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Free Group Algebras in Certain Division Rings

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Abstract

Let D be a division ring with centre k. We show that D contains the k-group algebra of the free group on two generators, when D is either the ring of fractions of a suitable skew polynomial ring, or when it is generated by a polycyclic-by-finite group which is not abelian-by-finite, or when it is the ring of fractions of the universal enveloping algebra of a finite-dimensional Lie algebra of characteristic zero.

1 Introduction

Let A_1 be the first Weyl algebra over the field of complex numbers, and let D be the field of fractions of A_1. In [9], in the course of computing the Gel'fand-Kirillov dimension of certain subalgebras of D, Makar-Limanov shows that D contains the complex group algebra of a (from now on always non-cyclic) free group. That this may be a general phenomenon, at least when the centre of a division ring is uncountable, is indicated by the following:

Theorem [1]. *If the centre* k *of a division ring* D *is uncountable, and if* D *contains a free* k*-algebra, then* D *contains the group algebra of a free group over* k. □

In this paper we address the question of the existence of free group algebras in certain families of division rings. More precisley, we prove

Theorem A *Let* K *be a field,* $\theta \in Aut(K)$ *of infinite order, and let* k *be the fixed field of* θ. *Define* δ *by* $c\delta = c - c\theta$ *for all* $c \in K$. *Let* $a, b \in K^*$ *be such that* a *has infinite orbit under* θ, *and* $K\delta \cap b\, k[a] = \{0\}$. *Let* N *be an integer. Then* $\{a, bx^N(1-x)^{-1}\}$ *is* k*-free.*

Theorem B. *Let* $K = k(t)$ *be the rational function field over* k, *and* $R = k(t)[x; \theta, \psi]$, *where* θ *is a* k*-automorphism of* K, *and* ψ *is a* θ*-derivation of*

K. *Let* D *be the field of fractions of* R.

(i) *If* θ *has infinite order, then* D *contains a* k*-free group algebra.*

(ii) *If* $\theta = 1$ *and* char $K = 0$, *then* D *contains a* k*-free group algebra.*

(iii) *In all other cases,* D *satisfies a polynomial identity.*

Theorem C. *Let* $D = k(G)$ *be the division* k*-algebra generated by a polycyclic-by-finite group* $G \leq D^*$. *Then* D *contains a* k*-free group algebra if and only if* G *is not abelian-by-finite.*

Theorem D. *The field of fractions of the universal enveloping algebra of a non-abelian, soluble or finite-dimensional Lie algebra over a field* k *of characteristic zero, contains a* k*-free group algebra.*

To prove the above results, we begin by producing k-free sets in the field of fractions of skew polynomial rings. If the free generators have valuation at least 1 (relative to whatever valuation happens to be present), then the following result of Lichtman's may be applied to produce elements that generate a free group algebra:

Proposition (Lichtman [4]). *Let* U *be an Ore domain with a discrete valuation* v. *Let* D *be the ring of fractions of* U *and* k *the centre of* D. *Then for any* k*-free set* x_1, x_2 *such that* $v(x_i) \geq 1$ *for* $i = 1, 2$, *the elements* $1 + x_i$ *generate a free group ring* $kF \subseteq D$. □

To consider the existence of free algebras in the field of fractions of skew polynomial rings, we introduce a polynomial ring with the property that the k-freeness of two elements is equivalent to the question of whether a certain family of polynomials is contained in an ideal in this new ring. We then use a suitable functional equation for a generating function of the family of polynomials, to show that the family is not contained in the ideal. In Section 4, we give a more combinatorial criterion (Corollary 4.1), deduce Theorem D, and indicate how the main freeness arguments of [7], [8], and [10] also follow from the criterion. The reader should also compare our approach with that of Lorenz [5]. Both ultimately rely on Makar-Limanov's ideas [7], [9], but ours seems to be simpler, and more suitable for generalization.

2 Free Group Algebras in the Ring of Fractions of Skew Polynomial Rings

Let K, θ, k, and N be as in the statement of Theorem A. Let $p = x^N \sum_{i=0}^{n} x^i a_i \in k[x;\theta]$, where $n \geq 1$, $a_i \in k$ for all i, and $a_0 a_n \neq 0$. Viewing $K[x;\theta]$ as a subset of the ring of fractions $K(x;\theta)$, and the latter as a subset of the ring of skew Laurent series $K((x;\theta))$, we may write

$$p^{-1} = x^{-N} \sum_{m=0}^{\infty} x^m \alpha_m$$

where $a_0\alpha_0 = 1$ and $\sum_{i+m=j} a_i \alpha_m = 0$ for all $j \geq 1$. It is more convenient to study the question of whether the set $\{a, bp^{-1}\}$ is k-free in a more general ring.

Let $A = k[X_i, Y_i, i \in Z]$ be the ring of polynomials in commuting indeterminates X_i and Y_i over k, and let σ be the automorphism of A defined by $X_i\sigma = X_{i+1}$ and $Y_i\sigma = Y_{i+1}$ for all i. In the ring of skew Laurent series $A((X;\sigma))$, consider the element $z = X^{-N}\sum_{m=0}^{\infty} X^m\alpha_m$. Given a, b in K, define $\varphi = \varphi_{a,b} : A((X;\sigma)) \to K((x;\theta))$ by $X\varphi = x$, $X_i\varphi = a\theta^i$, and $Y_i\varphi = b\theta^i$ for all i. Then $z\varphi = p^{-1}$.

Next, let W be the free k-algebra on $\{u, v\}$. For $w = w(u, v) \in W$, the element $w(X_0, Y_0 z) \in A((X;\sigma))$ may be written as

$$w(X_0, Y_0 z) = \sum_{i \in Z} X^i P_i(w)$$

with all $P_i(w) \in A$ (and the series has only a finite number of non-zero terms with $i < 0$). The idea, of course, is that $w(a, bp^{-1}) = w(X_0, Y_0 z)\varphi_{a,b}$, so the polynomials $P_i(w)$ contain the information on the k-freeness of $\{a, bp^{-1}\}$. Next, define the monomials $m_I \in W$ by: $m_\emptyset = 1$, and for a tuple $I = (i_0, \ldots, i_s)$ of non-negative integers, with $s \geq 0$, let

$$m_I = \begin{cases} u^{i_0} & \text{if} \quad s = 0 \\ u^{i_0}\prod_{r=1}^{s}(vu^{i_r}) & \text{if} \quad s \geq 1. \end{cases}$$

Clearly, every element of W is uniquely of the form $w = \sum \gamma_I m_I$, with $\gamma_I \in k$. Also, $P_i(w) = \sum \gamma_I P_i(m_I)$. We need the following result:

Lemma 2.1 *Let $I = (i_0, \ldots, i_s)$ with $s \geq 1$. Then for all integers t,*

$$P_t(m_I) = \left(\sum_{d \geq -N} P_{t-d}(m_{I'})^{\sigma^d} Y_d \alpha_{d+N} \right) X_0^{i_s}.$$

PROOF. By the definition of m_I, we have $m_I = m_{I'}(vu^{i_s})$, and so

$$\begin{aligned}
\sum_t X^t P_t(m_I) &= m_{I'}(X_0, Y_0 z) Y_0 z X_0^{i_s} \\
&= \left(\sum_j X^j P_j(m_{I'}) \right) \left(Y_0 X^{-N} \sum_{m=0}^{\infty} X^m \alpha_m \right) X_0^{i_s} \\
&= \sum_j \sum_{m \geq 0} X^{j-N} P_j(m_{I'})^{\sigma^{-N}} Y_{-N} X^m \alpha_m X_0^{i_s} \\
&= \sum_j \sum_{m \geq 0} X^{j-N+m} P_j(m_{I'})^{\sigma^{m-N}} Y_{m-N} \alpha_m X_0^{i_s}.
\end{aligned}$$

Now put $t = j + m - N$, $d = m - N$, and observe that $d \geq -N$ since $m \geq 0$. The result follows upon re-writing the last double summation, and equating the coefficients of the powers of X. □

As usual, for $I = (i_0, \ldots, i_s)$, we write $t(I) = i_s$, $I' = (i_0, \ldots, i_{s-1})$ and $\ell(I) = s + 1$. Let $A((\zeta))$ be the ring of Laurent series in a central indeterminate ζ over A, and let σ act on $A((\zeta))$ through the coefficients. For non-empty I, define

$$H(I') = \sum_{i \in \mathbf{Z}} P_i(m_I) X_0^{-t(I)} \zeta^i \in A((\zeta)).$$

We claim that the right-hand side only depends on I'. Indeed, if $I' = J'$, then $m_I(X_0, Y_0 z)X_0^{-t(I)} = m_{I'}(X_0, Y_0 z)Y_0 z = m_J(X_0, Y_0 z)X_0^{-t(J)}$, whence $P_i(m_I)X_0^{-t(I)} = P_i(m_J)X_0^{-t(J)}$ for all i. Define an equivalence relation $\sim$ on the set of tuples by: $I \sim J$ if and only if $I' = J'$, and let $\mathcal{I}$ be a complete set of representatives of the equivalence classes of $\sim$. Write $\Lambda_{a,b} = ker\varphi_{a,b|A}((\zeta))$, a prime ideal of $A((\zeta))$. We then have the following:

Proposition 2.1 *Let $w = \sum \gamma_I m_I \in W$, where almost all $\gamma_I = 0$. Then the following are equivalent:*

(a) $w(a, bp^{-1}) = 0$.

(b) $\sum_{J\in\mathcal{I}} \left(\sum_{I\sim J} \gamma_I X_0^{t(I)}\right) H(J') \in \Lambda_{a,b}$.

PROOF. Of course $0 = w(a, bp^{-1}) = w(X_0, Y_0 z)\varphi_{a,b}$ if and only if $\sum_i X^i P_i(w) = w(X_0, Y_0 z) \in ker\varphi_{a,b}$, if and only if each $P_i(w) \in ker\varphi_{a,b}$ (by the definiton of φ), if and only if $\sum_i P_i(w)\zeta^i \in \Lambda_{a,b}$ (again by definition). But

$$\begin{aligned}
\sum_i P_i(w)\zeta^i &= \sum_i \sum_I \gamma_I P_i(m_I)\zeta^i \\
&= \sum_I \left(\sum_i P_i(m_I)\zeta^i\right) \\
&= \sum_I \gamma_I H(I')X_0^{t(I)} \\
&= \sum_{J\in\mathcal{I}} \sum_{I\sim J} \left(\gamma_I X_0^{t(I)}\right) H(J'),
\end{aligned}$$

as required. □

The point is that condition (*b*) above is quite tractable, especially in view of a recurrence relation satisfied by the H. We also observe that, $H(\emptyset) = 1$, as follows immediately from the various definitions.

Theorem 2.1 . *For all I with $\ell(I) \geq 2$ we have $\zeta^N \sum_{r=0}^{n} a_r \zeta^r H(I)^{\sigma^{r+N}} = H(I')X_0^{\ell(I)} Y_0$.*

PROOF. Given J, Lemma 2.1 implies that

$$\begin{aligned}
H(J') &= \sum_{t} P_t(m_J) X_0^{-\ell(J)} \zeta^t \\
&= \sum_{t} \sum_{d \geq -N} P_{t-d}(m_{J'})^{\sigma^d} Y_d \alpha_{d+N} \zeta^t \\
&= \sum_{d \geq -N} Y_d \alpha_{d+N} \zeta^d \left(\sum_{t} P_{t-d}(m_{J'}) \zeta^{t-d} \right)^{\sigma^d} \\
&= \sum_{d \geq -N} Y_d \alpha_{d+N} \zeta^d \left(H(J'') X_0^{\ell(J')} \right)^{\sigma^d} \\
&= \sum_{d \geq -N} H(J'')^{\sigma^d} X_d^{\ell(J')} Y_d \alpha_{d+N} \zeta^d.
\end{aligned}$$

In particular, replacing J' by I gives $H(I) = \sum_{d \geq N} H(I')^{\sigma^d} X_d^{\ell(I)} Y_d \alpha_{d+N} \zeta^d$, whence

$$\zeta^N \sum_{r=0}^{n} a_r \zeta^r H(I)^{\sigma^{r+N}} = \sum_{r=0}^{n} a_r \zeta^{r+N} \left(\sum_{d \geq -N} H(I')^{\sigma^d} X_d^{\iota(I)} Y_d \alpha_{d+N} \zeta^d \right)^{\sigma^{r+N}}$$

$$= \sum_{r=0}^{n} \sum_{d \geq -N} H(I')^{\sigma^{r+N+d}} \zeta^{r+N+d} X_{r+N+d}^{\iota(I)} Y_{r+N+d} a_r \alpha_{d+N}.$$

Letting $N + d = m$ and $r + N + d = j$, the last double summation becomes $\sum_{j \geq 0} \sum_{r+m=j} a_r \alpha_m H(I')^{\sigma^j} \zeta^j X_j^{\iota(I')} Y_j$. Since $\sum_{r+m=j} a_r \alpha_m$ is 0 if $j \geq 1$, and is 1 when $j = 0$, the result follows. □

PROOF OF THEOREM A. In this case $p = X^N(1-X)^{-1}$, so $n = 1$, $a_0 = 1$, and $a_1 = -1$. Suppose first that $N > 0$. The formula in Theorem 2.1 becomes

$$\zeta^N H(J)^{\sigma^N} - \zeta^{N+1} H(J)^{\sigma^{N+1}} = Y_0 X_0^{\iota(J)} H(J'), \tag{1}$$

i.e.,

$$H(J)^{\sigma^{-1}} = \zeta H(J) + \zeta^{-N} Y_{-N-1} X_{-N-1}^{\iota(J)} H(J')^{\sigma^{-N-1}}. \tag{2}$$

Now $H(\emptyset) = 1$ is invariant under σ. Therefore, it can be proved by induction on $\ell(J)$, that

$$H(J)^{\sigma^{-1}} = \zeta H(J) + \sum_{i=1}^{\ell(J)} \lambda_i(J) H(J^{(i)}), \tag{3}$$

where $\lambda_i(J) \in A[\zeta, \zeta^{-1}]$, and $J^{(i)} = (J^{(i-1)})'$. To do this, use (2) and the inductive hypothesis to get a formula for $H(J')^{\sigma^{-N-1}}$, and then use (2) again.

From (1) we also have that

$$\zeta H(J)^{\sigma} = H(J) - \zeta^{-N} Y_{-N} X_{-N}^{\ell(J)} H(J')^{\sigma^{-N}},$$

and so using (2) with J' in place of J, it follows that

$$\zeta H(J)^{\sigma} = H(J) - Y_{-N} X_{-N}^{\ell(J)} H(J') + \sum_{i=2}^{\ell(J)} \tau_i(J^{(i)}) H(J^{(i)}), \tag{4}$$

where $\tau_i(J^{(i)}) \in A[\zeta, \zeta^{-1}]$. Now, by Proposition 2.1(b), it is enough to prove that if $\left(\sum_{J \in T} \sum_{I \sim J} \gamma_I X_0^{\ell(I)}\right) H(J) \in \Lambda_{a,b}$, then $\sum_{I \sim J} \gamma_I X_0^{\ell(I)} \in ker\varphi$ for each J, for then the transcendence of a over k implies that $\gamma_I = 0$ for all I.

We will work with a more general expression, proving

CLAIM. *If $g_J \in A[\zeta, \zeta^{-1}]$ are such that $\lambda = \sum_{i=0}^{t} \sum_{\ell(J)=i} g_J H(J) \in \Lambda_{a,b}$, then each $g_J \in (ker\varphi_{a,b|A})[\zeta, \zeta^{-1}]$.*

The claim finishes the proof of Theorem A, by letting $g_J = \sum_{I \sim J} \gamma_I X_0^{\ell(I)} \in A \subseteq A[\zeta, \zeta^{-1}]$.

We now turn to the proof of the claim. Suppose that the claim is not true. In a counterexample λ, we may clearly suppose that if each g_J occurring in λ is written in the form $g_J = \sum_{r=\ell}^{m} c(r, J)\zeta^r$ (where $\ell \leq m$ are integers and $c(r, J) \in A$), with $c(r, J) \notin ker\varphi_{a,b|A}$ if $c(r, J) \neq 0$. Evidently, this implies that $g_J \notin (ker\varphi_{a,b|A})[\zeta, \zeta^{-1}]$ for all J. In addition, choose the counterexample $\lambda \in \Lambda_{a,b}$ with the properties

(a) t is minimal.

(b) The total number of $c(r, I)$ with $\ell(I) = t$ is minimal.

Fix J_0 with $\ell(J_0) = t$. By multiplying λ by a suitable power of ζ, if necessary, we may suppose that $c(0, J_0) \notin ker\varphi_{a,b|A}$. Then $c(0, J_0)^\sigma \lambda - c(0, J_0)\zeta\lambda^\sigma \in \Lambda_{a,b}$, and

$$c(0, J_0)^\sigma \lambda - c(0, J_0)\zeta\lambda^\sigma = \sum_{i=0}^{t} \sum_{\ell(J)=i} (c(0, J_0)^\sigma g_J H(J) - c(0, J_0)\zeta H(J)^\sigma) =$$

$$\sum_{\ell(J)=t} (c(0, J_0)^\sigma g_J - c(0, J_0) g_J^\sigma) H(J) + \sum_{\ell(L)=t-1} \Big(c(0, J_0)^\sigma g_L - c(0, J_0) g_L^\sigma$$

$$+ \sum_{J'=L} c(0, J_0) g_J^\sigma Y_{-N} X_{-N}^{\ell(J)} \Big) H(L) + \sum_{i=0}^{t-2} \sum_{\ell(J)=i} \gamma_J H(J),$$

where $\gamma_J \in A[\zeta, \zeta^{-1}]$. Write $c = c(0, J_0)\varphi$ We now have to consider several possibilities.

Suppose first that $\{c(r, J) : \ell(J) = t\} = \{c(0, J_0)\}$. Then $g_{J_0} = c(0, J_0)$, and therefore the above becomes

$$c(0, J_0)^\sigma \lambda - c(0, J_0)\zeta\lambda^\sigma = \sum_{i=1}^{t-1} \sum_{\ell(J)=i} f_J H(J).$$

By the minimality of t, each $f_J \in (ker\varphi)[\zeta, \zeta^{-1}]$. In particular, for $L = J_0'$, we have

$$c(0, J_0)^\sigma g_{J_0'} - c(0, J_0) g_{J_0'}^\sigma + c(0, J_0) g_{J_0}^\sigma Y_{-N} X_{-N}^{\ell(J_0)} \in (ker\varphi)[\zeta, \zeta^{-1}].$$

Thus,

$$c(0,J_0)^\sigma c(0,J_0') - c(0,J_0)c(0,J_0')^\sigma + c(0,J_0)c(0,J_0)^\sigma Y_{-N}X_{-N}^{t(J_0)} \in ker\varphi.$$

Set $d = c(0,J_0')\varphi$. Then $c^\theta d - cd^\theta + cc^\theta(ba^{t(J_0)})^{\theta-N} = 0$. Multiplying by $(cc^\theta)^{-1}$, we obtain $c^{-1}d - (c^{-1}d)^\theta = -(ba^{t(J_0)})^{\theta-N} = 0$, contradicting the hypotheses.

Thus the number of $c(r,J)$ with $\ell(J) = t$ is at least two. In this case, $c(0,J_0)^\sigma\lambda - c(0,J_0)\zeta\lambda^\sigma$ has one less $c(r,J)$ with $\ell(J) = t$ than the original λ, so

$$c(0,J_0)^\sigma g_J - c(0,J_0)g_J^\sigma \in (ker\varphi)[\zeta,\zeta^{-1}]$$

for all J with $\ell(J) = t$, and

$$c(0,J_0)^\sigma g_L - c(0,J_0)g_L^\sigma + \sum_{J'=L} c(0,J_0)g_J^\sigma Y_{-N}X_{-N}^{t(J)} \in (ker\varphi)[\zeta,\zeta^{-1}]$$

for all L with $\ell(L) = t-1$.

We claim that there exists $J \neq J_0$ with $\ell(J) = t$. For if not, then there must exist $j \neq 0$ such that $c(j,J_0) \notin ker\varphi$. But

$$c(0,J_0)^\sigma c(j,J_0) - c(0,J_0)c(j,J_0)^\sigma \in ker\varphi,$$

and

$$c(0,J_0)^\sigma c(j,J_0') - c(0,J_0)c(j,J_0') + c(0,J_0)c(j,J_0)^\sigma Y_{-N}X_{-N}^{t(J)} \in (ker\varphi).$$

Let $d = c(j, J_0)\varphi$, and $e = c(j, J_0')\varphi$. Then, as before, we have $c^\theta d - cd^\theta = 0$, whence $c^{-1}d \in k$. Also, $c^\theta e - ce^\theta + cd^\theta(ba^{t(J_0)})^{\theta-N} = 0$. Multiplying both sides of the above equality by $(cc^\theta)^{-1}$, we obtain

$$\left(c^{-1}e - (c^{-1}e)^\theta\right) + c^{-1}d(ba^{t(J_0)})^{\theta-N} = 0.$$

The first term belongs to $K\delta$, and the second summand is a non-zero element of $bk[a]$. This is a contradiction.

Thus, there exists $J_1 \neq J_0$ and some $c(j_0, J_1) \notin ker\varphi$. Therefore,

$$c(0, J_0)^\sigma c(j_0, J) - c(0, J_0)c(j_0, J)^\sigma \in ker\varphi,$$

for all J with $\ell(J) = t$, and

$$c(0, J_0)^\sigma c(j_0, L) - c(0, J_0)c(j_0, L)^\sigma + \sum_{J'=L} c(0, J_0)c(j_0, J)Y_{-N}X_{-N}^{t(J)} \in ker\varphi.$$

for all L with $\ell(L) = t - 1$. Take $L = J_1'$, and set $d_J = c(j_0, J)\varphi$, and $e_L = c(j_0, L)\varphi$. Then, as before, $c^\theta d_J - cd_J^\theta = 0$ implies that $c^{-1}d_J \in k$ for all J with $\ell(J) = t$. We also have the equation

$$c^\theta e_L - ce_L^\theta + \sum_{J'=L} cd_J^\theta(ba^{t(J_0)})^{\theta-N} = 0,$$

which on multiplication by $(cc^\theta)^{-1}$ yields

$$\left(c^{-1}e_L - (c^{-1}e_L)^\theta\right) + \sum_{J'=L} (c^{-1}d_J^\theta)(ba^{t(J_0)})^{\theta-N} = 0.$$

Now, all the $t(J)$ in the above formula are distinct, and at least one of the coefficients of $a^{t(J)}$ is non-zero. Moreover, the first summand belongs to $\delta(K)$, and the second summand is a non-zero element of $bk[a]$, which is impossible. The theorem is, therefore, established for positive N. When $N \leq 0$, use the K-isomorphism $\psi : K(x;\theta) \longrightarrow K(y;\theta^{-1})$ defined by $x\psi = y^{-1}$. Then $\{a, bx^N(1-x)^{-1}\}\psi = \{a, -by^{-N+1}(1-y)^{-1}\}$, and the latter set is k-free by the first part of the proof, since $-N+1>0$, a has infinite orbit under θ^{-1}, and if δ' is defined on K by $c\delta' = c - c\theta^{-1}$, then $K\delta' \cap bk[a] = \{0\}$. □

Before we prove Theorem B, we turn to a generalization of a result of Lorenz. In [6], Lemma 2, Lorenz proves that for every infinite-order k-automorphism θ of the rational function field $k(t)$, there is a generator a of $k(t)$ satisfying the property stated in Theorem A in the form $K\delta \cap ak[a] = \{0\}$. We refer to such an a as a *distinguished* generator of $k(t)$.

Corollary 2.1 *Let k be a field, θ a k-automorphism of $k(t)$ of infinite order, and let a be a distinguished generator of $k(t)$ as above, and choose any $g \in k[t]$ with $g(0)=0$. Then for any integer N, $\{a, g(a)X^N(1-X)^{-1}\}$ is a k-free set in $D = k(t)(X;\theta)$. Moreover, $k(t)(X;\theta)$ contains a free group algebra.*

PROOF. Only the last statement requires proof. Put $c = aX(1-X)^{-1}$. Then $\{a,c\}$ is a k-free set in D. The elements ac and c are also k-free, and have valuation 1 relative to

$$v\left(\sum_{j\geq n}^{\infty} X^j a_j\right) = n \quad \text{where} \quad a_n \neq 0.$$

(where we regard D as a subset of the ring of skew Laurent series). The existence of the free group algebra now follows from Lichtman's Proposition. □

Corollary 2.2 *Let G be a torsion-free soluble-by-finite group, and let D be the ring of fractions of the group algebra kG. Then D contains a k-free group algebra if and only if G is not abelian-by-finite.*

PROOF. It is proved in [2] that kG has no zero divisors, and it is then well-known that kG is an Ore domain, so the existence of D is clear. Also, D is finite-dimensional over its centre if G is abelian-by- finite, so assume otherwise. By a theorem of Lichtman ([3]), if G contains no nilpotent normal subgroup of finite index, then D contains the group algebra of a free group. We are left with the case where G contains a normal nilpotent non-abelian subgroup N of finite index. Then kN contains the skew polynomial ring $k(t)[x;\theta]$, where $t^\theta = \lambda t$, and λ is not a root of unity. It follows from Corollary 2.1 that D contains the group algebra of a free group, as required. □

PROOF OF THEOREM B. To begin with, if $\theta \neq 1$, then by [12], Theorem 1, $R = k(t)[x;\theta,\psi] \simeq k(t)[y;\theta,0]$, where 0 denotes the zero derivation. The latter ring is finite-dimensional over its centre if θ has finite order, and contains a free group algebra if θ has infinite order, by (Corollary 2.1) above. This proves (i), and part of (iii). As for (ii), if ∂ denotes formal differentiation on $k(t)$, then it is known that every k-derivation of $k(t)$ is of the form $g\partial$ for some $g \in k(t)$. It follows that for any two k-derivations ψ_1 and ψ_2 of $k(t)$, the rings $k(t)[x_1;1,\psi_1]$ and $k(t)[x_2;1,\psi_2]$ are k-isomorphic (use the $k(t)$-map $x_1 \mapsto g_1g_2^{-1}x_2$). Therefore, it suffices to prove (ii) for any one derivation. Choose the k-derivation D with $tD = t$. Then in $k(t)[x;1,D]$ we have $tx = xt + t = (x+1)t$, so conjugation by t defines the automorphism $\alpha : x \mapsto x+1$ of the rational function field $k(x)$. Thus, $k(t)(x;1,D) = k(x)(t;\alpha)$. The latter is clearly finite-dimensional over its centre if $char k > 0$, and contains a free group algebra otherwise, again by Corollary 2.1. This proves (ii), and the remaining part of (iii). □

3 Free group algebras in division rings generated by polycyclic-by- finite groups

If G is abelian-by-finite, then D is finite-dimensional over its centre, so suppose that G is not abelian-by-finite. For the proof of Theorem C we require the following well-known result of Bergman (cf. [11] , 9.3.9).

Theorem (Bergman). *Let A be a finitely generated torsion-free abelian group, and let $\mathcal{U}$ be a group of automorphisms of A. Assume that $\mathcal{U}$ and all its subgroups of finite index act rationally irreducibly on A. If $I \neq A$ is a $\mathcal{U}$-invariant ideal of* kA, *then either* $I = 0$ *or* $\dim_k kA/I < \infty$. □

PROOF OF THEOREM C. Our argument is along the lines of Lorenz ([5], Theorem 2.3). Suppose first that G is not nilpotent-by-finite, and that all its nilpotent subgroups are abelian-by-finite, for otherwise D contains a skew polynomial ring of the form appearing in the proof of Theorem B.

Then, G contains a subgroup $H = A \quad < z >$, with A free abelian of rank at least 2, and z and all of its powers acting rationally irreducibly on A. Now, the inclusion $H \subset D^*$ induces an embedding $kH \hookrightarrow D$. For if the kernel I of the induced k-algebra map $kH \to D$ is non-zero, then I is completely prime and $I \cap kA = 0$. (This can be seen by choosing a non-zero element $\alpha = \sum_{i=0}^{q} \alpha_i z^i \in I$ with $\alpha_i \in kA$ and minimal q, and using the usual argument to find a non-zero element with smaller q on the assumption that $q > 0$.) Thus, by the above theorem of Bergman, the field $F = kA/(I \cap kA)$ is a finite extension of k, which implies that some power of z acts trivially on F. This contradicts the way H was chosen. Hence, kH embeds into D. But by Corollary 2.2, the field of fractions of kH contains a free group algebra, as required. □

REMARK. It follows from Theorem C that D^* always contains a free group. When G is abelian-by-finite, D is finite-dimensional over k, and the claim is a consequence of the Tits' Alternative for linear groups. Otherwise, Theorem C is complementary to the main result of [5]. Note, however, that the Tits' Alternative is not used in our proof when D is finite-dimensional over k.

4 Rings of Fractions of Universal Enveloping Algebras

The proof of Theorem D once again depends on the result of Lichtman, to go from suitable free k-sets to free group algebras. In order to produce the required free generators, we follow the construction of Makar-Limanov and Malcolmson, but use a different method to verify the freeness of the proposed generators. The following technical result is true in even greater generality, but already subsumes many of the freeness results in the literature.

Theorem 4.1 *Let F be a field, θ and φ commuting automorphisms of F, E the fixed field of φ, a a given element of F, and suppose that, for each multi-index $I \neq \emptyset$ and integer $r \in [0, \ell(I) - 1]$, we are given an element $d(a, I, r) \in F$. Write $d(a, I) = d(a, I, \ell(I) - 1)$, and let $B = \{0\}$ if $\theta = 1$, and $B = Z$ otherwise. Assume that the following hold:*

(a) E is contained in the fixed field of θ.

(b) $d(a,I,r) = d(a,I',r)^{\theta}$ for all I with $\ell(I) \geq 2$, and $r \in [0, \ell(I)-2]$.

(c) For a fixed I', the elements $d(a,I)^{\theta^j}$, $j \in B$ are linearly independent over E.

(d) For a fixed I', if $V(I')$ is the E-subspace of F spanned by all the $d(a,I)^{\theta^j}$ for $j \in B$, and δ is defined on F by $x\delta = x\varphi - x$ for all $x \in F$, then $V(I') \bigcap F\delta = \{0\}$.

Define $M(a,I)$ as follows: $M(a,\emptyset) = 1$, and for $I \neq \emptyset$

$$M(a,I) = \prod_{r=0}^{\ell(I)-1} \left[d(a,I,r)(1-t)^{-1}\right] \in F(t;\varphi).$$

Then, for different I, the elements $M(a,I)$ are linearly independent over F.

PROOF. Write $M(a,I) = \sum_{n=0}^{\infty} t^n f_n(a,I) \in F((t;\varphi))$, and consider the corresponding generating function $G(a,I) = \sum_{n=0}^{\infty} f_n(a,I)\zeta^n \in F((\zeta))$, where ζ is central over F. Clearly, it is enough to show that the $G(a,I)$ are linearly independent over F.

Let I be such that $\ell(I) \geq 2$. Then assumption (b) implies that $M(a,I) = M(a,I')^{\theta} d(a,I)(1-t)^{-1} = M(a,I')^{\theta} d(a,I)[1+t(1-t)^{-1}] = M(a,I')^{\theta} d(a,I) + tM(a,I')^{\theta\varphi} d(a,I)^{\varphi}(1-t)^{-1} = M(a,I')^{\theta} d(a,I) + tM(a,I)^{\varphi}$. On equating the powers of t, we obtain

$$f_n(a,I) = f_n(a,I')^{\theta} d(a,I) + f_{n-1}(a,I)^{\varphi} \quad \text{for all} \quad n \geq 0,$$

where $f_{-1} = 0$. Therefore

$$G(a,I) = G(a,I')^{\theta} d(a,I) + \zeta G(a,I)^{\varphi}. \tag{5}$$

So far, this has only been proven for $\ell(I) \geq 1$. However, it may be verified directly that (5) also holds when $\ell(I) = 1$, since $G(a,\emptyset) = 1$.

We prove that $G(a,\emptyset)$ and the $G(a,I)^{\theta^j}$, for all $\ell(I) \geq 2$ and $j \in B$, are linearly independent over F. If not, there exist non-trivial relations of the form

$$\sum_{\ell(I)=m} \sum_{j \in B} \lambda_j(I) G(a,I)^{\theta^j} = \sum_{1 \leq \ell(J) < m} \sum_{j \in B} \lambda_j(J) G(a,J)^{\theta^j} + \lambda, \tag{6}$$

with $\lambda \in F[\zeta]$ and all $\lambda_j(I), \lambda_j(J) \in F$. (There is clearly nothing lost by allowing λ to belong to $F[\zeta]$ rather than F.) Among all such relations, choose one with the minimal m, and among these, one with the least total number of $\lambda_j(I)$, with $\ell(I) = m$ occurring on the left. Since F is a field, we may suppose that for some $j_0 \in B$ and I_0 with $\ell(I_0) = m$, we have $\lambda_{j_0}(I_0) = 1$. Apply φ to (6), multiply by ζ, and substitute from (5) into the left-hand side of the resulting equation, and re-arrange. This yields

$$\sum_{\ell(I)=m} \sum_{j} \lambda_j(I)^{\varphi} G(a,I)^{\theta'} =$$

$$\sum_{j} \sum_{\ell(I)=m} \lambda_j(I)^{\varphi} G(a,I')^{\theta'^{+1}} d(a,I)^{\theta'} +$$

$$\sum_{1 \le \ell(J) < m} \sum_{j} \lambda_j(J)^{\varphi} (G(a,J)^{\varphi} \zeta)^{\theta'} + \lambda^{\varphi} \zeta. \quad (7)$$

Suppose first that $m = 1$. Then the second double sum on the right-hand side of (7) is vacuous. Also, each $G(a,I') = G(a,\emptyset) = 1$, so (7) reduces to

$$\sum_{I} \sum_{j} \lambda_j(I)^{\varphi} G(a,I)^{\theta'} = \sum_{I} \sum_{j} \lambda_j(I)^{\varphi} d(a,I)^{\theta'} + \lambda^{\varphi} \zeta. \quad (8)$$

Subtraction of (6) from (8) yields

$$\sum_{I} \sum_{j} (\lambda_j(I)^{\varphi} - \lambda_j(I)) G(a,I)^{\theta'} = \sum_{I} \sum_{j} \lambda_j(I)^{\varphi} d(a,I)^{\theta'} + \lambda^{\varphi} \zeta - \lambda. \quad (9)$$

Equation (9) has the same form as (6), but fewer terms on the left, and so must be the zero relation. It follows that

$$\lambda_j(I) \in E \quad \text{for all} \quad j \in B \;\text{ and }\; \ell(I) = m, \quad (10)$$

and

$$\lambda^{\varphi} \zeta - \lambda + \sum_{I} \sum_{j} \lambda_j(I)^{\varphi} d(a,I)^{\theta'} = 0. \quad (11)$$

Since the sum in (11) is in F, the equation is impossible unless λ is the zero polynomial. Thus we obtain $\sum_{I,j} \lambda_j(I) d(a,I)^{\theta^j} = 0$, which by assumption (c) implies that all $\lambda_j(I) = 0$, a contradiction.

Thus $m \geq 2$. In this case, the second summation on the right-hand side of (7) may also be expanded using (5). Do this, and subtract (6) from it to obtain an equation of the same form as (6), but with fewer terms on the left, so the coefficient of every G^{θ^j} must be the zero. Equation (10) then holds. Next, for each J with $\ell(J) = m-1$ and each $j \in B$, the coefficient of $G(a,J)^{\theta^j}$ on the right-hand side of the new equation is easily seen to be

$$\lambda_j(J)^{\varphi} - \lambda_j(J) + \sum_{I'=J} \lambda_{j-1}(I) d(a,I)^{\theta^{j-1}} = 0 \tag{12}$$

if $\theta \neq 1$, and

$$\lambda_0(J)^{\varphi} - \lambda_0(J) + \sum_{I'=J} \lambda_0(I) d(a,I) = 0 \tag{13}$$

if $\theta = 1$. Applying $\theta^{-(j-1)}$ to whichever equation we have, and using the assumptions (a) and (d), we find that $\sum_{I'=J} \lambda_{j-1}(I) d(a,I) = 0$ (respectively $\sum_{I'=J} \lambda_0(I) d(a,I) = 0$) if $\theta \neq 1$ (resp. $\theta = 1$). The E-linear independence of the $d(a,I)$ now implies that the coefficients are all zero, which is the final contradiction. The result follows. □

Corollary 4.1 *With the assumptions of the previous theorem, if a is transcendental over E, then the elements $G(a, I')a^{-\iota(I)}$ are linearly independent over E.*

PROOF. Recall the equivalence relation $\sim$ defined by $I \sim J$ if and only if $I' = J'$, with $\mathcal{I}$ a complete set of representatives of the equivalence classes. Suppose we are given elements $\alpha(I) \in E$ such that

$$\sum_I \alpha(I)G(a, I')a^{-\iota(I)} = 0.$$

Then $0 = \sum_I \alpha(I)G(a, I')a^{-\iota(I)} = \sum_{I\in\mathcal{I}} \left(\sum_{J\sim I} \alpha(I)a^{-\iota(I)}\right) G(a, I')$. The elements in the inner summation belong to F. By the previous theorem, each $\sum_{J\sim I} \alpha(I)a^{-\iota(I)} = 0$, and so all $\alpha(I) = 0$ by the transcendence of a over E. □

PROOF OF THEOREM D. There is nothing to prove in those cases where the universal enveloping algebra of L contains a copy of the first Weyl algebra (e.g., when L is soluble). In the remaining cases, we modify the proof of Malcolmson and Makar-Limanov ([10]) in order to be able to apply Lichtman's argument. We follow the notation of [10], pp. 319 *et seq.* We have an element a transcendental over the field K. Let us choose some $f(a) \in K(a) \setminus K$, and see whether $\{f(a), f(a)b^{-1}\}$ is a $\overline{K}$-free set. If not, repeat the proof from the introduction of the m_I on page 319 (there denoted m_ι). The final conclusion is that, for certain $I = (i_0, \ldots, i_k)$, the monomials

$$f(a-(k-1)\beta)^{-i_0}(1-t)^{-1}\ldots f(a-\beta)^{-i_{k-1}}(1-t)^{-1}f(a)^{-i_k}, \qquad (14)$$

are linearly independent over $\overline{K}$. Notė that $i_0, \ldots, i_{k-1} \geq 1$, and that t acts on $\overline{K}(a)$ as the $\overline{K}$- automorphism $\varphi : a \mapsto a - \gamma$. Recall also that $\beta, \gamma \in \overline{K}$ with $\gamma \neq 0$. Define

$$d(x, I, r) = f(x - (\ell(I) - r - 1)\beta)^{-i_r} \qquad (15)$$

for almost all $x \in \overline{K}(a)$. The conditions of Theorem 4.1 are satisfied: here $F = \overline{K}(a)$, the automorphisms are diefined by $a\varphi = a - \gamma$ and $a\theta = a - \beta$. Conditions (a) and (b) are trivial to verify. As for (c), we have $d(a, I) = f(a)^{-i_{k-1}}$, and $f(a)^{\theta^j} = f(a - j\beta)$, so, having fixed I', we require the $\overline{K}$-linear independence of rational functions of the form $f(a - j\beta)^{-n}$, for $j \in B$ and $n \geq 0$. If $1/f$ is not a polynomial, then the usual argument with poles (in case $\theta \neq 1$) and the transcendence of the powers of f over $\overline{K}$ imply the result. Finally, for condition (d) of the theorem, we need to choose f in such a way that the only solution to

$$x^\varphi - x = \sum_{m \geq 1} \alpha_m f^{-m},$$

for $x \in \overline{K}(a)$ and $\alpha_m \in \overline{K}$, is $\alpha_m = 0$. (Note that the summation excludes $m = 0$, since $i_{k-1} \geq 1$). The function f may be chosen in many ways, e.g., $f(a) = (1 + a^n)/a$ for all $n \geq 0$.

With this choice of f, note that (14) has the form of an $M(a, I)f(a)^{-t(I)}$ as defined in Theorem 4.1, so the set $\{f, fb^{-1}\}$ is $\overline{K}$-free by Corollary 4.1, which contradicts the assumption of linear depnedence. In other words, the set $\{f(a), f(a)b^{-1}\}$ *is* $\overline{K}$-free. Now use the degree function on $\mathcal{U}(L)$, with a large enough n in $f(a) = (1 + a^n)/a$, and Lichtman's result, to conclude the proof of Theorem D. □

Example.. Malcolmson and Makar-Limanov choose $f(a) = a^{-1}$ in [10], so the final part of their argument may be deduced from ours. Similarly in [8], one may deduce the main result by choosing $d(t, I, r) = (1 - t)^{-i_r}$, $t\theta = t$ and $t\varphi = \lambda t$, and applying Theorem 4.1, while the result of [7] may be obtained by choosing $d(t, I, r) = t^{-i_r}$, $t\theta = t$, and $t\varphi = t + 1$.

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