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ON THE GEODESICAL CONNECTEDNESS FOR A CLASS OF SEMI–RIEMANNIAN MANIFOLDS

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ABSTRACT. We prove a variational principle for geodesics on a semi-Riemannian manifold $\mathcal{M}$ of arbitrary index k and possessing k linearly independent Killing vector fields that generate a timelike distribution on $\mathcal{M}$. Using such principle and a suitable completeness condition for $\mathcal{M}$, we prove some existence and multiplicity results for geodesics joining two fixed points of $\mathcal{M}$.

1. INTRODUCTION

The geodesics on a Riemannian, or more in general a semi-Riemannian manifold, can be characterized by means of ordinary differential equations. However, in order to obtain results of existence of geodesics between two fixed points, the general theory of differential equations can only be used to develop a local theory, and it does not provide sufficient tools to prove global results. For this reason, and also for many other applications, it is used a variational characterization for geodesics, and to prove results of existence and multiplicity one uses all the machinery and techniques from Calculus of Variations, Global Analysis on Manifolds and Critical Point Theory.

The geodesics between two fixed points are stationary points (not necessarily minima) for the *length* functional in the Riemannian case, or, more in general, for the *action* functional for metrics of arbitrary signature, defined in the space of all curves joining the two given points and satisfying suitable regularity conditions.

For instance, in the case of a Riemannian manifold $\mathcal{M}$, it is proven in the classical literature that the action functional, which is bounded from below, satisfies the Palais–Smale compactness condition precisely when $\mathcal{M}$ is complete. This result, together with the theorem of Hopf and Rinow, can be applied in a number of situation

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to obtain precise information on the number of geodesics joining two points. Moreover, in the Riemannian case one proves that the Morse index of the action functional at its critical points is finite, and the Morse Index Theorem can be proven to obtain extra information on each geodesic, such as the presence of focal points, and also to obtain relations between the geodesical structure and the topological structure of $\mathcal{M}$ (see e.g. [7]).

When one passes to metric of arbitrary signature, the situation becomes quite complicated, due to the following main difficulties:

- the Hopf–Rinow Theorem does *not* hold;
- the action functional is *unbounded* both from above and from below;
- the Morse index of the action functional at its critical points is *infinite*.

It can be shown by several examples that all kinds of geodesic pathologies can occur, such as geodesical disconnectedness or incompleteness, of all causal character, even in the case of a compact semi-Riemannian manifold (see e.g. References [1, 2, 3, 4, 16, 18, 19]).

The problem of the geodesical connectedness and some related questions for Lorentzian manifolds, i.e., semi-Riemannian manifolds with metric tensor of index 1, has been studied using global variational methods mostly by V. Benci, D. Fortunato, F. Giannoni and A. Masiello, in a series of articles appeared in the last years starting from [6, 9]. We refer to [5] and [12] and the references therein for a reasonably complete panorama on the results available concerning the variational methods applied to the study of geodesics in (convex subsets of) Lorentzian manifolds.

A common assumption in all the papers mentioned above is that the Lorentzian manifold $\mathcal{M}$ admitted a *global* space-time splitting of the form $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, where, for all $t_0 \in \mathbb{R}$, $\mathcal{M}_0 \times \{t_0\}$ is a spacelike submanifold of $\mathcal{M}$ and, for all $x_0 \in \mathcal{M}_0$, $\{x_0\} \times \mathbb{R}$ is a timelike submanifold of $\mathcal{M}$. Then, the results proven were based on hypotheses of metric completeness for $\mathcal{M}_0$ and of growth of the metric coefficients and their derivatives with respect to the given splitting.

It should be observed that such results cover a great number of physically interesting Lorentzian manifolds; nevertheless, from both a mathematical and physical point of view, the weak point in the approach is in the fact that the hypotheses of metric growth are not invariant by change of coordinates, i.e., by what physicists consider a simple *rescaling* of the space-time variables.

The first attempt to give intrinsic, i.e., coordinate-free, results of geodesical connectedness for Lorentzian manifolds using global variational methods can be found in the paper by two of the authors in [10]. In this paper it is used a suitable completeness assumption discussed below, to replace the completeness of the Riemannian case, and a symmetry assumption. Namely, the Lorentzian manifold $\mathcal{M}$ was assumed to be *stationary*, i.e., to admit a timelike Killing vector field Y. The conservation law for geodesic induced by a Killing vector field was then used to prove an alternative variational principle for geodesics, yielding a functional bounded from below and satisfying the Palais–Smale compactness condition.

In this paper we want to extend the techniques and the results of [10] to the case of a semi-Riemannian manifold of arbitrary index $k \geq 1$. Even though some of the main ideas of [10] are carried over to this general case, we point out that the work presented is not a simple transliteration to the general case of the results of [10]. Namely, in the case of a metric of arbitrary index, in the construction of the functional framework one has to deal with non-singular matrix operators appearing inside integro-differential equations. The solutions of such equations contain integrals of the operators, and so, differently from the Lorentzian case, some singularities may appear in the operators, obstructing the solubility of certain linear systems.

Moreover, in the technical proof of the lower boundedness for the restricted action functional (see Section 2 and Lemma 3.1), given the vector valued integrals involved in the estimates, one is not able to use directly the Hölder's inequality, and sharper estimates to control the behavior of these integrals are needed.

We generalize the notion of stationarity for Lorentzian manifolds by requiring that our semi-Riemannian manifold $(\mathcal{M}, g)$, where g is a metric tensor of index k, admits a *timelike* k-dimensional distribution Δ, which is generated pointwise by k (linearly independent) Killing vector fields $Y_1, \dots, Y_k$. We recall that a subspace Δ_p of $T_p\mathcal{M}$ is said to be timelike if the restriction of g to Δ_p is negative definite. We also assume the vanishing of the Lie brackets $[Y_i, Y_j]$ and an intrinsic completeness condition on $\mathcal{M}$; non trivial examples of such structures are obtained by considering *warped products* of Riemannian manifolds.

For instance, the reader may find in [17], Example 2.4, a discussion of two examples of semi-Riemannian manifolds satisfying the above conditions. In Example 2.4-(1), it is considered a principal fiber bundle P over a Riemannian manifold B, with structural group G and a warped product of the metric of B and a bi-invariant metric of G. In this case, the vanishing of the Lie brackets holds precisely when the structural group G is abelian, i.e., isomorphic to $\mathbb{R}^k$ or $\mathbb{T}^k$.

An explicit example of a vector bundle over a Riemannian manifold is discussed in Appendix B.

Given such a structure, we prove the existence and, depending on the topology of $\mathcal{M}$, the multiplicity of geodesics between two fixed points of $\mathcal{M}$. In order to make our statements precise, we pass to a technical description of our setup.

We consider a smooth, connected, $(k+m)$-dimensional semi-Riemannian manifold $(\mathcal{M}, g)$, with $k, m \geq 1$, whose topology satisfies the second countability axiom and the Hausdorff separation axiom, and we assume that g is a semi-Riemannian C^∞ metric of index $\nu(g) = k$. We recall that the index of a bilinear form is the dimension of a maximal subspace on which the form is negative definite.

A vector field Y on $\mathcal{M}$ is a *Killing* vector field if the Lie derivative $L_Y g$ of the metric tensor g is everywhere vanishing. Equivalently, Y is a Killing vector field if and only if the stages of all its local flows are isometries, i.e., if the metric tensor g of $\mathcal{M}$ is invariant by the flow of Y. Hence, the Killing vector fields are seen as *infinitesimal isometries* of the manifold $\mathcal{M}$.

In this paper, we will often use the following well known characterization of Killing vector fields (see [14], Proposition 9.25). If $\mathcal{X}(\mathcal{M})$ denotes the Lie algebra of all C^1-vector fields on on $\mathcal{M}$, then $Y \in \mathcal{X}(\mathcal{M})$ is Killing if and only if for every pair $W_1, W_2 \in \mathcal{X}(\mathcal{M})$ it is:

$$\langle \nabla_{W_1} Y, W_2 \rangle = -\langle \nabla_{W_2} Y, W_1 \rangle, \tag{1}$$

where ∇ denotes the covariant derivative associated to the Levi–Civita connection of the metric g. In particular, if $z :]a, b[\longmapsto \mathcal{M}$ is an absolutely continuous curve and Y is Killing, then

$$\langle \dot z, \nabla_{\dot z} Y(z) \rangle \equiv 0, \quad \text{a.e..} \tag{2}$$

We make the following symmetry assumption on our manifold $\mathcal{M}$:

(Hp1) $\mathcal{M}$ admits k distinguished Killing vector fields $Y_1, Y_2, \ldots, Y_k$;

(Hp2) the Y_i's generate a k-dimensional timelike distribution Δ on $\mathcal{M}$, i.e., they are pointwise linearly independent, and the restriction of the metric tensor g on Δ is negative definite;

(Hp3) the Y_i's are pairwise *commuting*, i.e., the Lie brackets $[Y_i, Y_j]$ are everywhere vanishing on $\mathcal{M}$.

Recall that, for $W_1, W_2 \in \mathcal{X}(\mathcal{M})$, the Lie brackets $[W_1, W_2]$ can be written in terms of the covariant derivative as:

$$[W_1, W_2] = \nabla_{W_1} W_2 - \nabla_{W_2} W_1; \tag{3}$$

the equation (3) follows from the vanishing of the torsion of the Levi-Civita connection of g. We also remark here that the hypothesis (Hp3) implies, by Frobenius Theorem, that the distribution Δ is integrable (in the sense of Frobenius), i.e., through every point $p \in \mathcal{M}$ there exists an integral submanifold for Δ (see e.g. [11]).

Remark 1.1. The hypotheses (Hp1), (Hp2) and (Hp3) must be considered as the multi-dimensional counterpart of the stationarity assumption for a Lorentzian manifold used in [10]. Observe indeed that, if $k = 1$, then (Hp1), (Hp2) and (Hp3) reduce to the mere existence of a timelike Killing vector field Y on $\mathcal{M}$.

We now fix two points p and q in $\mathcal{M}$. We denote by $C^1([0,1], \mathcal{M}; p, q)$ the set of curves in $\mathcal{M}$ given by:

$$C^1([0,1], \mathcal{M}; p, q) = \Big\{ z \in C^1([0,1], \mathcal{M}) : z(0) = p,\ z(1) = q \Big\}.$$

The *action functional* $f : C^1([0,1],\mathcal{M};p,q) \longmapsto I\!R$ is given by the integral:

$$f(z) = \frac{1}{2}\int_0^1 \langle \dot z, \dot z\rangle \, \mathrm{d}s.$$

A classical bootstrap argument shows that the stationary points of the functional f in $C^1([0,1],\mathcal{M};p,q)$ are indeed smooth curves that satisfy the differential equation:

$$\nabla_{\dot z}\dot z \equiv 0; \tag{4}$$

the *geodesics* in $\mathcal{M}$ are precisely the curves satisfying the equation (4). Recalling (2), we have that, if Y is Killing, then for every geodesic z in $\mathcal{M}$ the quantity $\langle \dot z, Y(z)\rangle$ is constant:

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle \dot z, Y\rangle = \langle \nabla_{\dot z}\dot z, Y\rangle + \langle \dot z, \nabla_{\dot z} Y\rangle = 0.$$

We express this fact by saying that $\langle \dot z, Y\rangle$ = constant is a *natural* constraint for geodesics. Our variational principle for geodesics is based on this conservation law.

Suppose now that $\mathcal{M}$ satisfies (Hp1); let $Y_1, \ldots, Y_k$ be timelike Killing vector fields on $\mathcal{M}$. We introduce the following space:

$$\mathcal{C}_{p,q} = \Big\{ z \in C^1([0,1],\mathcal{M};p,q) \ : \ \exists\, c_1, c_2, \ldots, c_k \in I\!R \text{ such that } \langle \dot z, Y_i\rangle \equiv c_i, \quad \forall\, i = 1,\ldots,k \Big\} \tag{5}$$

Observe that, by the previous observation, if z is a geodesic joining p and q then $z \in \mathcal{C}_{p,q}$.

We give the following completeness condition:

Definition 1.2. Let c be a real number. The set $\mathcal{C}_{p,q}$ is said to be c-precompact if every sequence $\{z_n\}_{n\in I\!N} \subset \mathcal{C}_{p,q}$ with $f(z_n) \le c$ has a uniformly convergent subsequence in $\mathcal{M}$. We say that the restriction of f to $\mathcal{C}_{p,q}$ is *pseudo–coercive* if $\mathcal{C}_{p,q}$ is c-precompact for all $c \ge \inf\limits_{\mathcal{C}_{p,q}} f$.

The pseudo-coercivity for the restricted action functional replaces the completeness condition used in the case of Riemannian manifolds. We have the following result:

Theorem 1.3. *Suppose that (M, g) satisfies the hypotheses (Hp1), (Hp2) and (Hp3). Let p, q be two points in $\mathcal{M}$ such that $\mathcal{C}_{p,q}$ is non empty, and such that there exists $c > \inf\limits_{\mathcal{C}_{p,q}} f$ for which $\mathcal{C}_{p,q}$ is c-precompact. Then, there exists at least one geodesic in $\mathcal{M}$ joining p and q. In particular, if the last hypothesis holds for each pair p and q, then $\mathcal{M}$ is geodesically connected.*

Observe that it may happen that the set $C_{p,q}$ is empty; for instance, if $\mathcal{M} = \mathbb{R}^2 \setminus \{(0,0)\}$ is endowed with the Minkowski metric $\mathrm{d}x^2 - \mathrm{d}y^2$, $Y_1 = \frac{\partial}{\partial y}$ and $p = (-1,0)$, $q = (1,0)$, then $C_{p,q} = \emptyset$. Clearly, the hypothesis of non emptiness for $C_{p,q}$ is essential; for, if $C_{p,q} = \emptyset$, then obviously there exists no geodesic joining p and q. If the vector fields Y_i are *complete* in $\mathcal{M}$, then for every pair of points p and q, the $C_{p,q}$ is non empty (see Remark 5.3). We recall that a vector field Y on a differentiable manifold is said to be complete if all its integral lines are defined on the entire real line.

If we assume that the semi-Riemannian manifold $\mathcal{M}$ is non contractible, under the extra assumption of completeness for the Killing vector fields Y_i, then we can prove the analogue of Serre's Theorem for Riemannian manifolds (see [20]), that is a multiplicity results for (spacelike) geodesics.

Theorem 1.4. *Assume that (Hp1), (Hp2) and (Hp3) are satisfied. Suppose further that f is pseudo-coercive in $C_{p,q}$. If the vector fields Y_i are complete for $i = 1, \dots, k$ and $\mathcal{M}$ is non contractible, then there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ of geodesics in $\mathcal{M}$ joining p and q such that:*

$$\lim_{n\to\infty} f(z_n) = +\infty.$$

In [18], the authors prove the geodesical completeness for a semi-Riemannian manifold $\mathcal{M}$ satisfying (Hp1), (Hp2), a boundedness condition on the metric coefficients and a suitable Riemannian completeness property for $\mathcal{M}$.

Theorems 1.3 and 1.4 will be proven in the rest of the paper, which is organized as follows. In Section 2 we present our functional framework, we introduce the restricted action functional J and we discuss and prove a variational principle for semi-Riemannian geodesics. In Section 3 we prove that, under the pseudo-coercivity assumption, the functional J is bounded from below; in Section 4 we prove that J satisfies the Palais–Smale compactness condition and we prove Theorem 1.3. Finally, in Section 5 we develop a Ljusternik–Schnirelman Theory for the critical points of J and we prove the multiplicity result of Theorem 1.4. For the reader's convenience, in Appendix A we present some elementary facts about the local metric structure of our semi-Riemannian manifold $\mathcal{M}$, that were used in Section 3, and in Appendix B we discuss a concrete example for our theory.

2. The Functional Setup and the Variational Principle

In this section we fix some notation and we prove some preliminary results concerning the variational structure of our problem. We will assume henceforth that that the hypotheses (Hp1), (Hp2) and (Hp3) are satisfied.

We refer to the textbook [11] for the basic notions of the geometry of infinite dimensional manifolds.

For shortness, for all $p \in \mathcal{M}$ we denote by $\langle \cdot, \cdot \rangle$ the bilinear form induced by $g(p)$ on $T_p\mathcal{M}$; moreover, let Δ_p denote the k-dimensional vector subspace of $T_p\mathcal{M}$ generated by $Y_1(p), Y_2(p), \dots, Y_k(p)$. If $V \subset T_p\mathcal{M}$ is a vector subspace, we denote by $V^\perp = \{w \in T_p\mathcal{M} : \langle w, v \rangle = 0 \text{ for all } v \in V\}$ the orthogonal subspace. A subspace $V \subset T_p\mathcal{M}$ is said to be *nondegenerate* if the restriction of $g(p)$ to V is a nondegenerate bilinear form.

By the hypothesis (Hp2), for all $p \in \mathcal{M}$, the space $\Delta_p \subset T_p\mathcal{M}$ is non degenerate, hence $T_p\mathcal{M} = \Delta_p^\perp \oplus \Delta_p$; then any $\zeta \in T_p\mathcal{M}$ can be decomposed uniquely as $\zeta = \zeta_1 + \zeta_2$, with $\zeta_1 \in \Delta_p^\perp$ and $\zeta_2 \in \Delta_p$. By the *wrong-way Schwartz's inequality*, $\Delta_p^\perp$ is a spacelike subspace of $T_p\mathcal{M}$, i.e., the restriction of $g(p)$ to $\Delta_p^\perp$ is positive definite.

This allows to introduce an auxiliary Riemannian metric $g_{(\mathrm{R})}$ defined on $\mathcal{M}$ as follows

$$g_{(\mathrm{R})}(z)[\zeta, \bar\zeta] = \langle \zeta, \bar\zeta \rangle_{(\mathrm{R})} = \langle \zeta_1, \bar\zeta_1 \rangle - \langle \zeta_2, \bar\zeta_2 \rangle, \tag{6}$$

for every $z \in \mathcal{M}$ and every ζ, $\bar\zeta \in T_z\mathcal{M}$. It is straightforward to see that

$$|\langle \zeta, \bar\zeta \rangle| \le \langle \zeta, \bar\zeta \rangle_{(\mathrm{R})}, \quad \forall \zeta, \bar\zeta \in T_z\mathcal{M}. \tag{7}$$

If p and q are any two fixed points in $\mathcal{M}$ we denote by $\Omega^{1,2}_{p,q}(\mathcal{M})$ the space of $H^{1,2}$-curves in $\mathcal{M}$ joining p and q :

$$\Omega^{1,2}_{p,q} = \Omega^{1,2}_{p,q}(\mathcal{M}) = \Big\{ z\colon [0,1] \to \mathcal{M} \;\Big|\; z \text{ absolutely continuous}, \; z(0) = p, \;\; z(1) = q, \int_0^1 \langle \dot z, \dot z \rangle_{(\mathrm{R})} \, \mathrm{d}s < +\infty \Big\}.$$

It is well known that $\Omega^{1,2}_{p,q}(\mathcal{M})$ has a natural structure of an infinite dimensional Hilbert manifold (see [15]) and for $z \in \Omega^{1,2}_{p,q}(\mathcal{M})$ the tangent space $T_z\Omega^{1,2}_{p,q}$ can be identified with the space of $H^{1,2}_0$-vector fields along z :

$$T_z\Omega^{1,2}_{p,q}(\mathcal{M}) = \Big\{ \zeta \in H^{1,2}([0,1], T\mathcal{M}), \; \zeta(0) = \zeta(1) = 0, \; \zeta \in T_{z(s)}\mathcal{M} \; \forall s \Big\},$$

where

$$H^{1,2}([0,1], T\mathcal{M}) = \Big\{ z\colon [0,1] \to T\mathcal{M} \;\Big|\; \zeta \;\; \text{absolutely continuous}, \; \|\zeta\|_* =< +\infty \Big\},$$

and

$$\|\zeta\|_* = \left(\int_0^1 \langle \nabla^{(\mathrm{R})}_{\dot z} \zeta, \nabla^{(\mathrm{R})}_{\dot z} \zeta \rangle_{(\mathrm{R})} \, \mathrm{d}s \right)^{\frac{1}{2}} \tag{8}$$

Note that $T_z\Omega^{1,2}_{p,q}$ is an Hilbert space with respect to the norm $\|\zeta\|_*$.

Let $L^r([0,1],T\mathcal{M})$, $r \geq 1$, denote the set of all r-integrable vector valued functions from $[0,1]$ to $T\mathcal{M}$, and, for $\zeta \in L^r([0,1],T\mathcal{M})$, we define

$$\|\zeta\|_r = \Big(\int_0^1 (\langle \zeta(s), \zeta(s)\rangle_{(\mathrm{R})})^{\frac{r}{2}} \, \mathrm{d}s\Big)^{\frac{1}{r}}.$$

Similarly, one can define the set $L^\infty([0,1],T\mathcal{M})$ and, for $\zeta \in L^\infty([0,1],T\mathcal{M})$, we set

$$\|\zeta\|_\infty = \operatorname{ess\,sup} \sqrt{\langle \zeta(s), \zeta(s)\rangle_{(\mathrm{R})}}.$$

The functions $\|\cdot\|_r$, $r \in [1,+\infty]$, when restricted to the vector space of continuous vector fields along a fixed curve z, define Banach norms.

The semi-Riemannian *action functional* f on $\Omega^{1,2}_{p,q}(\mathcal{M})$ is defined by:

$$f(z) = \frac{1}{2}\int_0^1 \langle \dot z(s), \dot z(s)\rangle \, \mathrm{d}s; \tag{9}$$

from (7), it follows that the integral (9) is finite for all $z \in \Omega^{1,2}_{p,q}(\mathcal{M})$. The action functional is smooth on $\Omega^{1,2}_{p,q}(\mathcal{M})$, and its differential is given by:

$$f'(z)[\zeta] = \int_0^1 \langle \dot z, \nabla_{\dot z}\zeta\rangle \, \mathrm{d}s, \tag{10}$$

for $\zeta \in T_z\Omega^{1,2}_{p,q}(\mathcal{M})$. Its critical points are smooth curves that satisfy (4), hence they are geodesics.

We denote by $\mathcal{W}$ the smooth distribution on the manifold $\Omega^{1,2}_{p,q}(\mathcal{M})$ consisting of vector fields taking values in Δ:

$$\mathcal{W} = \Big\{(z,\zeta) \in T\Omega^{1,2}_{p,q}(\mathcal{M}) \;\Big|\; \zeta(s) \in \Delta_{z(s)},\ \forall s \in [0,1]\Big\}. \tag{11}$$

Let $\Pi(z,\zeta) = z$ be the projection of $\mathcal{W}$ onto $\Omega^{1,2}_{p,q}(\mathcal{M})$, and for $z \in \Omega^{1,2}_{p,q}(\mathcal{M})$, let $\mathcal{W}_z$ denote the subspace of $T_z\Omega^{1,2}_{p,q}(\mathcal{M})$ given by $\Pi^{-1}(z)$.

We denote by $H^{1,2}_0([0,1],\mathbb{R})$ the Hilbert space of functions $z\colon [0,1] \to \mathbb{R}$ of class $H^{1,2}$ such that $\mu(0) = \mu(1) = 0$.

Observe that a pair $(z,\zeta) \in T\Omega^{1,2}_{p,q}$ belongs to $\mathcal{W}$ if and only if

$$\zeta = \sum_{i=1}^k \mu_i Y_i(z),$$

for some $\mu_i \in H^{1,2}_0([0,1],\mathbb{R})$, $i = 1, \dots, k$.

Finally, we introduce the space $\mathcal{N}_{p,q}(\mathcal{M})$ of curves z in $\Omega^{1,2}_{p,q}(\mathcal{M})$ such that the scalar product $\langle \dot{z}, Y_i\rangle$ is constant for each $i = 1, ..., k$:

$$\mathcal{N}_{p,q} = \mathcal{N}_{p,q}(\mathcal{M}) = \Big\{ z \in \Omega^{1,2}_{p,q}(\mathcal{M}) \quad \Big| \ \langle \dot{z}(s), Y_i(z(s))\rangle \text{ is constant a.e. on } [0,1] \tag{12}$$
$$\text{for each } i = 1, ..., k \Big\}$$

We introduce the space $\mathcal{N}_{p,q}$ because it is the natural space in which it is possible to prove the Palais-Smale compactness condition for the action functional. The details of the proof will be given in section 4.

The space $\mathcal{N}_{p,q}$ can be characterized as the set of the curves $z \in \Omega^{1,2}_{p,q}$ such that the derivative of $f'(z)$ vanishes in the directions of $\mathcal{W}_z$:

Proposition 2.1. *It is*

$$\mathcal{N}_{p,q} = \Big\{ z \in \Omega^{1,2}_{p,q}(\mathcal{M}) \mid f'(z)[\zeta] = 0 \ \ \forall \zeta \in \mathcal{W}_z \Big\}. \tag{13}$$

Proof. Let (z, ζ) be an element of $\mathcal{W}$, then ζ has the form

$$\zeta(s) = \sum_{i=1}^{n} \mu_i Y_i$$

for some $\mu_i \in H^{1,2}_0([0,1], \mathbb{R}),\ i = 1, ..., k.$

Since Y_i is killing, then $\langle \dot{z}, \nabla_{\dot{z}} Y_i\rangle$ vanishes identically on $[0,1]$, hence

$$f'(z)[\mu_i Y_i] \ = \ \int_0^1 \langle \dot{z}, \nabla_{\dot{z}}(\mu_i Y_i)\rangle ds = \int_0^1 (\mu_i \langle \dot{z}, \nabla_{\dot{z}} Y_i(z)\rangle + \mu_i' \langle \dot{z}, Y_i(z)\rangle)\, ds$$
$$= \int_0^1 \mu_i' \langle \dot{z}, Y_i(z)\rangle \, ds. \tag{14}$$

The last integral in (14) vanishes for every $\mu_i \in H^{1,2}_0([0,1], \mathbb{R})$ if and only if $\langle \dot{z}, Y_i(z)\rangle$ is constant a.e., and so we get the claim. □

In order to prove a variational principle for geodesics, we need to show that $\mathcal{N}_{p,q}$ is a regular manifold.

Proposition 2.2. *The set $\mathcal{N}_{p,q}$ is a C^1-submanifold of $\Omega^{1,2}_{p,q}$.*

Proof. Let us consider the following map

$$F : \Omega^{1,2}_{p,q} \longmapsto L^2([0,1], \mathbb{R}^k)$$
$$z \longmapsto (\langle \dot{z}, Y_1(z)\rangle, \langle \dot{z}, Y_2(z)\rangle, ..., \langle \dot{z}, Y_k(z)\rangle) \tag{15}$$

and the closed subspace $\mathcal{K}$ of $L^2([0,1], \mathbb{R}^k)$ made of k–tuples of constant functions from $[0,1]$ to $\mathbb{R}$:

$$\mathcal{K} = \big\{ (c_1, c_2, ..., c_k), \quad c_i \in \mathbb{R},\ i = 1, ..., k \big\}.$$

It is clear that $\mathcal{N}_{p,q} = F^{-1}(\mathcal{K})$, and the map F defined above is of class C^1. The Gateaux derivative of F is easily computed as

$$(16) \qquad F'(z)[\zeta] = (\langle \nabla_{\dot z}\zeta, Y_i(z)\rangle + \langle \dot z, \nabla_\zeta Y_i(z)\rangle)_i, \; i = 1, ..., k,$$

where $z \in \Omega^{1,2}_{p,q}$, $\zeta \in T_z\Omega^{1,2}_{p,q}$ and $\nabla_\zeta Y_i(z)$ is the covariant derivative of Y_i in the direction of the vector field ζ.

Using a generalization of the Implicit Function Theorem (see Proposition 3.II.2 of [11]), in order to prove that $\mathcal{N}_{p,q}$ is a regular submanifold of $\Omega^{1,2}_{p,q}$ it suffices to show that the composite map:

$$(17) \qquad T_z\Omega^{1,2}_{p,q} \xrightarrow{F'(z)} T_{F(z)}L^2([0,1], I\!R^k) \overset{\pi}{\longmapsto} T_{F(z)}L^2([0,1], I\!R^k)/T_{F(z)}\mathcal{K},$$

is surjective, where $T_{F(z)}L^2([0,1], I\!R)/T_{F(z)}\mathcal{K}$ is the quotient Banach space and π is the canonical projection onto the quotient.

This is equivalent to proving that, for every $z \in \mathcal{N}_{p,q}$ and for all $h \in L^2([0,1], I\!R^k)$, $h = (h_1, ..., h_k)$, the equation in ζ:

$$(18) \qquad F'(z)[\zeta] = h + c, \quad c = (c_1, ..., c_k) \in \mathcal{K}$$

can be solved in $T_z\Omega^{1,2}_{p,q}$.

Then, let $z \in \mathcal{N}_{p,q}$ and $h \in L^2([0,1], I\!R^k)$ be fixed and consider the vector field along z

$$\zeta = \sum_{i=1}^k \mu_i(s) Y_i(s);$$

if $\mu_i \in H^{1,2}_0([0,1], I\!R)$ for all i, then $\zeta \in T_z\Omega^{1,2}_{p,q}$.

Substituting such ζ in (18) we obtain a system of k equations of the following kind

$$\langle \nabla_{\dot z}\zeta, Y_i(z)\rangle + \langle \dot z, \nabla_\zeta Y_i(z)\rangle = h_i + c_i.$$

Recalling that, since Y_i is Killing, it is

$$(19) \qquad \langle \dot z, \nabla_{Y_i} Y_i(z)\rangle = -\langle Y_i(z), \nabla_{\dot z} Y_i(z)\rangle$$

if ζ satisfies (19), one has:

$$\begin{aligned}
\langle\nabla_{\dot z}\zeta, Y_i(z)\rangle + \langle\dot z, \nabla_\zeta Y_i(z)\rangle &= \langle\sum_{j=1}^k \mu_j' Y_j + \sum_{j=1}^k \mu_j \nabla_{\dot z} Y_j, Y_i\rangle - \sum_{j=1}^k \mu_j \langle Y_j, \nabla_{\dot z} Y_i\rangle \\
&= \sum_{j=1}^k \mu_j'\langle Y_j, Y_i\rangle + \sum_{j=1}^k \mu_j\Big(\langle\nabla_{\dot z} Y_j, Y_i\rangle - \langle Y_j, \nabla_{\dot z} Y_i\rangle\Big) \\
&= \sum_{j=1}^k \mu_j'\langle Y_j, Y_i\rangle + \sum_{j=1}^k \mu_j\Big(-\langle\nabla_{Y_i} Y_j, \dot z\rangle + \langle\dot z, \nabla_{Y_j} Y_i\rangle\Big) = \qquad (20) \\
&= \sum_{j=1}^k \mu_j'\langle Y_j, Y_i\rangle + \sum_{j=1}^k \mu_j\,\langle[Y_i, Y_j], \dot z\rangle = \\
= \text{(by (Hp3))} \quad &= \sum_{j=1}^k \mu_j'\langle Y_j, Y_i\rangle = h_i + c_i.
\end{aligned}$$

Now, setting

$$\bar\mu = (\mu_1, \mu_2, ..., \mu_k),$$

we consider the resulting equation in the entries of the vector $\bar\mu$:

$$\sum_{j=1}^k \mu_j'\langle Y_j, Y_i\rangle = h_i + c_i. \tag{21}$$

If we set $a_{ij} = \langle Y_i, Y_j\rangle$, by the linear independence assumption on the set of vector fields $\{Y_1, Y_2, ..., Y_k\}$, we have that the symmetric matrix $A = (a_{ij})$ is invertible, and (21) can be rewritten as

$$A\bar\mu' = h + c.$$

We can solve explicitly this equation by setting

$$\bar\mu(s) = \int_0^s A^{-1}(h+c)\,\mathrm{d}r.$$

Clearly, $\bar\mu(0) = 0$ and we only need to find a constant vector c such that

$$\bar\mu(1) = 0 \tag{22}$$

Since

$$\bar\mu(1) = \int_0^1 A^{-1}(h+c)\mathrm{d}s = 0,$$

if we set

$$c = -\Big(\int_0^1 A^{-1}\mathrm{d}s\Big)^{-1}\int_0^1 A^{-1}h\,\mathrm{d}s,$$

then (22) is satisfied and the proof is concluded. □

Remark 2.3. Observe that in the proof of Proposition 2.2 we have used the fact that the integral $\left(\int_0^1 A^{-1}\mathrm{d}s\right)$ gives an invertible matrix. Indeed, by (Hp2), $A = (a_{ij}) = (\langle Y_i, Y_j\rangle)$ is negative definite and so is its inverse matrix A^{-1}. In particular, $\forall s \in [0,1]$ $\int_0^s A(r)^{-1}\mathrm{d}r$ is still negative definite and thus invertible. For, given any $v \in \mathbb{R}^k$, we have

$$0 > \int_0^s \langle A(r)^{-1}v, v\rangle \mathrm{d}r = \langle (\int_0^s A(r)^{-1}\mathrm{d}r)v, v\rangle.$$

The tangent space $T_z\mathcal{N}_{p,q}$ can now be easily characterized by means of the same Implicit Function Theorem as follows:

Corollary 2.4. *If $z \in \mathcal{N}_{p,q}$, the tangent space $T_z\mathcal{N}_{p,q}$ can be identified with the set:*

$$T_z\mathcal{N}_{p,q} = \Big\{\zeta \in T_z\Omega^{1,2}_{p,q} \;\Big|\; \langle \nabla_{\dot z}\zeta, Y_i(z)\rangle + \langle \dot z, \nabla_\zeta Y_i(z)\rangle \text{ is constant a.e. on } [0,1] \text{ for } i = 1,\dots,k\Big\}.$$

In what follows the restriction of the action functional f on $\mathcal{N}_{p,q}$ will be denoted by J :

$$J = f\,|_{\mathcal{N}_{p,q}}.$$

We can now settle our variational principle for geodesics on semi-Riemannian manifolds

Theorem 2.5. *A curve $z \in \Omega^{1,2}_{p,q}$ is a geodesic in $\mathcal{M}$ if and only if $z \in \mathcal{N}_{p,q}$ and z is a critical point for the functional J.*

Proof. If z is a geodesic in $\Omega^{1,2}_{p,q}$, then $\langle \dot z, Y_i\rangle$ $i = 1,\dots,k$ is constant and $z \in \mathcal{N}_{p,q}$. Conversely, if $z \in \mathcal{N}_{p,q}$ is a critical point for the functional J, then $f'(z)$ clearly vanishes on all vectors $\zeta \in T_z\mathcal{N}_{p,q}$ and it vanishes also on all vectors $\zeta \in T_z\Omega^{1,2}_{p,q}$ of the form $\sum_{i=1}^k \mu_i Y_i$, with $\mu_i \in H^{1,2}_0([0,1],\mathbb{R})$.

In order to obtain the thesis, it suffices to show that the space $T_z\Omega^{1,2}_{p,q}$ is the direct sum of the two spaces $\{\sum_{i=1}^k \mu_i Y_i,\ \mu_i \in H^{1,2}_0([0,1],\mathbb{R})\}$ and $T_z\mathcal{N}_{p,q}$, i.e., that every $\zeta \in T_z\Omega^{1,2}_{p,q}$ can be written as

$$\zeta = \sum_{i=1}^k \mu_i Y_i + \tilde\zeta, \tag{23}$$

where $\tilde\zeta \in T_z\mathcal{N}_{p,q}$.

To prove this, we consider an arbitrary $\zeta \in T_z\Omega^{1,2}_{p,q}$ and we search k functions $\mu_i \in H_0^{1,2}([0,1], I\!R)$ such that the vector field $\tilde{\zeta} = \zeta - \sum_{i=1}^{k} \mu_i Y_i$ belongs to $T_z\mathcal{N}_{p,q}$.

By Corollary 2.4, one has to find a constant c_i such that

$$\langle \nabla_{\dot z}\tilde{\zeta}, Y_i(z)\rangle + \langle \dot z, \nabla_{\tilde{\zeta}} Y_i(z)\rangle = c_i,$$

that is

$$\begin{aligned}
\langle \nabla_{\dot z}(\zeta - \sum_{j=1}^{k}\mu_j Y_j), Y_i(z)\rangle \quad - \quad & \langle \zeta - \sum_{j=1}^{k}\mu_j Y_j, \nabla_{\dot z} Y_i(z)\rangle = \\
&= \langle \nabla_{\dot z}\zeta, Y_i\rangle - \sum_{j=1}^{k}\mu_j'\langle Y_j, Y_i\rangle - \sum_{j=1}^{k}\mu_j\langle \nabla_{\dot z}Y_j, Y_i\rangle - \langle \zeta, \nabla_{\dot z}Y_i\rangle + \\
&\quad + \sum_{j=1}^{k}\mu_j\langle Y_j, \nabla_{\dot z}Y_i\rangle = \\
&= \langle \nabla_{\dot z}\zeta, Y_i\rangle - \sum_{j=1}^{k}\mu_j'\langle Y_j, Y_i\rangle + \sum_{j=1}^{k}\mu_j\langle \nabla_{Y_i}Y_j, \dot z\rangle + \\
&\quad - \sum_{j=1}^{k}\mu_j\langle \dot z, \nabla_{Y_j}Y_i\rangle - \langle \zeta, \nabla_{\dot z}Y_i\rangle = \\
&= \langle \nabla_{\dot z}\zeta, Y_i\rangle - \sum_{j=1}^{k}\mu_j'\langle Y_j, Y_i\rangle - \langle \zeta, \nabla_{\dot z}Y_i\rangle + \\
&\quad + \sum_{j=1}^{k}\mu_j\langle [Y_i, Y_j], \dot z\rangle = c_i.
\end{aligned} \tag{24}$$

From (Hp3) we obtain the following system of k differential equations that the vector $\mu = (\mu_i, ..., \mu_k)$ has to satisfy, namely if $A = (a_{ij}) = (\langle Y_i, Y_j\rangle)$, $\mathcal{K} = (c_1, ..., c_k)$ and $L = (l_i) = (\langle \nabla_{\dot z}\zeta, Y_i\rangle - \langle \zeta, \nabla_{\dot z}Y_i\rangle)$ one has to solve

$$A\mu' = L - \mathcal{K}.$$

Since A is invertible, we can set

$$\mu' = A^{-1}(L - \mathcal{K})$$

and then

$$\mu(s) = \int_0^s A^{-1}(L - \mathcal{K})\mathrm{d}r. \tag{25}$$

As $\mu(0) = 0$, in order to get our claim we only have to determine $\mathcal{K}$ in such a way that

$$\mu(1) = \int_0^1 A^{-1}(L - \mathcal{K})\mathrm{d}s = 0.$$

By Remark 2.3, the matrix $\left(\int_0^1 A^{-1}\mathrm{d}s\right)$ is negative definite and then invertible, therefore, in order to finish the proof, we only need to set:

$$\mathcal{K} = \Big(\int_0^1 A^{-1}\mathrm{d}s\Big)^{-1}\Big(\int_0^1 A^{-1}L\,\mathrm{d}s\Big). \tag{26}$$

□

Observe that the curves in $\mathcal{N}_{p,q}$ have less regularity of the curves in $\mathcal{C}_{p,q}$. Using standard arguments in Sobolev spaces, one sees that the set $\mathcal{C}_{p,q}$ is contained as a dense subset of $\mathcal{N}_{p,q}$. Thus, in the statements of Definition 1.2 and Theorems 1.3 and 1.4 we can replace the space $\mathcal{C}_{p,q}$ with $\mathcal{N}_{p,q}$.

3. The Lower Boundedness Condition for the Restricted Action Functional

In this section we show that, if $\mathcal{N}_{p,q}$ is c-precompact for some $c > \inf\limits_{\mathcal{N}_{p,q}} J$, then the restricted action functional J is bounded from below in $\mathcal{N}_{p,q}$.

To this aim, for $z \in \mathcal{N}_{p,q}$, we denote by $C(z) \in \mathbb{R}^k$ the constant vector:

$$C(z) = (\langle \dot z, Y_1\rangle, \langle \dot z, Y_2\rangle, \ldots, \langle \dot z, Y_k\rangle).$$

For $c \in \mathbb{R}$, we also denote by J^c the c-sublevel of J in $\mathcal{N}_{p,q}$:

$$J^c = \Big\{ z \in \mathcal{N}_{p,q} : J(z) \le c \Big\}.$$

The crucial fact for the lower boundedness of the functional J is given by the boundedness of the quantity $\|C(z)\|$, which is proven in the following Lemma:

Lemma 3.1. *Suppose that $\mathcal{N}_{p,q}$ is c-precompact for some $c \in \mathbb{R}$. Then, there exists a positive constant H such that $\|C(z)\| \le H$ for all $z \in J^c$.*

Proof. Let z_n be a sequence in J^c which is maximizing for the quantity $C(z)$, i.e.:

$$\lim_{n\to\infty} \|C(z_n)\| = \sup_{\mathcal{N}_{p,q}} \|C(z)\|.$$

By the c-precompactness, there exists a compact subset K of $\mathcal{M}$ containing the image of all the curves z_n; the set K is covered by a finite number of local charts

$$(U^i, x_1^i, x_2^i, \ldots, x_m^i, t_1^i, t_2^i, \ldots, t_k^i), \quad i = 1, 2, \ldots, N,$$

adapted to the k-tuple $(Y_1, Y_2, \ldots, Y_k)$ (see Appendix A). We recall that this means that, for each $i = 1, 2, \ldots, N$, the sets U^i are open subsets of $\mathcal{M}$ with compact closure in $\mathcal{M}$, and $(x_1^i, \ldots, x_m^i, t_1^i, \ldots, t_k^i) : U^i \longmapsto \mathbb{R}^m \times \mathbb{R}^k$ are coordinate functions on U^i satisfying the following:

(a) $Y_j = \dfrac{\partial}{\partial t_j^i}$ on U^i, for all $j = 1, \ldots, k$;

(b) the metric coefficients $g_{\alpha\beta} = g_{\alpha\beta}(x_1, \ldots, x_m)$ only depend on the first m-variables for all $\alpha, \beta = 1, \ldots, k + m$;

(c) writing the metric tensor g in matrix form, we have:

$$g = \begin{pmatrix} P & D^T \\ D & -Q \end{pmatrix}, \tag{27}$$

where P and Q are (square) positive definite matrices of size $m \times m$ and $k \times k$ respectively. Here, D^T denotes the transpose matrix of D.

Clearly, we can assume that all the coefficients $g_{\alpha\beta}$ are bounded, and so is the operator norm of each of the three matrices P, Q and D.

Moreover, by the uniform convergence of the z_n, we can also assume the following:

(d) there exists a partition of the interval $[0, 1]$, given by a finite sequence $0 = a_0 < a_1 < \ldots < a_{n_0} = 1$, such that $z_n([a_{i-1}, a_i]) \subset U^i$ for all $i = 1, \ldots, n_0$ and for n sufficiently large;

moreover, we will also assume, without loss of generality that:

$$\sup_{\substack{j=1,\ldots,k \\ i=1,\ldots,N \\ r_1, r_2 \in U_i}} |t_j^i(r_1) - t_j^i(r_2)| \leq T_0 < +\infty, \tag{28}$$

and

$$\sup_{\substack{r \in U_i \\ i=1,\ldots,N}} \|D(r)\| \leq D_0 < +\infty, \tag{29}$$

for some positive constant D_0.

For $s \in [a_{i-1}, a_i]$ we write:

$$z_n(s) = (\mathrm{x}_n^i(s), \mathrm{t}_n^i(s)),$$

where

$$\mathrm{x}_n^i(s) = (x_1^i(z_n(s)), \ldots, x_m^i(z_n(s))), \text{ and } \mathrm{t}_n^i(s) = (t_1^i(z_n(s)), \ldots, t_k^i(z_n(s))).$$

If we denote by $(\cdot|\cdot)$ the Euclidean inner product, from (27) we can write:

$$\begin{aligned} 2J(z_n) &= \sum_{i=1}^{n_0} \int_{a_{i-1}}^{a_i} \langle \dot z_n, \dot z_n \rangle \, ds = \\ &= \sum_{i=1}^{n_0} \int_{a_{i-1}}^{a_i} \Big((P\dot{\mathrm{x}}_n^i | \dot{\mathrm{x}}_n^i) + 2(D\dot{\mathrm{x}}_n^i | \dot{\mathrm{t}}_n^i) - (Q\dot{\mathrm{t}}_n^i | \dot{\mathrm{t}}_n^i) \Big) \, ds. \end{aligned} \tag{30}$$

The condition $\langle \dot{z}_n, Y_j \rangle = C_j(z_n)$ gives:

(31) $$C(z_n) = D\dot{x}_n^i - Q\dot{t}_n^i, \quad \forall\, i = 1, \ldots, n_0;$$

and substituting in (30), considering that $Q^{-1} = (Q^{-1})^T$, gives:

(32)
$$2J(z_n) = \sum_{i=1}^{n_0} \int_{a_{i-1}}^{a_i} \Big((P\dot{x}_n^i | \dot{x}_n^i) + (Q^{-1}D\dot{x}_n^i | D\dot{x}_n^i) - (Q^{-1}C(z_n) | C(z_n)) \Big)\, ds.$$

Since $J(z_n)$, P, D and Q^{-1} are bounded, in order to prove that $\|C(z_n)\|$ is bounded, from (32) it suffices to show that:

(33) $$\int_{a_{i-1}}^{a_i} (\dot{x}_n^i | \dot{x}_n^i)\, ds$$

is bounded on n for all $i = 1, \ldots, k$.

From (31) we get:

(34) $$\dot{t}_n^i = Q^{-1}D\dot{x}_n^i - Q^{-1}C(z_n),$$

and, integrating on $[a_{i-a}, a_i]$, we get:

(35) $$\eta_n^i = t_n^i(a_i) - t_n^i(a_{i-1}) = \int_{a_{i-1}}^{a_i} Q^{-1}D\dot{x}_n^i\, ds - \left(\int_{a_{i-1}}^{a_i} Q^{-1}\, ds \right) C(z_n),$$

and so

(36) $$C(z_n) = \left(\int_{a_{i-1}}^{a_i} Q^{-1}\, ds \right)^{-1} \left(\int_{a_{i-1}}^{a_i} Q^{-1}D\dot{x}_n^i\, ds - \eta_n^i \right).$$

Observe indeed that, since Q^{-1} is positive definite, then the integral $\int_{a_{i-1}}^{a_i} Q^{-1}\, ds$ is also positive definite, hence this matrix is invertible.

Recalling that $C(z_n)$ is constant, substituting (36) into (32) gives:

(37)
$$2c \geq J(z_n) = \sum_{i=1}^{n_0} \int_{a_{i-1}}^{a_i} \Big((P\dot{x}_n^i | \dot{x}_n^i) + (Q^{-1}D\dot{x}_n^i | D\dot{x}_n^i) \Big)\, ds +$$
$$-\Big(\int_{a_{i-1}}^{a_i} Q^{-1}D\dot{x}_n^i\, ds - \eta_n^i \;\Big|\; \left(\int_{a_{i-1}}^{a_i} Q^{-1}\, ds \right)^{-1} \left(\int_{a_{i-1}}^{a_i} Q^{-1}D\dot{x}_n^i\, ds - \eta_n^i \right) \Big).$$

From (28) we get that $\|\eta_n^i\|$ is bounded; hence, to prove the boundedness of the integrals (33), we only need to prove that, in (37), the sum of the terms which are quadratic in $\dot{x}_n^i$ is bounded.

To prove this, first of all observe that, given the positivity of P, we have:

$$(38)\qquad \int_{a_{i-1}}^{a_i} (P\dot{x}_n^i|\dot{x}_n^i)\ \mathrm{d}s \geq \nu_0 \int_{a_{i-1}}^{a_i} (\dot{x}_n^i|\dot{x}_n^i)\ \mathrm{d}s, \quad \forall i,\ n,$$

for some positive number ν_0.

We now consider the sum:

$$(39)\qquad \Gamma_n^i = \int_{a_{i-1}}^{a_i} (Q^{-1}D\dot{x}_n^i|D\dot{x}_n^i)\ \mathrm{d}s + \\ -\left(\int_{a_{i-1}}^{a_i} Q^{-1}D\dot{x}_n^i\ \mathrm{d}s \,\Big|\, \left(\int_{a_{i-1}}^{a_i} Q^{-1}\ \mathrm{d}s\right)^{-1} \int_{a_{i-1}}^{a_i} Q^{-1}D\dot{x}_n^i\ \mathrm{d}s \right).$$

If M is a fixed $k \times k$ positive definite constant matrix and $y : [a_{i-1}, a_i] \longmapsto \mathbb{R}^k$ is an L^2-function, then, by *Jensen's inequality*, it is:

$$(40)\qquad \left(\left(\int_{a_{i-1}}^{a_i} M\ \mathrm{d}s\right)^{-1} \int_{a_{i-1}}^{a_i} My\ \mathrm{d}s \,\Big|\, \int_{a_{i-1}}^{a_i} My\ \mathrm{d}s \right) \leq \int_{a_{i-1}}^{a_i} (My|y)\ \mathrm{d}s.$$

We recall that Jensen's inequality states that, if $y : [\alpha, \beta] \longmapsto \mathcal{K}$ is a continuous function with values in the convex subset $\mathcal{K} \subseteq \mathbb{R}^n$ and $\varphi : \mathcal{K} \longmapsto \mathbb{R}$ is a convex function, then

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \varphi(y(s))\ \mathrm{d}s \geq \varphi\left(\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} y(s)\ \mathrm{d}s\right);$$

in our case, to obtain (40), we set $\varphi(y) = (My \mid y)$.

Now, on the interval $[a_{i-1}, a_i]$, we can write:

$$(41)\qquad Q^{-1} = M + B_1,$$

where $M = Q^{-1}(a_{i-1})$ is a fixed positive definite matrix and the operator norm $\|B_1\|$ can be made arbitrarily small by reducing the size of the interval $[a_{i-1}, a_i]$, say:

$$(42)\qquad \|B_1\| \leq \delta_1,$$

for some *small* $\delta_1 > 0$. From (41), we compute as follows:

$$\left(\int_{a_{i-1}}^{a_i} Q^{-1}\,\mathrm{d}s\right)^{-1} = \left(\int_{a_{i-1}}^{a_i} M\,\mathrm{d}s + \int_{a_{i-1}}^{a_i} B_1\,\mathrm{d}s\right)^{-1} =$$

$$(43) \qquad = \left((a_i - a_{i-1})M + \int_{a_{i-1}}^{a_i} B_1\,\mathrm{d}s\right)^{-1} =$$

$$= \frac{1}{a_i - a_{i-1}}\left(M + \frac{1}{a_i - a_{i-1}}\int_{a_{i-1}}^{a_i} B_1\,\mathrm{d}s\right)^{-1} =$$

$$= \frac{1}{a_i - a_{i-1}}\,(M + B_2)^{-1},$$

where

$$B_2 = \frac{1}{a_i - a_{i-1}}\int_{a_{i-1}}^{a_i} B_1\,\mathrm{d}s$$

is such that:

$$\|B_2\| \le \delta_1.$$

Observe that, if δ_1 is small enough, then the matrix $(1+M^{-1}B_2)$ is invertible; hence, it is:

$$\left(\int_{a_{i-1}}^{a_i} Q^{-1}\,\mathrm{d}s\right)^{-1} = \frac{1}{a_i - a_{i-1}}\,(1+M^{-1}B_2)^{-1}M^{-1} =$$

$$= \frac{1}{a_i - a_{i-1}}\,(1+B_3)^{-1}\,M^{-1},$$

where

$$B_3 = M^{-1}B_2$$

satisfies:

$$\|B_3\| \le \delta_2 > 0,$$

and δ_2 ban be made arbitrarily small with δ_1. Moreover, using the Neumann series for the quantity $(1+B_3)^{-1}$, we get:

$$(1+B_3)^{-1} = 1 - B_3(1 - B_3 + B_3^2 - \cdots) = 1 + B_4,$$

where:

$$(44) \qquad \|B_4\| \le \delta_3 > 0,$$

where δ_3 can be made arbitrarily small with δ_1. Then, we can write:

$$\left(\int_{a_{i-1}}^{a_i} Q^{-1}\,\mathrm{d}s\right)^{-1} = \frac{1}{a_i - a_{i-1}}M^{-1} + \frac{1}{a_i - a_{i-1}}B_4 = \tag{45}$$

$$= \left(\int_{a_{i-1}}^{a_i} M\,\mathrm{d}s\right)^{-1} + \frac{1}{a_i - a_{i-1}}B_4.$$

By (41) and (45), since M is constant, we have:

$$\Big(\int_{a_{i-1}}^{a_i} Q^{-1}D\dot{x}_n^i\,\mathrm{d}s \;\Big|\; \left(\int_{a_{i-1}}^{a_i} Q^{-1}\,\mathrm{d}s\right)^{-1}\int_{a_{i-1}}^{a_i} Q^{-1}D\dot{x}_n^i\,\mathrm{d}s\Big) =$$

$$= \Big(\int_{a_{i-1}}^{a_i} MD\dot{x}_n^i\,\mathrm{d}s \;\Big|\; \left(\int_{a_{i-1}}^{a_i} M\,\mathrm{d}s\right)^{-1}\int_{a_{i-1}}^{a_i} (M+B_1)D\dot{x}_n^i\,\mathrm{d}s\Big) +$$

$$+\Big(\int_{a_{i-1}}^{a_i} MD\dot{x}_n^i\,\mathrm{d}s \;\Big|\; \frac{B_4}{a_i - a_{i-1}}\int_{a_{i-1}}^{a_i} (M+B_1)D\dot{x}_n^i\,\mathrm{d}s\Big) + \tag{46}$$

$$+\Big(\int_{a_{i-1}}^{a_i} B_1D\dot{x}_n^i\,\mathrm{d}s \;\Big|\; \left(\int_{a_{i-1}}^{a_i} M\,\mathrm{d}s\right)^{-1}\int_{a_{i-1}}^{a_i} (M+B_1)D\dot{x}_n^i\,\mathrm{d}s\Big) +$$

$$+\Big(\int_{a_{i-1}}^{a_i} B_1D\dot{x}_n^i\,\mathrm{d}s \;\Big|\; \frac{B_4}{a_i - a_{i-1}}\int_{a_{i-1}}^{a_i} (M+B_1)D\dot{x}_n^i\,\mathrm{d}s\Big).$$

Now, by Hölder's inequality, we have:

$$\frac{1}{a_i - a_{i-1}}\cdot\left(\int_{a_i}^{a_{i-1}} (D\dot{x}_n^i \,|\, D\dot{x}_n^i)^{\frac{1}{2}}\,\mathrm{d}s\right)^2 \le \int_{a_{i-1}}^{a_i} (D\dot{x}_n^i \,|\, D\dot{x}_n^i)\,\mathrm{d}s. \tag{47}$$

Then, using the c-precompactness, from (46) we have the existence of a positive constant k_0 such that:

$$\Big(\int_{a_{i-1}}^{a_i} Q^{-1}D\dot{x}_n^i\,\mathrm{d}s \;\Big|\; \left(\int_{a_{i-1}}^{a_i} Q^{-1}\,\mathrm{d}s\right)^{-1}\int_{a_{i-1}}^{a_i} Q^{-1}D\dot{x}_n^i\,\mathrm{d}s\Big) \le$$

$$\le \Big(\int_{a_{i-1}}^{a_i} MD\dot{x}_n^i\,\mathrm{d}s \;\Big|\; \left(\int_{a_{i-1}}^{a_i} M\,\mathrm{d}s\right)^{-1}\int_{a_{i-1}}^{a_i} MD\dot{x}_n^i\,\mathrm{d}s\Big) + \tag{48}$$

$$+\frac{\delta_1+\delta_3}{a_i - a_{i-1}}\cdot k_0\cdot\left(\int_{a_i}^{a_{i-1}} (D\dot{x}_n^i \,|\, D\dot{x}_n^i)^{\frac{1}{2}}\,\mathrm{d}s\right)^2 \le$$

$$\le (\delta_1+\delta_3)\,k_0\int_{a_{i-1}}^{a_i} (D\dot{x}_n^i, D\dot{x}_n^i)\,\mathrm{d}s.$$

If we set

$$\delta = (\delta_1 + \delta_3) \cdot k_0, \tag{49}$$

from (39) and (40) we obtain

$$\Gamma_n^i \geq -\delta \int_{a_{i-1}}^{a_i} (D\dot{x}_n^i \mid D\dot{x}_n^i)\, ds, \tag{50}$$

Finally, from (29), we obtain:

$$\Gamma_n^i \geq -\delta D_0^2 \int_{a_{i-1}}^{a_i} (\dot{x}_n^i \mid \dot{x}_n^i)\, ds. \tag{51}$$

Now, each interval $[a_{i-1}, a_i]$ can be chosen small enough so that the constant δ of (49) satisfies:

$$\delta < \frac{\nu_0}{D_0^2},$$

then, from (38) and (51) we see that the term $\int_{a_{i-1}}^{a_i} (P\dot{x}_n^i | \dot{x}_n^i)\, ds$ dominates Γ_n^i. Hence, the inequality (37) implies that the sum of the integrals $\int_{a_{i-1}}^{a_i} (\dot{x}_n^i | \dot{x}_n^i)\, ds$ is bounded, which concludes the proof. □

Remark 3.2. Observe that, if $\mathcal{N}_{p,q}$ is c-precompact for some $c \in \mathbb{R}$, then, by definition, all the curves $z \in J^c$ have image in a compact subset K of $\mathcal{M}$. Hence, by continuity, there exists positive constants ν_1, ν_2 such that:

$$|\langle Y_i(z(s)), Y_j(z(s))\rangle| \leq \nu_1, \quad i, j = 1, 2, \ldots, k, \tag{52}$$

and, denoting by $A(q)$ the matrix $(a_{ij}(q)) = (\langle Y_i(q), Y_j(q)\rangle)$,

$$|\det(A(z(s)))| \geq \nu_2 > 0, \tag{53}$$

for all $z \in J^c$ and $s \in [0, 1]$.

Proposition 3.3. *If $\mathcal{N}_{p,q}$ is c-precompact for some $c > \inf\limits_{\mathcal{N}_{p,q}} J$, then J is bounded from below in $\mathcal{N}_{p,q}$.*

Proof. Suppose $\mathcal{N}_{p,q}$ c-precompact and let $z \in J^c$ be fixed. For (almost) all $s \in [0, 1]$, we decompose the tangent vector $\dot{z}(s)$ as:

$$\dot{z}(s) = \zeta_1(s) + \zeta_2(s),$$

with $\zeta_1(s) \in \Delta_{z(s)}^{\perp}$ and $\zeta_2(s) \in \Delta_{z(s)}$.

By definition of the Riemannian metric $g_{(\mathrm{R})}$, we have:

$$J(z) = \int_0^1 \langle \dot{z}, \dot{z}\rangle_{(\mathrm{R})}\, ds + 2\int_0^1 \langle \zeta_2(s), \zeta_2(s)\rangle\, ds. \tag{54}$$

Since the first integral in (54) is non negative, it suffices to show the lower boundedness of the second integral in (54). To prove this, we write:

$$\zeta_2(s) = \sum_{i=1}^{k} \lambda_i(s) \cdot Y_i(z(s));$$

and from (52) it is enough to prove that the coefficients $\lambda_i(s)$ are bounded on $[0,1]$. These coefficients are related to the constants $C_j(z) = \langle \dot{z}, Y_j \rangle$ and to the matrix $A = (a_{ij})$ by the system of linear equations:

$$C_j(z) = \sum_{i=1}^{k} \lambda_i(s) \cdot \langle Y_i(z(s)), Y_j(z(s)) \rangle, \quad j = 1, \ldots, k.$$

Thus, the boundedness of the λ_i's follows at once from Lemma 3.1 and from (53). □

4. The Palais–Smale Condition for the Restricted Action Functional

In this section we will prove Theorem 1.3, using standard critical point theory for functionals satisfying the *Palais–Smale* condition. Observe that the necessity to consider Palais–Smale sequences, rather than minimizing sequences for the functional J, is the fact that the constraint $z \in \mathcal{N}_{p,q}$ is not weakly closed in $\Omega^{1,2}_{p,q}$.

We recall that if (X, h) is an Hilbert manifold and $F : X \longmapsto \mathbb{R}$ is a C^1-functional on X, then F is said to satisfy the *Palais–Smale* condition at level $c \in \mathbb{R}$ if every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ satisfying

(PS1)$_c$ $\lim_{n\to\infty} F(x_n) = c$,

(PS2)$_c$ $\lim_{n\to\infty} \|F'(x_n)\| = 0$,

has a subsequence converging in X. The norm considered in (PS2) is the operator norm in the Hilbert space $T_{x_n}X$.

A sequence x_n in X that satisfies (PS1)$_c$ and (PS2)$_c$ will be called a *Palais–Smale* sequence ((PS)$_c$ for short) at level c for the functional F.

Theorem 4.1. *If $\mathcal{N}_{p,q}$ is c–precompact, then J satisfies the Palais–Smale condition at every level $c' < c$.*

Proof. Let $c' < c$ be fixed and z_n a Palais–Smale sequence at the level c'. Arguing as in Lemma 3.1, we obtain a subsequence of z_n, still denoted by z_n, that converges weakly to some $z \in \Omega^{1,2}_{p,q}$. Now we prove that this convergence is strong by using the fact that $J'(z_n)$ is infinitesimal. Let $\zeta_n \in T_{z_n}\Omega^{1,2}_{p,q}$ be any bounded sequence in $H^{1,2}([0,1], T\mathcal{M})$, by (23), we can write

$$\zeta_n = \sum_{i=1}^{k} \mu_i Y_i(z_n) + \tilde{\zeta}_n,$$

where $\mu = (\mu_1, ..., \mu_k)$ is given by (25) and (26), while $\tilde{\zeta}_n \in T_{z_n}\mathcal{N}_{p,q}$.

Since ζ_n is bounded in $H^{1,2}([0,1], T\mathcal{M})$, from (25) and (26), it can be easily seen that also $\tilde{\zeta}_n$ is bounded in $H^{1,2}([0,1], T\mathcal{M})$, hence:

$$\lim_{n\to\infty} J'(z_n)[\tilde{\zeta}_n] = \lim_{n\to\infty} \int_0^1 \langle \dot{z}_n, \nabla_{\dot{z}_n}\tilde{\zeta}_n \rangle \mathrm{d}s = 0. \tag{55}$$

Recalling that $\langle \dot{z}_n, Y_i \rangle$ is constant and that $\langle \dot{z}_n, \nabla_{\dot{z}_n} Y \rangle = 0$, it is

$$\begin{aligned} \int_0^1 \langle \dot{z}_n, \nabla_{\dot{z}_n}(\sum_{i=1}^{k} \mu_i Y_i) \rangle \mathrm{d}s &= \sum_{i=1}^{k} \int_0^1 \mu_i' \langle \dot{z}_n, Y_i \rangle \mathrm{d}s + \\ &+ \sum_{i=1}^{k} \int_0^1 \mu_i \langle \dot{z}_n, \nabla_{\dot{z}_n} Y_i \rangle \mathrm{d}s = 0. \end{aligned} \tag{56}$$

Putting together (55) and (56), we obtain:

$$\lim_{n\to\infty} \int_0^1 \langle \dot{z}_n, \nabla_{\dot{z}_n}\zeta_n \rangle \mathrm{d}s = 0. \tag{57}$$

We need the following technical result:

Lemma 4.2. *In the above notations, there exists a sequence α_n in $T_{z_n}\Omega^{1,2}_{p,q}$ that tends to 0 in $L^2([0,1], T\mathcal{M})$ and such that:*

$$\int_0^1 \langle \dot{z}_n, \nabla_{\dot{z}_n}\zeta_n \rangle \ \mathrm{d}s = \int_0^1 \langle \alpha_n, \nabla_{\dot{z}_n}\zeta_n \rangle \ \mathrm{d}s. \tag{58}$$

Proof. The proof is done for the Lorentzian case in [10]. The case of a semi-Riemannian manifold of arbitrary index is treated analogously. □

We can now consider the sequence of vector fields

$$\omega_n = \dot{z}_n - \alpha_n, \tag{59}$$

from (58) we deduce that ω_n is of class C^1 and that

$$\nabla_{\dot{z}_n}\omega_n = 0. \tag{60}$$

Since $\|\dot{z}_n\|_2$ is bounded and α_n tends to 0 in $L^2([0,1], T\mathcal{M})$, the L^2-norm $\|\omega_n\|_2$ of ω_n is bounded. Then it is possible to find a sequence $\{s_n\} \subset [0,1]$, and a constant

c_0 such that

$$|\omega_n(s_n)| \le c_0, \quad \forall n \in \mathbb{N}. \tag{61}$$

Gronwall's Lemma applied to the differential equation (60) and the boundedness condition (61) gives the existence of $\gamma_0 > 0$ such that:

$$|\omega_n(s_n)| \le c_0 \cdot e^{\gamma_0 \int_0^1 |\dot{z}_n|\, dr}, \quad \forall s \in [0,1].$$

It follows that ω_n is bounded in L^∞.

From (59) it follows that $\dot{z}_n$ is bounded in L^2, and since $z_n(0)$ is fixed the sequence z_n is uniformly bounded.

Writing equation (60) in coordinates, it becomes:

$$\omega_n' + \Gamma(z_n)[\dot{z}_n, \omega_n] = 0, \tag{62}$$

where Γ is a continuous function in z_n (that can be expressed using the Christoffel symbols of g), which is linear in the arguments $\dot{z}_n$ and ω_n. From (62), we obtain that ω_n' is bounded in L^2, and thus ω_n is bounded in $H^{1,2}$.

It follows that a subsequence of ω_n still denoted by ω_n, is weakly convergent in $H^{1,2}$, and, in particular, ω_n is convergent in $L^2([0,1], T\mathcal{M})$.

Therefore, there exists a subsequence of z_n that tends to z strongly in $\Omega^{1,2}_{p,q}$.

By the L^2-convergence, a subsequence of $\langle \dot{z}_n, Y_i \rangle$ converges pointwise to $\langle \dot{z}, Y_i \rangle$ almost everywhere for every $i = 1, \ldots, k$, this implies that $\langle \dot{z}, Y_i \rangle$ is constant a.e., and then that $z \in \mathcal{N}_{p,q}$. □

We prove now the completeness of the c-sublevels of J using the c-precompactness condition:

Proposition 4.3. *Let $c \in \mathbb{R}$ be fixed. If $\mathcal{N}_{p,q}$ is c-precompact, then $J^{c'}$ is a complete metric subspace of $\mathcal{N}_{p,q}$ for all $c' \le c$.*

Proof. It suffices to consider the c-sublevel. Since all the curves in J^c lie in a compact set (see remark 3.2), we can assume that $\mathcal{M}$ is complete with respect to the Riemannian metric $g_{(\mathrm{R})}$. This implies that $\Omega^{1,2}_{p,q}$ is a complete Hilbertian manifold. If z_n is a Cauchy sequence in J^c then z_n converges to some z in $\Omega^{1,2}_{p,q}$ and, up to passing to a subsequence, $\langle \dot{z}_n, Y_i \rangle$ converges pointwise to $\langle \dot{z}, Y_i \rangle$ almost everywhere for every $i = 1, \ldots, k$. Then $\langle \dot{z}_n, Y_i \rangle$ is constant a.e. on $[0,1]$ and $z \in \mathcal{N}_{p,q}$. By the continuity of J, it is $J(z) \le c$ and then J^c is complete. □

We can now prove Theorem 1.3:

Proof of Theorem 1.3. Once the Palais–Smale condition, the completeness of the sublevels of J and the boundedness property J are proved, the claim is an immediate application of the classical deformation Lemmas for Palais–Smale functionals (see [13]). □

Remark 4.4. Note that we need the existence of a minimizing Palais–Smale sequence in order to obtain the existence of a minimal point for J. Indeed, one cannot use any minimizing sequence because our constraint is not closed with respect to the weak convergence.

5. Multiplicity of Geodesics

The goal of this section is to give a proof of Theorem 1.4 by means of the Ljusternik–Schnirelman theory for Palais–Smale functionals.

We recall the following definition:

Definition 5.1. If X is a topological space and B any subset of X, the *Ljusternik–Schnirelman category* $\mathrm{cat}_X(B)$ of B in X is the minimal number (possibly infinite) of closed, contractible subsets of X that cover B.

The Ljusternik–Schnirelman category of B in X is a homotopical invariant, in the sense that $\mathrm{cat}_X(B) = \mathrm{cat}_{\mathcal{F}(X)}(\mathcal{F}(B))$ for every continuous map $\mathcal{F} : X \longmapsto \mathcal{F}(X)$ which is a homotopy equivalence.

A well known result by Fadell and Husseini (see [8]) states that, if $\mathcal{M}$ is non contractible, then the category of the space $\Omega^{1,2}_{p,q}(\mathcal{M})$ is infinite.

We show now that, if the Y_i's are complete, then $\mathcal{N}_{p,q}$ and $\Omega^{1,2}_{p,q}$ have the same homotopy type:

Proposition 5.2. *Suppose that the Killing vector fields Y_i are complete, $i = 1, \dots, k$. Then, there exists a smooth map $\mathcal{F} : \Omega^{1,2}_{p,q} \longmapsto \mathcal{N}_{p,q}$ which is a homotopy equivalence.*

Proof. For all $i = 1, \dots, k$ let $\psi^i : \mathcal{M} \times \mathbb{R} \longmapsto \mathcal{M}$ denote the flow of the vector field Y_i; we define a map $\mathcal{F}_i : \Omega^{1,2}_{p,q} \longmapsto \Omega^{1,2}_{p,q}$ by:

$$\mathcal{F}_i(z)(s) = \psi^i(z(s), \phi_i(s)),$$

where $\phi_i : [0,1] \longmapsto \mathbb{R}$ is a function to be determined. Observe that, in order for $\mathcal{F}_i$ to take values in $\Omega^{1,2}_{p,q}$, the function ϕ_i must be of class $H^{1,2}$ and it must satisfy the boundary conditions:

$$\phi_i(0) = \phi_i(1) = 0. \tag{63}$$

The relation $[Y_i, Y_j] \equiv 0$ implies that the flows $\psi^i(\cdot, t)$ and $\psi^j(\cdot, s)$ commute, and so do the maps $\mathcal{F}_i$ and $\mathcal{F}_j$:

$$\mathcal{F}_i \circ \mathcal{F}_j = \mathcal{F}_j \circ \mathcal{F}_i, \quad \forall\, i, j. \tag{64}$$

Moreover, denoting by $\mathrm{d}_x\psi^i$ the derivative of the flow ψ^i with respect to the first variable (which is an isometry by the Killing property of Y_i), the commuting relation $[Y_i, Y_j] \equiv 0$ yields:

$$\mathrm{d}_x\psi^i(x,t)[Y_j(x)] = Y_j(\psi^i(x,t)). \tag{65}$$

The formulas (64) and (65) are easily proven passing in local coordinates (see Appendix A), as the vector fields Y_i can be taken to be coordinate fields.

Let $\mathcal{F} : \Omega^{1,2}_{p,q} \longmapsto \Omega^{1,2}_{p,q}$ denote the map:

$$\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2 \circ \cdots \mathcal{F}_k;$$

for $z \in \Omega^{1,2}_{p,q}$, let's take $w = \mathcal{F}(z)$. From (64) and (65), we compute easily:

$$\dot{w}(s) = (\mathrm{d}_x\psi^1 \circ \mathrm{d}_x\psi^2 \circ \cdots \circ \mathrm{d}_x\psi^k)[\dot{z}(s)] + \sum_{i=1}^{k} \phi_i'(s) \cdot Y_i(w(s)), \tag{66}$$

and, for all i, using (65) and the isometry property of $\mathrm{d}_x\psi^j$, it is:

$$\langle \dot{w}, Y_i \rangle = \langle \dot{z}, Y_i \rangle + \sum_{j=1}^{k} \langle Y_i, Y_j \rangle \phi_j'.$$

In the notation of Section 3, we denote by $A = (a_{ij})$ the $k \times k$ matrix with coefficients $a_{ij} = \langle Y_i, Y_j \rangle$; moreover, we denote by Φ and Z the column vectors:

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_k \end{pmatrix}, \quad Z = \begin{pmatrix} \langle \dot{z}, Y_1 \rangle \\ \langle \dot{z}, Y_2 \rangle \\ \vdots \\ \langle \dot{z}, Y_k \rangle \end{pmatrix}$$

and by C a generic column vector with constant entries (c_i) to be determined. Using this notation, the conditions $\langle \dot{w}, Y_i \rangle \equiv c_i$ translate into the system of differential equations:

$$A\Phi' = C - Z. \tag{67}$$

Observe that all the solutions Φ of (67) are of class $H^{1,2}$. If we solve (67) with the initial condition $\Phi(0) = 0$, and if we set:

$$C = \left(\int_0^1 A^{-1} \,\mathrm{d}s \right)^{-1} \cdot \int_0^1 A^{-1} Z \,\mathrm{d}s, \tag{68}$$

we obtain $\Phi(1) = 0$, i.e. that the boundary conditions (63) are satisfied, and so $\mathcal{F}$ is a well defined map on $\Omega^{1,2}_{p,q}$ with image in $\mathcal{N}_{p,q}$.

By standard theorems on the regular dependence on the data for ordinary differential equations, it follows easily that the maps ϕ_i depend smoothly on z, hence $\mathcal{F}_i$ is smooth on $\Omega^{1,2}_{p,q}$. Moreover, the map $\mathcal{H}_i : \Omega^{1,2}_{p,q} \times [0,1] \longmapsto \Omega^{1,2}_{p,q}$ given by:

$$\mathcal{H}_i(z,\sigma)(s) = \psi^i(z(s), \sigma \cdot \phi_i(s))$$

is a smooth homotopy between $\mathcal{F}_i$ and the the identity map on $\Omega^{1,2}_{p,q}$, hence each $\mathcal{F}_i$ is a strong deformation retract, and so is $\mathcal{F}$. Observe that, if $z \in \mathcal{N}_{p,q}$, then from (68) it follows $C = Z$, and from (67) we obtain $\Phi \equiv 0$ and $\mathcal{F}(z) = z$, hence $\mathcal{F}$ is the identity on $\mathcal{N}_{p,q}$. □

Remark 5.3. Observe that, from Proposition 5.2 it follows in particular that the spaces $\mathcal{N}_{p,q}$ and $\mathcal{C}_{p,q}$ are non empty.

We now prove the upper unboundedness of the restricted action functional J on $\mathcal{N}_{p,q}$.

Lemma 5.4. *The restricted action functional J is unbounded (from above) in $\mathcal{N}_{p,q}$.*

Proof. Let $z \in \Omega^{1,2}_{p,q}$ be fixed and $w = \mathcal{F}(z)$. In the notation of Proposition 5.2, recalling that the matrix A is symmetric, from (66) we compute directly:

$$\begin{aligned} \langle \dot w, \dot w\rangle - \langle \dot z, \dot z\rangle &= 2\sum_{i=1}^{k} \phi_i'\langle \dot z, Y_i\rangle + \sum_{i,j=1}^{k} \phi_i'\phi_j'\langle Y_i, Y_j\rangle = \\ &= 2(\Phi'|Z) + (A\Phi'|\Phi') = \\ &= (A^{-1}C|C) - (A^{-1}Z|Z), \end{aligned} \tag{69}$$

where $(\cdot|\cdot)$ denotes the Euclidean product in $\mathbb{R}^k$. Substituting (68) in (69) and integrating on $[0,1]$, we obtain:

$$\begin{aligned} 2(J(w) - f(z)) &= \Big(\int_0^1 A^{-1}Z \, ds \,\Big|\, \Big(\int_0^1 A^{-1}\, ds\Big)^{-1} \int_0^1 A^{-1}Z\, ds\Big) + \\ &\quad - \int_0^1 (A^{-1}Z\,|Z)\, ds. \end{aligned} \tag{70}$$

We use the following construction to build a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $\Omega^{1,2}_{p,q}$. Let z be any fixed curve in $\Omega^{1,2}_{p,q}$ and $a, b \in [0,1]$ be close enough, so that the corresponding points $z(a)$ and $z(b)$ lie in an open set U of $\mathcal{M}$ which is the domain of a local coordinate system adapted to the k-tuple $(Y_1,\dots,Y_k)$. We use the same notations adopted in the proof of Lemma 3.1 (see also Appendix A); the coordinate functions will be denoted $(\mathbf{x},\mathbf{t}) : U \longmapsto \mathbb{R}^{m+k}$. We also assume that U has compact closure in $\mathcal{M}$.

Let $\gamma_n = (\mathbf{x}_n, \mathbf{t}_n) : [a,b] \longmapsto U$ be a sequence of smooth curves satisfying the following properties:

(1) $\gamma_n(a) \equiv z(a)$, $\gamma_n(b) \equiv z(b)$ for all n;
(2) $\mathbf{t}_n \equiv \mathbf{t}_*$ is a fixed curve joining the points $\mathbf{t}(z(a))$ and $\mathbf{t}(z(b))$;
(3) $\mathbf{x}_n$ is *bounded* in $H^{1,1}([a,b], \mathbb{R}^m)$;
(4) $\mathbf{x}_n$ is *unbounded* in $H^{1,2}([a,b], \mathbb{R}^m)$.

Finally, we denote by z_n the sequence in $\Omega^{1,2}_{p,q}$ defined by:

$$z_n(s) = z(s) \quad \text{if } s \in [0,a] \cup [b,1] \quad \text{and} \quad z_n(s) = \gamma_n(s) \quad \text{for } s \in]a,b[.$$

By construction, the sequence $\{z_n\}_{n\in\mathbb{N}}$ is made of curves having image in a fixed compact subset of $\mathcal{M}$; moreover it is bounded in $H^{1,1}([0,1], \mathcal{M})$. It follows that the family of functions $\langle \dot{z}_n, Y_i \rangle$ is bounded in $L^1([0,1], \mathbb{R})$; denoting by Z_n the column vector with entries $\langle \dot{z}_n, Y_i \rangle$, we have:

$$\left| \left(\int_0^1 A^{-1} Z_n \, \mathrm{d}s \;\middle|\; \left(\int_0^1 A^{-1} \, \mathrm{d}s \right)^{-1} \int_0^1 A^{-1} Z_n \, \mathrm{d}s \right) \right| \leq a_0 < +\infty, \tag{71}$$

for some $a_0 > 0$.

Moreover, by the properties (2) and (4) of γ_n, it follows easily:

$$\lim_{n\to\infty} f(z_n) = +\infty. \tag{72}$$

Setting $w_n = \mathcal{F}(z_n)$, since A^{-1} is negative definite, formulas (70), (71) and (72) imply immediately that:

$$\lim_{n\to\infty} J(w_n) = +\infty, \tag{73}$$

and we are done. □

The proof of Theorem 1.4 is based on the following result of the classical Ljusternik Schnirelman theory on infinite dimensional manifolds (see e.g. [12, 13]):

Theorem 5.5. *Let M be a Hilbert manifold and $F : M \longmapsto \mathbb{R}$ be a C^1-functional on M. Suppose that the following hypotheses are satisfied:*

(1) F is bounded from below;
(2) F satisfies the Palais–Smale condition at every level $c \geq \inf_M F$;
(3) for all $c \geq \inf_M F$, the sublevel F^c is a complete metric subspace of M.

Then, there exists at least $\mathrm{cat}_M(M)$ critical points of F in M. Moreover, if the category $\mathrm{cat}_M(M) = +\infty$, there exists a sequence x_n of critical points of F in M such that:

$$\lim_{n\to\infty} F(x_n) = \sup_M F.$$ □

Proof of Theorem 1.4. By a well known result of Fadell and Husseini (see [8]), if $\mathcal{M}$ is non contractible, then the category of the space $\Omega^{1,2}_{p,q}(\mathcal{M})$ is infinite. By Proposition 5.2, it is:

$$\mathrm{cat}(\mathcal{N}_{p,q}(\mathcal{M})) = \mathrm{cat}(\Omega^{1,2}_{p,q}(\mathcal{M})) = +\infty.$$

Hence, the proof follows at once from Theorem 5.5, whose hypotheses are proven in Theorem 4.1, Proposition 4.3 and Lemma 5.4. □

Appendix A. About the Local Structure of $(\mathcal{M}, g)$

In this section we describe the local metric structure of a $(m+k)$-dimensional semi-Riemannian manifold $(\mathcal{M}, g)$ satisfying the hypotheses (Hp1), (Hp2) and (Hp3) introduced in Section 1.

Given k non zero vector fields $Y_1, \dots, Y_k$ on $\mathcal{M}$ satisfying $[Y_i, Y_j] \equiv 0$ for all $i, j = 1, \dots, k$, by standard results in Differential Geometry (see e.g. [11]) around every point p_0 of $\mathcal{M}$ there exists a neighborhood U and a coordinate system on U given by functions:

$$(x_1, \dots, x_m, t_1, \dots, t_k) : U \longmapsto \mathbb{R}^{m+k}$$

such that, on U, it is:

$$Y_i = \frac{\partial}{\partial t_i}, \quad i = 1, \dots, k.$$

We can also choose the functions x_j in such a way that the subspace $\Sigma(p_0)$ of $T_{p_0}\mathcal{M}$ generated by the vectors $\frac{\partial}{\partial x_j}|_{p_0}$, is spacelike, i.e., the restriction of the metric tensor g to $\Sigma(p_0)$ is positive definite. Since such condition is open (it is given by the positivity of a finite number of determinants in the coefficients of the metric tensor g), by restricting the neighborhood U, we can assume that the distribution generated by the $\frac{\partial}{\partial x_j}$'s is spacelike on U. We will say that such a coordinate system is *adapted* to the k-tuple $Y_1, \dots, Y_k$.

Using these coordinates, we can therefore write the metric tensor g in matrix form:

$$g = \begin{pmatrix} P & D^T \\ D & -Q \end{pmatrix},$$

where P is a $m \times m$ positive definite matrix, Q is a $k \times k$ positive definite matrix, D is a $k \times m$ matrix and D^T is its transpose.

If $q \in \mathcal{M}$ and $(\xi, \tau) \in \mathbb{R}^m \times \mathbb{R}^k$ is a tangent vector in $T_q\mathcal{M}$, then:

$$g(q)[(\xi,\tau),(\xi,\tau)] = (P(q)\,\xi \mid \xi) + 2(D(q)\,\xi \mid \tau) - (Q(q)\,\tau \mid \tau),$$

where $(\cdot \mid \cdot)$ denotes the Euclidean inner product.

It is an easy observation that the Killing property of the vector fields Y_i, $i = 1, \dots, k$ is expressed by the fact that the metric coefficients g_{ij} of g with respect to any coordinate system adapted to the k-tuple $Y_1, \dots, Y_k$ do not depend on the variables t_i. Namely, denoting by r_i, $i = 1, \dots, m+k$, any coordinate system such that $\frac{\partial}{\partial r_i} = Y_{i-m}$ for $i > m$, and writing $g = (g_{ij})$, $g_{ij} = \langle \frac{\partial}{\partial r_i}, \frac{\partial}{\partial r_j} \rangle$, it is:

$$\frac{\partial g_{ij}}{\partial t_k} = \langle \nabla_{\frac{\partial}{\partial r_i}} Y_k, \frac{\partial}{\partial r_j} \rangle + \langle \nabla_{\frac{\partial}{\partial r_j}} Y_k, \frac{\partial}{\partial r_i} \rangle = 0.$$

Appendix B. An Example

In this appendix we give an example to illustrate our abstract theory.

We consider a manifold $\mathcal{M}$ given by a global splitting $\mathcal{M}_0 \times \mathbb{R}^k$ (or $\mathcal{M}_0 \times \mathbb{T}^k$, with $\mathbb{T}^k = S^1 \times S^1 \times \cdots \times S^1$ the k-dimensional torus) where $\mathcal{M}_0$ is a closed m-dimensional submanifold of the Euclidean space $\mathbb{R}^N$. We denote by $(\cdot|\cdot)$ the Euclidean scalar product. We consider the following semi-Riemannian metric on $\mathcal{M}$.

$$g(\mathbf{x}, \mathbf{t})[(\xi, \tau), (\xi, \tau)] = (P\xi|\xi) + 2(D\xi|\tau) - (Q\tau|\tau), \tag{74}$$

where $\mathbf{x} \in \mathcal{M}_0$, $\mathbf{t} \in \mathbb{R}^k$ (or $\mathbb{T}^k$), $\xi \in T_{\mathbf{x}}\mathcal{M}_0$ and $\tau \in \mathbb{R}^k$.

Here, $P = P(\mathbf{x})$ and $Q(\mathbf{x})$ are (square) positive definite matrices of size $m \times m$ and $k \times k$ respectively, and $D = D(\mathbf{x}) : T_{\mathbf{x}}\mathcal{M}_0 \longmapsto \mathbb{R}^k$ is a matrix operator. We assume that P, Q and D depend smoothly on $\mathbf{x}$; moreover the following boundedness assumptions are made:

$$\sup_{\mathbf{x}\in\mathcal{M}_0} \|Q(\mathbf{x})\| = N < +\infty, \quad \sup_{\mathbf{x}\in\mathcal{M}_0} \|Q^{-1}(\mathbf{x})\| = \nu < +\infty,$$
$$\sup_{\mathbf{x}\in\mathcal{M}_0} \|P^{-1}(\mathbf{x})\| = P_0 < +\infty, \quad \text{and} \quad \sup_{\mathbf{x}\in\mathcal{M}_0} \|D(\mathbf{x})\| = D_0 < +\infty.$$

Finally, we make the assumption that the following inequality be satisfied:

$$\nu^2 N D_0^2 < P_0. \tag{75}$$

Let us consider the timelike Killing vector fields on $\mathcal{M}$ defined by $Y_i = \frac{\partial}{\partial t_i}$, $i = 1, ..., k$. Being coordinate vector fields one has that $[Y_i, Y_j] = 0$ for $i, j = 1, ..., k$. So, the hypotheses (Hp1), (Hp2) and (Hp3) of Theorem 1.3 are satisfied.

As to the c-precompactness hypothesis, if $z_n = (\mathbf{x}_n, \mathbf{t}_n)$ is a sequence in J^c, then, arguing as in the proof of Lemma 3.1 (see formula (37)), we prove easily that:

$$2c \geq J(z_n) = \int_0^1 \Big((P\dot{\mathbf{x}}_n|\dot{\mathbf{x}}_n) + (Q^{-1}D\dot{\mathbf{x}}_n|D\dot{\mathbf{x}}_n) \Big)\, ds +$$
$$- \Big(\int_0^1 Q^{-1}D\dot{\mathbf{x}}_n\, ds - \eta \;\Big|\; \Big(\int_0^1 Q^{-1}\, ds\Big)^{-1} \Big(\int_0^1 Q^{-1}D\dot{\mathbf{x}}_n\, ds - \eta\Big) \Big), \tag{76}$$

where $\eta = \mathbf{t}_n(q) - \mathbf{t}_n(p)$. Using the inequality (75), from (76) and the boundedness of $\|Q^{-1}\|$ we obtain that the integral:

$$\int_0^1 (\,\dot{\mathbf{x}}_n \,|\, \dot{\mathbf{x}}_n\,)\,\mathrm{d}s$$

is bounded. From the completeness of $\mathcal{M}_0$ it follows that $\mathbf{x}_n$ is uniformly convergent to a curve $x \in H^{1,2}([0,1], \mathcal{M}_0)$. Moreover, x_n is bounded in $H^{1,2}([0,1], \mathcal{M}_0)$.

From the equality:

$$\dot{\mathbf{t}}_n = Q^{-1} D \dot{\mathbf{x}}_n - Q^{-1} C(z_n)$$

(see formula (34)) it follows in first place that $C(z_n)$ is bounded (integrating over $[0,1]$), and then that $\mathbf{t}_n$ is bounded in $H^{1,2}([0,1], \mathbb{R}^k)$. Hence, $\mathbf{t}_n$ has a uniformly convergent subsequence, and so does z_n.

Thus, $C_{p,q}$ is c-precompact for all $c \in \mathbb{R}$.

By Theorem 1.3, the manifold $\mathcal{M}$ is geodesically connected; moreover, by Theorem 1.4, if $\mathcal{M}_0$ is not contractible, then, there exist infinitely many geodesics z_n, of arbitrary large $f(z_n)$, joining every pair of points p and q.

References

[1] J. K. Beem, *Conformal Changes and Geodesic Completeness*, Comm. Math. Phys. **49** (1976), 179–186.

[2] J. K. Beem, H. Buseman, *Axiom for Indefinite Metrics*, Cir. Mat. Palermo **15** (1966), 223–246.

[3] J. K. Beem, P. E. Ehrlich, K. L. Easly, *Global Lorentzian Geometry*, Marcel Dekker, Inc., New York and Basel, 1996.

[4] J. K. Beem, P. E. Parker, *Pseudoconvexity and Geodesic Connectedness*, Ann. Mat. Pura e Applicata **155**, No. 4, 1989, 137–142.

[5] V. Benci, *Metodi Variazionali nella Geometria dello Spazio-Tempo*, Boll. Un. Mat. It. (7) 11-A (1997), 297–321.

[6] V. Benci, D. Fortunato, F. Giannoni, *On the Existence of Infinitely Many Geodesics on Space-Time Manifolds*, Ann. Inst. Henri Poincaré, Analyse Nonlinéaire, **8** (1991), 79–102.

[7] R. Bott, *Lectures on Morse Theory old and new*, Bull. Amer. Math. Soc. 7 (1982), 331–358.

[8] E. Fadell, S. Husseini, *Category of Loop Spaces of Open Subsets in Euclidean Spaces*, Nonlinear Analysis: T.M.A. 17 (1991), 1153–1161.

[9] F. Giannoni, A. Masiello, *On the Existence of Geodesics on Stationary Lorentz Manifolds with Convex Boundary*, J. Funct. Anal. **101**, No. 2, 1991.

[10] F. Giannoni, P. Piccione, *An Intrinsic Approach to the Geodesical Connectedness for Stationary Lorentzian Manifolds*, to appear on Comm. Anal. Geom., 1997.

[11] S. Lang, *Differential Manifolds*, Springer-Verlag, Berlin, 1985.

[12] A. Masiello, *Variational Methods in Lorentzian Geometry*, Pitman Research Notes in Mathematics Series 59, 1994.

[13] J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer–Verlag, Berlin, 1989.

[14] B. O'Neill, *Semi–Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[15] R. Palais, *Morse Theory on Hilbert Manifolds*, Topology **2** (1963), 299–340.

[16] P. Piccione, R. Sampalmieri, *Geodesical Connectedness of Compact Lorentzian Manifolds*, Dynam. Sys. Appl. 5 (1996), 479–502.

[17] A. Romero, M. Sánchez, *On Completeness of Certain Families of Semi-Riemannian Manifolds*, Geometriae Dedicata 53 (1994), 103–117.

[18] A. Romero, M. Sánchez, *New Properties and Examples of Incomplete Lorentzian Tori*, J. Math. Phys. 34, 4 (1994), 1992–1997.

[19] A. Romero, M. Sánchez, *Completeness of Compact Lorentz Manifolds Admitting a Timelike Conformal Killing Vector Field*, Proc. AMS 123, 9 (1995), 2831–2833.

[20] J. P. Serre, *Homologie singuliere des espaces fibres*, Ann. Math. 54 (1951), 425–505.

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