous. Some properties of ψ are analogues to those of the usual characteristic function. The most remarkable is that a similar result to

$$\phi'(t) = \int_{-\infty}^{\infty} e^{itx}(ix)\,dF(x)$$

holds for ψ, where ϕ is the usual characteristic function.

But the definition of derivative has to be changed. If ψ is a real-valued function of a continuous, non-negative real variable, we attach to ψ a function $\psi^{[1]}$ given by

$$\psi^{[1]}(t) = \sum_k [\psi(t) - \psi(t \dotplus 2^{-k}]2^{k-2}. \qquad (8.2)$$

If $\psi^{[1]}$ exists we call it the *first dyadic derivative of* ψ. See Gibbs and Millard (1969), Pichler (1970) and Butzer & Wagner (1973). It follows in particular that

$$W^{[1]}(t,x) = xW(t,x), \qquad (8.3)$$

$t,x \in \mathbb{R}_+$, which is similar to

$$\frac{d}{dt}e^{itx} = ix\,e^{itx}.$$

Equation (8.3) shows that the WF are eigenfunctions of the dyadic differential operator, satisfying

$$\psi^{[1]} - x\psi = 0,$$

Then, it can be shown that if ψ is defined by (8.1), we have

$$\psi^{[1]}(t) = \int_0^\infty xW(t,x)\,dF(x). \qquad (8.4)$$

For details, see Morettin (1980b).

Another point that should be mentioned is that follows. It is known that a way to estimate the Fourier spectrum of a stationary time series is through the auto-regressive (or maximum entropy) spectral estimator. We fit to the process an auto-regressive process of order p (Ar(p)-process).

$$X(t) = a_1X(t-1) + \ldots + a_pX(t-p) + \varepsilon(t), \qquad (8.5)$$

where $\varepsilon(t)$ is white noise. Then we estimate the spectrum $g(x)$ of X by

$$\hat{g}(x) = \frac{\hat{\sigma}^2}{2a}\cdot\frac{1}{|1-\sum_{j=1}^{p}\hat{a}_j e^{-ixj}|^2}, \qquad (8.6)$$

where $\hat{\sigma}^2$ and $\hat{a}_j$ are estimators of the white noise variance σ^2 and of the coefficients a_j, respectively. See Parzen (1969) and Akaike (1969,1970,1978).

A question that arises is then: could we do the same to estimate the Walsh spectrum? A *dyadic-auto-regressive process of order* p (DAR(p)-process) is a dyadic-stationary process satisfying

$$\sum_{k=0}^{p} b_k X_{n \dot{+} k} = \varepsilon_n, \qquad (8.7)$$

where ε_n is white noise and $b_0 = 1$. The idea is to fit a process of the form (8.7) to the data and estimate the Walsh spectrum f(x) through the estimated spectrum of (8.7). A problem that remains is to find a procedure to estimate the coefficients b_k efficiently (for the process (8.5) we have a procedure through the Yule-Walker equations). See Nagai (1977) for some considerations on DMA(q)-processes and DAR(p)-processes.

REFERENCES

[1] - Akaike,H. (1969), Fitting autoregressive models for prediction, *Ann. Inst. Statist. Math.*, *21*, 243-247.

[2] - Akaike,H. (1970), Statistical predictor identification, *Ann. Inst. Statist. Math.*, *22*, 203-217.

[3] - Akaike,H. (1978), Time series analysis and control through parametric models. In *Applied Time Series Analysis*, D.F.Findley (Editor), 1-24, New York, Academic Press.

[4] - Beauchamp,K.G. (1975), *Walsh Functions and Their Applications*, London, Academic Press.

[5] - Blum,J.R. & Eisenberg,B. (1972), Conditions for metric transitivity for stationary Gaussian processes on groups, *The Ann. of Math. Statist.*, *43*, 1937-1741.

[6] - Brillinger, D.R. (1975), *Time-Series – Data Analysis and Theory*, New York, Holt, Hinehart & Winston.

[7] - Brillinger, D.R. & Rosenblatt, M. (1967), Asymptotic theory of k-th order spectra. In *Spectral Analysis of Time Series*, B. Harris (Editor), 153-158, New York, Wiley.

[8] - Butzer, P.L. & Wagner, H.J. (1973), Walsh-Fourier series and the concept of a derivative, *Applicable Analysis*, 3, 29-46.

[9] - Cairns, T.W. (1971), On the fast Fourier transform on finite Abelian groups. *I.E.E.E. Trans. on Computers*, May, 1971, 569-571.

[10] - Chrestenson, N.E. (1955), A class of generalized Walsh functions, *Pacific J. of Math.*, 5, 17-31.

[11] - Crittenden, R.B. (1970), Walsh-Fourier transform, In *Proc. Sympos. Appl. Walsh Functions*, 170-174, Washington, D.C.

[12] - Fine, N.J. (1949), On the Walsh functions. *Trans. Amer. Math. Soc.*, *65*, 372-414.

[13] - Fine, N.J. (1950), The generalized Walsh functions, *Trans. Amer. Math. Soc.*, *69*, 66-77.

[14] - Gibbs, J.E. & Millard, M.J. (1969), Some methods of solution of linear ordinary logical differential equations, *National Physical Laboratory*, DES Report nº 2, England.

[15] - Harmuth, H. (1972), *Transmission of Information by Orthogonal Functions*, Berlin, Springer.

[16] - Harmuth, H. (1977), *Sequency Theory-Foundations and Applications*, New York, Academic Press.

[17] - Jajte, R., (1967), On stochastic processes on an Abelian locally compact group, *Colloq. Mathem.*, *XVII*, 351-355.

[18] - Jenkins, G.M. & Watts, D.G. (1968), *Spectral Analysis and Its Applications*, San Francisco, Holden-Day.

[19] - Jones, R.H. (1963), Stochastic processes on a sphere, *Ann. Math. Statist.*, *34*, 213-215.

[20] - Koopmans, L.H. (1974), *The Spectral Analysis of Time Series*, New York, Academic Press.

[21] - Morettin, P.A. (1972), Walsh-Fourier analysis of time series, Ph.D. Dissertation, Univ. of California, Berkeley.

[22] - Morettin, P.A. (1973a), Stochastic dyadic systems, In *1973 Proc. Symp. Appl. Walsh Functions*, 290-293, Washington, D.C.

[23] - Morettin, P.A. (1973b), A note on a central limit theorem for dependent random variables, *Bol.Soc.Brasil.Mat.*, *4*, 47-49.

[24] - Morettin, P.A. (1974a) Limit theorems for stationary and dyadic-stationary processes, *Bol. Soc. Brasil. Mat.*, *5*, 97-104.

[25] - Morettin, P.A. (1974b), Walsh-function analysis of a certain class of time series, *Stoch. Processes and their Applic.* *2*, 183-193.

[26] - Morettin, P.A. (1976), Estimation of the Walsh spectrum, *I. E. E. E. Trans. on Inf. Theory*, Jan. 1976, 106-107.

[27] - Morettin, P.A. (1978a), Estimation of the spectrum and covariance function of a dyadic-stationary series, *Bol. Soc. Brasil. Mat.*, *9*, 83-88.

[28] - Morettin P.A. (1978b), On homogeneous stochastic processes on compact Abelian groups, *Ann. Inst. Statist. Math.*, *30*, 465-472.

[29] - Morettin, P.A. (1980a), Homogeneous random processes on locally compact Abelian groups, *Annais Academia Brasileira de Ciências*, *51*, to appear.

[30] - Morettin, P.A. (1980b), On Walsh characteristic functions, *Ciência e Cultura* (J.Braz. Soc. Advanc. Sci.), *32*, to appear.

[31] - Morgenthaler, G.W. (1957), On Walsh-Fourier series, *Trans.Amer.Math. Soc.*, *84*, 472-507.

[32] - Nagai, T. (1976a), Dyadic stationary processes and their spectral representations. *Bull. Math. Statist.*, *17*, 65-73.

[33] - Nagai, T. (1976b), On finite Walsh transforms of a dyadic stationary time series. *The Res.Rep.Faculty Eng., Oita Univ.*, 63-66.

[34] - Nagai,T. (1977), On finite parametric linear models of dyadic stationary processes. Unpublished manuscript.

[35] - Paley,R.E.A.C. (1932), A remarkable series of orthogonal functions, I and II, *Proc. London Math. Soc.*, *34*, 241-279.

[36] - Parzen,E. (1969), Multiple time series modelling. In *Multivariate Analysis - II*, P.R.Krishnaiah (Editor), 389-409, New York,Academic Press.

[37] - Ponomarenko,A.I.(1974),Harmonic analysis of generalized wide-sense homogeneous random fields on a locally compact commutative group, *Theory Probab. and Math. Statist.*, nº 3, 119-137.

[38] - Pichler,F. (1967), Das system der sal-und-cal-Funktionen als Erweiterung des systems der Walsh-Functionen und Theorie der sal-und -cal-Fouriertransformation, Dissertation, Univ. of Innsbruck.

[39] - Pichler,F. (1970a), Walsh functions and linear system theory.In *Proc. Symp. Appl. Walsh Functions*, 175-182, Washington, D.C.

[40] - Pichler,F. (1970b), Walsh-Fourier-synthese optimaler filter. *A.E.U.*, *vol. 24*, 350-360.

[41] - Roy,R. (1972), Spectral analysis for a random process on the circle, *J. Appl. Prob.*, *9*, 745-757.

[42] - Roy,R. (1976), Spectral analysis for a random process on the sphere, *Ann. Inst. Statist. Math.*, *28*, 91-97.

[43] - Shanks,J.L. (1969), Computation of the fast Walsh transform. *I.E.E.E. Trans. on Computers*, May 1969, 457-459.

[44] - Selfridge,R.G. (1955), Generalized Walsh transform,*Pacific.J.Math.*, *5*, 451-480.

[45] - Tanigushi,M. (1977), On a generalization of a statistical spectrum analysis, *Res. Reports on Inf. Sci.*, *nº B - 47*, Tokyo Inst. of Technology.

[46] - Ulman,L.J. (1970), Computation of the Hadmard transform and the R-transform in ordered form, *I.E.E.E. Trans. on Computers*, April 1970, 359-360.

[47] - Walsh,J.L. (1923),A closed set of normal orthogonal functions, *Amer. J. Math.*, 45, 5-24.

[48] - Weiser,F.E. (1964), Walsh function analysis of instataneous nonlinear stochastic problems, Ph.D. dissertation, Politecnic Institue of Brooklyn.

[49] - Weron,A. (1972), Interpolation of multivariate stationary processes over any locally compact Abelian group. *Bull. Acad. Polonaise des Sciences*, XX, 949-953.

[50] - Yaglon,A.M. (1961), Second-order homogeneous random fields, *Proc. Fourth-Berkeley Symp.*, 593-622.

- □ -

RELATÓRIO TÉCNICO DO DEPARTAMENTO DE ESTATÍSTICA

Títulos Publicados

RT-MAE-7901 - Wagner de Souza Borges, On the Limiting Distribution of the Failure Time of Composite Material, outubro de 1979.

RT-MAE-7902 - A.Galves, J.G.Leite & M.Roussignol, The Invariance Principle For The One-Dimensional Symmetric Simple Exclusion Process, novembro de 1979.

RT-MAE-8001 - Raúl P.Mentz, Juan C.Abril, Zulema N.Cardozo & Maria B. Ceballos, Exploratory Fitting of Autoregressive and Moving Models to Well-behaved Time Series Data, fevereiro de 1980.