

The anti-Ramsey threshold of complete graphs [☆]

Yoshiharu Kohayakawa ^a, Guilherme Oliveira Mota ^{a,*}, Olaf Parczyk ^b,
Jakob Schnitzer ^c

^a Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brazil

^b Fachbereich Mathematik und Informatik, Freie Universität Berlin, Berlin, Germany

^c Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany

ARTICLE INFO

Article history:

Received 15 September 2020

Received in revised form 3 January 2023

Accepted 17 January 2023

Available online 1 February 2023

Keywords:

Anti-Ramsey property

Proper colourings

Random graphs

Thresholds

ABSTRACT

For graphs G and H , let $G \xrightarrow{\text{rb}} H$ denote the property that, for every proper edge-colouring of G , there is a rainbow H in G . For every graph H , the threshold function $p_H^{\text{rb}} = p_H^{\text{rb}}(n)$ of this property in the random graph $G(n, p)$ satisfies $p_H^{\text{rb}} = O(n^{-1/m^{(2)}(H)})$, where $m^{(2)}(H)$ denotes the so-called maximum 2-density of H . Completing a result of Nenadov, Person, Škorić, and Steger [J. Combin. Theory Ser. B **124** (2017), 1–38], we prove a matching lower bound for $p_{K_k}^{\text{rb}}$ for $k \geq 5$. Furthermore, we show that $p_{K_4}^{\text{rb}} = n^{-7/15} \ll n^{-1/m^{(2)}(K_4)}$.

© 2023 Elsevier B.V. All rights reserved.

1. Introduction

As usual, we write $G(n, p)$ for the binomial random graph and, given a monotone increasing property \mathcal{P} of graphs, we call a function $\hat{p} = \hat{p}(n)$ a *threshold function* for \mathcal{P} if

$$\lim_{n \rightarrow \infty} \mathbb{P}(G(n, p) \in \mathcal{P}) = \begin{cases} 0 & \text{if } p \ll \hat{p} \\ 1 & \text{if } p \gg \hat{p}. \end{cases} \quad (1)$$

In (1) and in what follows, $f \ll g$ stands for $\lim_n f(n)/g(n) = 0$ and $f \gg g$ stands for $g \ll f$. Following standard practice, we refer to any \hat{p} as in (1) as *the* threshold function for \mathcal{P} even though threshold functions are not uniquely defined. When $\lim_n \mathbb{P}(G(n, p) \in \mathcal{P}) = 1$, we say that $G(n, p)$ has \mathcal{P} with *high probability*. For notation and terminology not explicitly defined here, see [4,6].

In a major breakthrough in the area of random graphs, Rödl and Ruciński [10,11] established the threshold function for the property $G(n, p) \rightarrow (H)_r$ for any given graph H , that is, the property that, for every edge-colouring of $G(n, p)$ with at most r colours, there exists a monochromatic copy of H in $G(n, p)$. Their result is in fact more refined, but it implies that,

[☆] The first author was partially supported by CNPq (311412/2018-1, 423833/2018-9, 406248/2021-4) and FAPESP (2018/04876-1, 2019/13364-7). The second author was partially supported by FAPESP (2018/04876-1, 2019/13364-7) and CNPq (304733/2017-2, 428385/2018-4). The third author was supported by DFG grant PE 2299/1-1. The cooperation of the authors was supported by a joint CAPES/DAAD PROBRAL project (Proj. 430/15, 57350402, 57391197). This research was partially supported by CAPES (Finance Code 001). FAPESP is the São Paulo Research Foundation. CNPq is the National Council for Scientific and Technological Development of Brazil.

* Corresponding author.

E-mail addresses: yoshi@ime.usp.br (Y. Kohayakawa), mota@ime.usp.br (G.O. Mota), parczyk@mi.fu-berlin.de (O. Parczyk), jakob.schnitzer@uni-hamburg.de (J. Schnitzer).

as long as H is not a star-forest, the threshold is $n^{-1/m^{(2)}(H)}$, where $m^{(2)}(H)$ stands for the *maximum 2-density* of H , given by

$$m^{(2)}(H) = \max \left\{ \frac{|E(J)| - 1}{|V(J)| - 2} : J \subseteq H, |V(J)| \geq 3 \right\} \quad (2)$$

(for simplicity, we suppose $|V(H)| \geq 3$ in (2)). Since this result was obtained, this line of research developed into a very rich area. We refer the reader to [6, Chapter 8] and Conlon [5] for an overview of this line of research.

In this paper we study a variation of the classical Ramsey property described above. Given graphs G and H , we denote by $G \xrightarrow{\text{rb}} H$ the so-called *anti-Ramsey* property of G with respect to H : for every *proper* edge-colouring of G there exists a *rainbow* copy of H in G , i.e., a copy of H where no two edges are assigned the same colour. Note that the property $G \xrightarrow{\text{rb}} H$ is monotone in G for every fixed graph H , and hence it admits a threshold [2], which we denote by p_H^{rb} .

The study of anti-Ramsey properties of random graphs was initiated by Rödl and Tuza [12], who proved that for every integer $\ell \geq 4$ we have $G(n, p) \xrightarrow{\text{rb}} C_\ell$ with high probability if $p \geq n^{-(\ell-2)/(\ell-1)} = n^{-1/m^{(2)}(C_\ell)}$. The general case was studied in [7], where the following result was proved.

Theorem 1. *Let H be a graph. Then there exists a constant $C > 0$ such that for $p = p(n) \geq Cn^{-1/m^{(2)}(H)}$ we have $G(n, p) \xrightarrow{\text{rb}} H$ with high probability.*

Note that Theorem 1 implies that $p_H^{\text{rb}} \leq n^{-1/m^{(2)}(H)}$. However, in contrast to the Rödl-Ruciński case, it was established in [8] that there are infinitely many “non-trivial” graphs H for which $p_H^{\text{rb}} \ll n^{-1/m^{(2)}(H)}$, highlighting the interest on the problem of establishing *lower bounds* for p_H^{rb} .

Nenadov, Person, Škorić and Steger [9] developed a general framework for proving lower bounds for various Ramsey type problems within random settings. Applying this framework to the anti-Ramsey problem, they proved that the upper bound given in Theorem 1 is sharp for sufficiently large cycles and complete graphs.

Theorem 2 (Nenadov, Person, Škorić and Steger [9]). *Let H be a cycle on at least 7 vertices or a complete graph on at least 19 vertices. Then there exists a constant $c > 0$ such that for $p = p(n) \leq cn^{-1/m^{(2)}(H)}$ we have $G(n, p) \not\xrightarrow{\text{rb}} H$ with high probability.*

In [1], Theorem 2 was extended to every cycle, as it was proved that $p_{C_\ell}^{\text{rb}} = n^{-1/m^{(2)}(C_\ell)}$ when $\ell \geq 5$ and that $p_{C_4}^{\text{rb}} = n^{-3/4} \ll n^{-1/m^{(2)}(C_4)}$ (when $\ell = 3$, it is easily seen that $p_{C_3}^{\text{rb}} = n^{-1}$). In this paper we prove that Theorem 2 can be extended to complete graphs of order $k \geq 5$ ($p_{K_k}^{\text{rb}} = n^{-2/(k+1)} = n^{-1/m^{(2)}(K_k)}$) and that $p_{K_4}^{\text{rb}} = n^{-7/15} \ll n^{-2/5} = n^{-1/m^{(2)}(K_4)}$.

In the framework developed in [9], the probabilistic problem of showing that with high probability the anti-Ramsey property does not hold for $G(n, p)$ when $p \ll n^{-1/m^{(2)}(H)}$ is reduced to a certain deterministic problem involving graphs with bounded “density”. We will rely on this reduction to prove the following theorem.

Theorem 3. *For every $k \geq 5$, there exists a constant $c > 0$ such that for $p = p(n) \leq cn^{-1/m^{(2)}(K_k)}$ we have $G(n, p) \not\xrightarrow{\text{rb}} K_k$ with high probability. Furthermore, $p_{K_4}^{\text{rb}} = n^{-7/15} \ll n^{-1/m^{(2)}(K_4)}$.*

Note that Theorems 1 and 3 together determine $p_{K_k}^{\text{rb}}$ for every $k \geq 4$ (note that $p_{K_3}^{\text{rb}} = p_{C_3}^{\text{rb}} = n^{-1}$). Before moving on, we remark that, using ideas similar to those used to determine the threshold for K_4 in Theorem 3, one can show that $p_{K_4^-}^{\text{rb}} = n^{-2/3} \ll n^{-1/2} = n^{-1/m^{(2)}(K_4^-)}$, where K_4^- denotes the graph obtained from K_4 by removing an edge.

1.1. Reduction to a deterministic problem

In view of Theorem 1, to obtain the threshold $p_{K_k}^{\text{rb}}$ we must prove that for $p \ll n^{-1/m^{(2)}(K_k)}$ with high probability there exists an edge colouring of $G \sim G(n, p)$ with no rainbow copies of K_k .

As mentioned before, the framework of Nenadov, Person, Škorić, and Steger [9] reduces random Ramsey lower bound problems to certain deterministic problems for graphs with bounded “density”, where by “density” we mean the so called maximum density. For a graph H , the *maximum density* $m(H)$ of H is given by

$$m(H) = \max \left\{ \frac{|E(J)|}{|V(J)|} : J \subseteq H, |V(J)| \geq 1 \right\}$$

and we remark that the threshold for the appearance of a graph H in $G(n, p)$ is given by $n^{-1/m(H)}$ (see Theorem 15), which is a trivial lower bound for p_H^{rb} and for the threshold for the property $G(n, p) \rightarrow (H)_r$.

Roughly speaking, to obtain a proper edge-colouring of $G \sim G(n, p)$ with no rainbow K_k , the procedure in [9] is as follows: for every pair of disjoint edges that lie in precisely the same K_k 's, give a new colour (the same colour to both edges). Then give new colours to edges not contained in K_k 's. Let \hat{G} be the graph obtained by removing all edges coloured by this procedure. In [9],¹ the authors prove that, if $p \ll n^{-1/m^{(2)}(K_k)}$, then with high probability \hat{G} is composed of a family \mathcal{F} of disjoint subgraphs such that we have $m(F) < m^{(2)}(K_k)$ for every $F \in \mathcal{F}$. Therefore, to find the desired colouring of $G(n, p)$, it suffices to find such a colouring for every graph F with $m(F) < m^{(2)}(K_k)$. In view of the above discussion, the part of Theorem 2 concerning complete graphs follows from the following result.

Lemma 4 (Nenadov, Person, Škorić and Steger [9]). *Let H be a complete graph on at least 19 vertices. Then for any graph G with $m(G) < m^{(2)}(H)$ we have $G \xrightarrow{\text{rb}} H$.*

The bound of 19 in Theorem 2 is imposed only by their Lemma 4, and hence to extend Theorem 2 to complete graphs with at least 5 vertices, we give a proof of Lemma 4 that works for all K_k with $k \geq 5$. Since the proof of Lemma 4 in [9] does not extend to smaller complete graphs, we employ a new strategy. Our proof will involve separating G with $m(G) < m^{(2)}(K_k)$ into “chains” of K_k 's and carefully constructing an edge colouring that avoids rainbow copies of K_k .

Given a graph G , we write $|G|$ for the number of vertices and $e(G)$ for the number of edges of G . Section 2 contains the proof of Theorem 3 for complete graphs with at least 5 vertices. We remark that our proof works uniformly for all complete graphs on at least 5 vertices, i.e., not only for complete graphs on fewer than 19 vertices. The proof of Theorem 3 for K_4 is given in Section 3.

2. Proof of Theorem 3 for $k \geq 5$

In this section we prove the following lemma, which gives a lower bound for p_H^{rb} when H is a complete graph with at least five vertices. Together with Theorem 1, this implies that $p_{K_k}^{\text{rb}} = n^{-1/m^{(2)}(H)}$.

Lemma 5. *Let H be a complete graph on at least 5 vertices. If $p = p(n) \ll n^{-1/m^{(2)}(H)}$, then $G(n, p) \xrightarrow{\text{rb}} K_k$ with high probability.*

As discussed in the previous section, we accomplish this by proving the following lemma.

Lemma 6. *Let H be a complete graph on at least five vertices, then for any G with $m(G) < m^{(2)}(H)$ we have $G \xrightarrow{\text{rb}} H$.*

In the remainder of this section we prove Lemma 6. In what follows we outline the ideas of our proof, analysing the structure of some subgraphs that will be important in our proof strategy (see Proposition 8 and Definition 9). We finish by proving an inductive result (Lemma 10) that directly implies Lemma 6.

In what follows, let $k \geq 5$ and let G be a connected graph with $m(G) < m^{(2)}(K_k) = (k+1)/2$. Since we are interested in obtaining a colouring such that every copy of K_k is non-rainbow, we may assume that all vertices and edges of G are contained in a copy of K_k . In this direction, we say that two copies of K_k are K_k -connected if they are connected in the auxiliary graph that has every copy of K_k as vertices and edge-intersecting copies of K_k as edges. Furthermore, a subgraph of G is a K_k -component if any edge and vertex is contained in a copy of K_k and any pair of copies of K_k is K_k -connected. Clearly, we may assume that G contains only a single K_k -component, as K_k -components are edge disjoint and the colourings of all its K_k -components induce a colouring of G .

Let v be a vertex of minimum degree. A simple but important observation is that since $m(G) < (k+1)/2$, the average degree in G is less than $k+1$. Thus, v has degree at most k . The following induced subgraphs of G on v and some of its neighbours play a special role in our proof:

- $K(v)$: induced subgraph of G on $\{v\} \cup N(v)$;
- $R(v)$: induced subgraph of G on $\{v\} \cup \{w \in N(v) : \text{every copy of } K_k \text{ containing } w \text{ also contains } v\}$;
- $S(v)$: induced subgraph of G on $V(K(v)) \setminus V(R(v))$.

Furthermore, we define the following graphs:

- G_v^* : the induced graph on the vertices $V(G) \setminus V(R(v))$;
- G_v : the graph obtained from G_v^* by removing all edges not contained in a copy of K_k in G_v^* .

¹ In fact they prove a more general and stronger result.

In the inductive colouring strategy for Lemma 10, the induction step will assume a colouring of G_v that does not contain a rainbow copy of K_k and produce a colouring of G with the same property. The following simple fact provides useful information about the structure of G_v .

Fact 7. Let $k \geq 5$ and let G be a graph on at least $k+1$ vertices with $m(G) < (k+1)/2$ such that all vertices and edges of G are contained in a copy of K_k . Let v be a vertex of minimum degree in G . Then the following hold:

- (i) If $|G_v| \leq k$ then G_v is isomorphic to K_k ;
- (ii) $|R(v)| \leq k-1$.

Proof. First suppose that $|R(v)| = k+1$. Thus, since $d(v) \leq k$ (recall that v is a vertex of minimum degree), we know that v has exactly k neighbours. A clique K_k on $N(v)$ would contradict the definition of $R(v)$ so there is a non-edge in $R(v)$, say between vertices u and w . Since w has degree at least k , there is an edge $\{w, z\}$ between w and a vertex z outside of $N(v)$. However, $\{w, z\}$ is also contained in a K_k , so w cannot be in $R(v)$, a contradiction, so $|R(v)| \leq k$.

For item (i), it is enough to show that any vertex that is contained in G_v is contained in a K_k . Since $|G| \geq k+1$ and at most k vertices are removed, at least one vertex is left. This vertex is contained in a copy of K_k that v is not contained in and, therefore, no vertex of this copy is in $R(v)$. If G_v contains at most k vertices, then it is actually isomorphic to K_k .

For item (ii), suppose for a contradiction that $|R(v)| = k$. Then, since $d(v) \leq k$, no vertex in $R(v)$ has neighbours outside of $R(v)$. If $d(v) = k-1$, then G is a K_k , a contradiction with the fact that $|G| \geq k+1$. If $d(v) = k$, then since all vertices in $R(v)$ have degree at least k , the neighbourhood of v induces a K_k , contradicting the assumption that $|R(v)| = k$. Therefore, $|R(v)| \leq k-1$. \square

Note that since $d(v) \leq k$ and all vertices and edges are in a copy of K_k , the subgraph $K(v)$ is isomorphic to either a K_k , K_{k+1}^- , or K_{k+1} . Indeed, if $|K(v)| = k+1$ and two edges uw and wz are missing, then the edge vw would not be contained in a copy of K_k , and if two parallel edges are missing, then v would not be contained in a copy of K_k at all. In the following proposition we categorise $K(v)$ according to its structure. For brevity, in what follows, we will abuse notation and write $K(v) = K_k$ or say that $K(v)$ is a K_k , while we mean that $K(v)$ is isomorphic to K_k .

Proposition 8. Let $k \geq 5$ and let G be a connected graph on at least $k+1$ vertices with $m(G) < (k+1)/2$ such that all vertices and edges of G are contained in a single K_k -component. Let v be a vertex of minimum degree in G . Then the subgraph $K(v)$ of G has one of the configurations X_ℓ , Y_ℓ or U_1 , defined as follows:

- X_ℓ : $K(v) = K_k$, and $R(v) = K_\ell$, and $S(v) = K_{k-\ell}$ ($1 \leq \ell \leq k-2$);
- Y_ℓ : $K(v) = K_{k+1}^-$, and $R(v) = K_\ell$, and $S(v) = K_{k-\ell+1}^-$ ($1 \leq \ell \leq k-2$);
- U_1 : $K(v) = K_{k+1}$, and $R(v) = K_1$, and $S(v) = K_k$.

Proof. Clearly, if $K(v) = K_{k+1}$, then $R(v) = K_1$ and $S(v) = K_k$, which gives us the configuration U_1 . So it remains to discuss the cases $K(v) = K_k$ and $K(v) = K_{k+1}^-$.

First, we will show that since G is a single K_k -component on at least $k+1$ vertices, we have $|R(v)| \leq k-2$. In fact, from 7(ii) we already know that $|R(v)| \leq k-1$. Suppose that $|R(v)| = k-1$. In this case, $K(v)$ cannot be a K_{k+1} , so $K(v)$ is either a K_k or a K_{k+1}^- . If $K(v) = K_k$, then any edge incident to vertices of $R(v)$ is in $K(v)$. Then, $K(v)$ is not K_k -connected with the other copies of K_k in G , which implies that G is not a single K_k -component, a contradiction. If $K(v) = K_{k+1}^-$, then $|S(v)| = 2$. Moreover, $S(v)$ is an edge as otherwise G would not be a single K_k -component. But then, there is a missing edge xy with $x \in R(v)$ and $y \in S(v)$, as otherwise there will be a copy of K_k containing the vertices of $R(v)$ and not containing v . But this implies $d(x) < d(v)$, a contradiction. Therefore, we conclude that $|R(v)| \leq k-2$.

Since $|R(v)| \leq k-2$, note that if $K(v) = K_k$, then $R(v) = K_\ell$, and $S(v) = K_{k-\ell}$ for some $1 \leq \ell \leq k-2$, which is the configuration X_ℓ .

It is left to show that if $K(v) = K_{k+1}^-$ and $R(v)$ has ℓ vertices (for any $1 \leq \ell \leq k-2$), then $R(v) = K_\ell$, and $S(v) = K_{k-\ell+1}^-$. Suppose that $R(v)$ is not a K_ℓ . Then, since there is only one missing edge in $K(v)$, we have $R(v) = K_\ell^-$, from where we conclude that there is a vertex in $R(v)$ with degree smaller than $d(v)$, a contradiction. Then, $R(v) = K_\ell$. Now, we just note that if $S(v)$ is not a $K_{k-\ell+1}^-$, then the missing edge xy of $K(v)$ is such that $x \in R(v)$ and $y \in S(v)$, which implies $d(x) < d(v)$, a contradiction, which concludes the proof. \square

In Fig. 1 we show all possible structures for $K(v)$ when $k=5$. In our proof we will use the fact that $m(G) < (k+1)/2$ to bound the number of occurrences of the configurations X_ℓ , Y_ℓ , and U_1 as $K(v)$ in the induction.

Using the characterisation given in Proposition 8, the number of vertices removed from G to obtain G_v is given in the subscripts of X_ℓ , Y_ℓ and U_1 , and the difference in the number of edges between G_v^* and G is as follows.

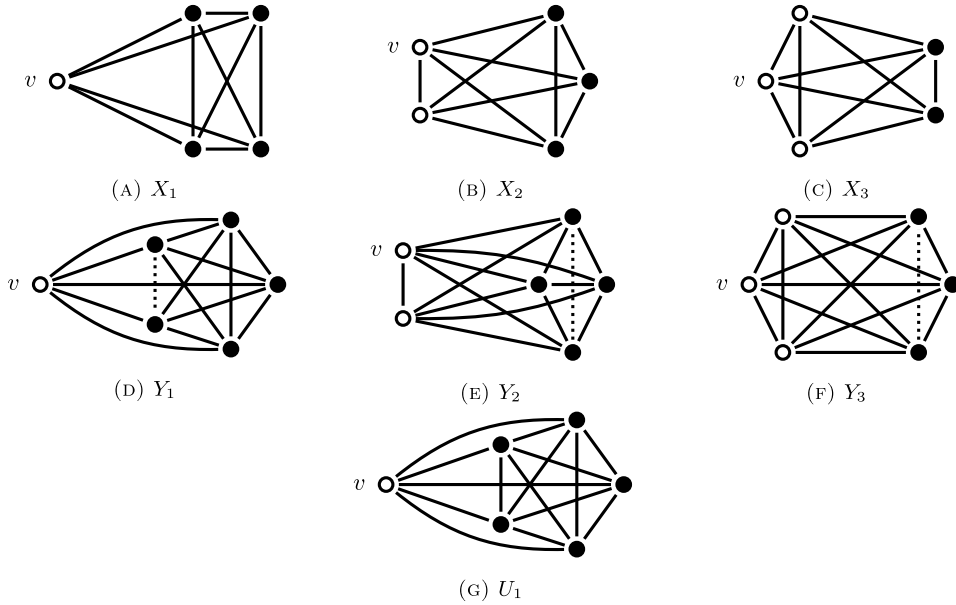


Fig. 1. Possible configurations of $K(v)$ for $k=5$. Dotted lines represent non-edges, the vertices of $R(v)$ are white and the vertices of $S(v)$ are black.

$$e(G) - e(G_v^*) = \begin{cases} k & \text{if } K(v) \text{ is } U_1, \\ \binom{\ell}{2} + \ell(k - \ell) & \text{if } K(v) \text{ is } X_\ell, \\ \binom{\ell}{2} + \ell(k - \ell + 1) & \text{if } K(v) \text{ is } Y_\ell. \end{cases} \quad (3)$$

We will use the following measure of how close G is to the allowed upper bound $(k+1)/2$ on the density $m(G)$. Set

$$b(G) := 2e(G) - (k+1)|G| + 2k.$$

The term $2k$ in $b(G)$ is chosen so that $b(K_k) = 0$. Moreover, from $e(G)/|G| \leq m(G) < (k+1)/2$, we know that

$$b(G) < 2k. \quad (4)$$

Using (3) we get

$$b(G) - b(G_v^*) = \begin{cases} k-1 & \text{if } K(v) \text{ is } U_1, \\ (k-\ell-2)\ell & \text{if } K(v) \text{ is } X_\ell, \text{ for } (1 \leq \ell \leq k-2), \\ (k-\ell)\ell & \text{if } K(v) \text{ is } Y_\ell, \text{ for } (1 \leq \ell \leq k-2). \end{cases} \quad (5)$$

Note that there can be an arbitrary number of X_{k-2} 's in G (they contribute 0 to $b(G)$), but because of the upper bound in (4) we know that all other types of $K(v)$ are limited to a small number of occurrences. Since $(k-\ell-2)\ell \geq k-3$ for $1 \leq \ell \leq k-3$, and $(k-\ell)\ell \geq k-1$ for $1 \leq \ell \leq k-2$, the following follows directly from (5).

$$b(G) \geq \begin{cases} b(G_v^*) + k-1 & \text{if } K(v) \text{ is } U_1 \text{ or } K(v) \text{ is } Y_\ell, \text{ for } (1 \leq \ell \leq k-2), \\ b(G_v^*) + k-3 & \text{if } K(v) \text{ is } X_\ell, \text{ for } (1 \leq \ell \leq k-2). \end{cases} \quad (6)$$

We will describe an inductive colouring strategy, which will always lead to an edge-colouring of G with no rainbow copy of K_k . To keep track of some additional properties of the colouring that will help us during the induction, we introduce five stages P_0, \dots, P_4 , which guarantee the existence of a partial colouring of G with some useful properties.

Definition 9 (Stages). Let $0 \leq j \leq 4$. We say that G is in stage P_j or $G \in P_j$ if there exists a partial proper colouring of G such that the following properties hold.

- (i) Any copy of K_k in G is non-rainbow;
- (ii) If $G \in P_0$ then each colour is used exactly twice, in any copy of K_k there are exactly two coloured edges, and any 4 vertices span at most 3 coloured edges; also, any two copies of K_k intersect in at most one edge;
- (iii) If $G \in P_j$ ($1 \leq j \leq 3$) then any 4 vertices span at most $j+2$ coloured edges;

Property (i) is the main property of the colouring we want to ensure. Properties (ii) and (iii) will allow us to keep the induction proof for Lemma 10 going. Note that, for $0 \leq i \leq 3$, if $G \in P_i$ then $G \in P_{i+1}$. We will inductively extend a partial colouring of G_v^* to a partial colouring of G . To allow such induction, we will prove that if G_v^* is not in some stage P_i , then $b(G_v^*)$ is already “large”, which implies that some configurations are forbidden for $K(v)$, as otherwise $b(G)$ would be too large (recall that $b(G) < 2k$).

Lemma 6 follows trivially from Definition 9(i) and Lemma 10 below.

Lemma 10. *Let $k \geq 5$ and let G be a connected graph on at least k vertices with $m(G) < (k+1)/2$ such that all vertices and edges of G are contained in a single K_k -component. There exists $0 \leq j \leq 4$ and a proper partial edge-colouring of G such that G is in stage P_j under this colouring. Furthermore,*

- (a) $b(G) \geq 0$;
- (b) If $G \notin P_0$, then $b(G) \geq k-3$;
- (c) If $G \notin P_1$, then $b(G) \geq k-1$;
- (d) If $G \notin P_2$, then $b(G) \geq k+1$;
- (e) If $G \notin P_3$, then $b(G) \geq 2k-2$.

Proof. We prove the lemma by induction on the number of vertices of the graph G . If $|G| = k$ then Fact 7 implies that G is a K_k and then we can colour two non-adjacent edges of G with the same colour, from where we conclude that G is in P_0 . Also, $b(K_k) = 0$, so the lemma holds.

Now consider a graph G on at least $k+1$ vertices satisfying the assumptions of the lemma. Depending on $b(G)$, we have to show that G is in a certain stage. Let v be a vertex of minimum degree in G . Fact 7(i) implies that G_v has at least k vertices. We will first handle the case that G_v is a single K_k -component (Claim 11). The case where there are multiple K_k -components will be considered in Claim 12.

Claim 11. *If G_v contains a single K_k -component, then Lemma 10 holds.*

Proof of Claim 11. By the inductive hypothesis, the lemma holds for G_v . Let j_v be the smaller index such that $G_v \in P_{j_v}$ and note that, for $1 \leq j_v \leq 4$, if $G_v \in P_{j_v}$ then $G_v^* \in P_{j_v}$. For $j_v = 0$, it could be that $G_v \in P_0$ does not imply $G_v^* \in P_0$. In fact, if $e(G_v^* \setminus G_v) > 0$, then it could be that the edges in $G_v^* \setminus G_v$ form a K_4 in G_v^* with 3 coloured edges, because 4 vertices of G_v could span 3 coloured edges. Then, property P_0 would not hold for G_v^* . But it is not hard to check that $G_v^* \in P_1$. In view of this, we define

$$j_v^* = \begin{cases} j_v & \text{if } 1 \leq j_v \leq 4, \\ 1 & \text{if } j_v = 0. \end{cases} \quad (7)$$

For $1 \leq j_v \leq 4$, as no edge in $E(G_v^*) \setminus E(G_v)$ is contained in a copy of K_k , the partial edge-colouring that guarantees that $G_v \in P_{j_v}$ ensures that $G_v^* \in P_{j_v^*}$. Thus, one can view j_v^* as the smaller index such that one can always ensure that $G_v^* \in P_{j_v^*}$.

We will prove that G is in some of the stages described in Definition 9. More precisely, we will prove that

- if $K(v) = X_{k-2}$, then G is in the same stage as G_v^* ,
 - if $K(v) = X_\ell$ ($1 \leq \ell \leq k-3$), then we advance at most one stage from G_v^* to G ,
 - if $K(v) = Y_\ell$ ($1 \leq \ell \leq k-2$), then we advance at most two stages from G_v^* to G ,
 - if $K(v) = U_1$, then we advance at most two stages from G_v^* to G .
- (8)

Note that, in the statement of the lemma, the difference in the bound on $b(G)$ between two consecutive stages P_i and P_{i+1} is at most $k-3$ and between two stages P_i and P_{i+2} it is at most $k-1$. From (6) and (8) it is not hard to check that items (a)–(e) in the statement of the lemma hold.

Note that the only case that we are sure there will be no change in the stage from G_v^* to G is when $K(v) = X_{k-2}$ (recall that it contributes zero to $b(G)$). By the difference $b(G) - b(G_v^*)$ described in (5) and the fact that $b(G) < 2k$, the statement of the lemma applied on G_v^* implies that

- if $K(v) = X_\ell$ ($1 \leq \ell \leq k-3$), then $b(G_v^*) \leq 2k-2$, which implies $G_v^* \in P_3$, so $j_v^* \leq 3$,
 - if $K(v) = Y_\ell$ ($1 \leq \ell \leq k-2$), then $b(G_v^*) \leq k+1$, which implies $G_v^* \in P_2$, so $j_v^* \leq 2$,
 - if $K(v) = U_1$, then $b(G_v^*) \leq k+1$, which implies $G_v^* \in P_2$, so $j_v^* \leq 2$.
- (9)

We will now give the desired partial edge-colouring that extends the edge-colouring of G_v^* to G and advances the stages in the promised way. We split our proof into a few cases depending on the structure of $K(v)$.

Case $K(v) = X_{k-2}$.

Since $K(v) = X_{k-2}$, the graph G_v^* intersects $K(v)$ in exactly one edge. We colour two disjoint edges, one contained in $R(v)$ and the other with one endpoint in $R(v)$, with a new colour. These two edges do not close a coloured triangle and clearly all sets of four vertices and copies of K_k in $K(v)$ contain at most two coloured edges. Also any four vertices containing one of the newly coloured edges can contain at most three coloured edges, so if $G_v^* \in P_0$ then Property 9(ii) holds for G , which implies that $G \in P_0$. By the last part of this argument, Property 9(iii) holds in G if it did in G_v^* , so G is in $P_{j_v^*}$.

Case $K(v) = X_\ell$ for $2 \leq \ell \leq k-3$.

By (9), we have $j_v^* \leq 3$. We extend the current colouring in the following way: if there is any coloured edge in $S(v)$, then we give this colour to one of the edges in $R(v)$, which contains an edge since $\ell \geq 2$. Otherwise we choose a new colour and colour two disjoint edges that both intersect $R(v)$ with this colour. In the first case it is trivial that G is in $P_{j_v^*+1}$ and in the second it is easy to see that G is in $P_{j_v^*+1}$ as the only set of four vertices that contains the two new coloured edges has no other coloured edge.

Case $K(v) = X_1$.

By (9), we have $j_v^* \leq 3$. First suppose that $G_v^* \in P_0$. If $S(v)$ already contains two coloured edges with the same colour, then we are done. So assume this is not the case. Since in P_0 any two copies of K_k in G_v^* intersect in at most one edge, any K_{k-1} must be contained in a copy of K_k . If $S(v)$ is a K_k , then there is a coloured edge in $S(v)$, say e , and its colour is used exactly twice. Then, there is an edge incident to v that we can colour with the colour of the edge e . Note that any four vertices containing this newly coloured edge can contain at most three coloured edges, from where we conclude that G is in P_1 . However, if $S(v)$ is not a K_k and no edge of $S(v)$ is coloured then $e(G_v^* \setminus G_v) > 0$ and it is enough to colour any edge of $S(v)$ and another edge incident to v with a new colour and note that G is in P_2 .

Now suppose that $1 \leq j_v^* \leq 3$. Then, G_v^* is in P_1 , P_2 or P_3 . We choose an uncoloured edge e in $S(v)$ and an edge e' that is incident to v and disjoint from e . We colour e and e' with the same new colour. On any four vertices not containing v we increase the number of coloured edges by at most one, and any four vertices containing v have at most four coloured edges. Therefore, $G \in P_{j_v^*+1}$.

In the remaining cases ($K(v) = Y_\ell$ or $K(v) = U_1$), we always have $j_v^* \leq 2$ and we will show that we advance at most two stages. Also note that unless G_v^* is in stage P_0 , we are allowed to colour two or three disjoint edges (in what follows we will use this fact repeatedly): on any four vertices the number of coloured edges can increase by at most two, which is fine with Property 9(iii) as j_v^* increases by two. In case that G_v^* is in P_0 we will separately verify that Property 9(iii) holds for $j = 2$ in G , i.e., that any four vertices contain at most three edges.

Case $K(v) = Y_\ell$ for $1 \leq \ell \leq k-2$.

By (9), we have $j_v^* \leq 2$. We have to deal with the two copies of K_k contained in $K(v)$. If $K(v) = Y_\ell$ for $2 \leq \ell \leq k-2$, then there exist two disjoint edges e and e' incident to vertices of $R(v)$ that are contained in both copies of K_k . Thus, we just give the same new colour to e and e' , concluding that $G \in P_{j_v^*+2}$.

If $K(v) = Y_1$, then $S(v) = K_k^-$. Now the two copies of K_k contained in $K(v)$ intersect in a K_{k-2} . If G_v^* is in stage P_0 , then this K_{k-2} contains a triangle with vertices $\{x, y, z\}$ and hence an uncoloured edge, say xy . We colour xy and the edge vz with the same new colour, which ensures that both copies of K_k contained in $K(v)$ are non-rainbow. Now any four vertices that contain v contain at most three coloured edges; also, for any four vertices that do not contain v we only added one coloured edge so the number of coloured edges might have increased from three to four, so G is in P_2 . Now suppose that $G_v^* \in P_1$. Thus, no matter how the coloured edges are distributed, only using Property 9(iii), we can always find three disjoint uncoloured edges in $K(v)$ such that each of the copies of K_k in $K(v)$ contains two of them. Then, we colour these three disjoint edges with the same new colour. Finally, suppose that $G_v^* \in P_2$. It follows from Property 9(iii) that there are two uncoloured edges e and e' not incident to v (but not necessarily disjoint) that belong to the two copies of K_k contained in $K(v)$. For both copies of K_k we can choose an additional edge incident to v (say, edges f and f') such that colouring the edges e and f with the same new colour, and e' and f' with the same new colour (different from the colour given to e and f) makes both copies of K_k non-rainbow. Recall that the only property we have to ensure to show that G is in P_4 is that every copy of K_k is non-rainbow. Therefore, $G \in P_4$, which completes the case that $K(v) = Y_1$.

Case $K(v) = U_1$.

By (9), we have $j_v^* \leq 2$. It is easy to show that if any four vertices contain at most four coloured edges then five or more vertices contain two disjoint uncoloured edges. Recall that for $K(v) = U_1$, the graph $S(v)$ is a K_k obtained by removing v from $K(v)$. Then, Property 9(iii) implies that there are two disjoint uncoloured edges e and e' in $S(v)$, which together with an edge f incident to v that is disjoint from e and e' , form a set of three disjoint uncoloured edges. We colour e , e' and f with the same new colour. Note that if $G_v^* \in P_0$, all four-sets of vertices that contain two of the new coloured edges (e , e' and f) either contain v or is a K_4 's in G_v^* , so they contain at most four coloured edges in G , which implies that $G \in P_2$. If G_v^* is in P_1 or in P_2 , then we observe as before that $G \in P_{j_v^*+2}$. \square

It remains to prove that Lemma 10 holds when G_v has multiple K_k -components.

Claim 12. *If G_v contains more than one K_k -component, then Lemma 10 holds.*

Proof of Claim 12. Since G_v contains more than one K_k -component, removing $R(v)$ from G splits into edge-disjoint K_k -components G_1, \dots, G_m for $m \geq 2$. In this situation there is an extra complication, which is the fact that the colouring we give to the edges of $S(v)$ must be consistent with the colouring of the graphs G_1, \dots, G_m , which all contain some edges of $S(v)$. On the other hand, large intersections of some G_i with $S(v)$ contribute a lot to $b(G)$, which we will take advantage of.

Note that since G_v contains more than one K_k -component, $K(v)$ is neither X_{k-2} nor U_1 , so we only have to deal with the other configurations. We apply the induction hypothesis to each one of these K_k -components and, without loss of generality, we may assume that the components are vertex-disjoint in $G_v \setminus K(v)$: intersecting in vertices would yield a denser graph and since all K_k -components can use different colours, combining the partial colourings would still yield a proper colouring at the vertices in which they intersect.

Recall that $b(G) = 2e(G) - (k+1)|G| + 2k$. Let $G_{i,S(v)}$ be the subgraph of G induced by the vertices that are in $S(v) \cap V(G_i)$. Note that since $G_{i,S(v)}$ either has less than k vertices or is a K_k^- , we have $b(G_{i,S(v)}) \leq 0$. As any component G_i intersects $K(v)$ in at least one edge we get the following lower bound on $b(G)$.

$$\begin{aligned} b(G) &\geq b(K(v)) + \sum_{i=1}^m (b(G_i) - b(G_{i,S(v)})) \\ &= b(K(v)) + \sum_{i=1}^m (b(G_i) + |b(G_{i,S(v)})|). \end{aligned}$$

We will use this bound on $b(G)$ to limit the contributions of each G_i to $b(G)$. The following observations are helpful. If $G_{i,S(v)}$ consists of a single edge, then by definition $|b(G_{i,S(v)})| = 0$. Moreover, one can check that

$$\text{if } G_{i,S(v)} \text{ contains more than one edge, then } |b(G_{i,S(v)})| \geq k - 3. \quad (10)$$

If one of the K_k -components, say G_1 , is in P_0 , we will use the induction hypothesis in a slightly stronger version. Note that for G_1 in P_0 , if we repeatedly remove graphs $K(w)$ for minimum degree vertices w of G_1 , then we know that all such $K(w)$'s are X_{k-2} , as otherwise there would be two copies of K_k sharing more than one edge, which contradicts the fact that G_1 is in stage P_0 . Then, by the colouring procedure we described before for extending G_v^* to G in case $K(v) = X_{k-2}$, we may always pick any edge $e \in G_1$ and ensure that e is uncoloured and any K_3 containing e also contains another uncoloured edge. Thus for all components G_i that are in P_0 and intersect $S(v)$ in a single edge e , we know how to give a partial colouring of G_i that respects Definition 9 and ensure that e is uncoloured. Alternatively, we can also guarantee that a given edge e is coloured.

Furthermore, the stronger induction hypothesis also applies to the case when G_v contains more than one K_k -component and $b(G) \leq k - 4$. In general if $b(G) \leq k - 4$, then any two copies of K_k intersect in at most a single edge and there is no cycle chain of copies of K_k . This implies that there is another choice of v such that G_v only contains a single K_k -component and the above argument gives the desired statement.

Before we take care of the copies of K_k that are contained in G but not in G_v (i.e., the copies of K_k contained in $K(v)$) we deal with the combination of the colourings of the G_i in $S(v)$. For that, since there is no copy of K_k in $S(v)$, we only have to check conditions (ii) and (iii) of Definition 9.

In view of items (a)–(e) of the statement of the lemma, we define j_{\min} as follows, where we use $b_{\text{sum}}(G)$ for $\sum_{i=1}^m (b(G_i) + |b(G_{i,S(v)})|)$.

$$j_{\min} = \begin{cases} 0 & \text{if } 0 \leq b_{\text{sum}}(G) < k - 3, \\ 1 & \text{if } k - 3 \leq b_{\text{sum}}(G) < k - 1, \\ 2 & \text{if } k - 1 \leq b_{\text{sum}}(G) < k, \\ 3 & \text{if } k \leq b_{\text{sum}}(G) < 2k - 2. \end{cases} \quad (11)$$

We will show that $S(v) \in P_{j_{\min}}$ (Sub-Claim 13). Then, we deal with the copies of K_k in $K(v)$ to prove that in fact we have $G \in P_{j_{\min}}$ (Sub-Claim 14). In view of (11) it is clear that Sub-Claim 14 implies the statement of Claim 12.

Sub-Claim 13. *The graph $S(v)$ is in $P_{j_{\min}}$.*

Proof of Sub-Claim 13. If $j_{\min} = 0$, then $b_{\text{sum}}(G) < k - 3$, which from (10) implies that all K_k -components are in P_0 . Therefore,

there are no coloured edges within $S(v)$, (12)

which trivially implies $S(v) \in P_0$.

For $j_{\min} \in \{1, 2, 3\}$, we only need to show that Property (iii) of Definition 9 holds in $S(v)$, which says that any 4 vertices span at most $j_{\min} + 2$ coloured edges. We now argue that in $S(v)$ we can not have too many coloured edges, as any coloured edge in $K(v)$ belongs to a K_k -component. In fact, from the induction hypothesis,

if $G_i \notin P_0$, then $b(G_i) \geq k - 3$, (13)

and in case $G_i \in P_0$, the graph $G_{i,S(v)}$ contains more than one edge if one is coloured. Then, from (10), we know that

if $G_i \in P_0$, then $|b(G_{i,S(v)})| \geq k - 3$, (14)

In conclusion, every K_k -component G_i that shares a coloured edge with $S(v)$ contributes at least $k - 3$ to $b_{\text{sum}}(G)$.

If $j_{\min} = 1$ then, in view of (11),

at most one of the graphs G_i contributes with coloured edges to $S(v)$. (15)

In fact, if a K_k -component G_i contributes with coloured edges to $S(v)$, then G_i is not in P_0 (because of the stronger induction hypothesis). But then, in case there are at least two K_k -components that contribute with coloured edges to $S(v)$, we know from (10) that $b_{\text{sum}}(G) \geq 2(k - 3) \geq k - 1$, a contradiction with (11). Therefore, since $b_{\text{sum}}(G) \leq k - 1$, the induction hypothesis implies that $G_i \in P_1$ and we are done.

If $j_{\min} = 2$, then we have to argue that there can not be more than 4 coloured edges spanned by 4 vertices in $S(v)$. Thus, suppose for a contradiction that $S(v)$ contains a set S_4 of 4 vertices that spans 5 coloured edges. Since all G_i 's are in P_2 , which implies that any 4 vertices span at most 4 coloured edges (see Definition 9), if there is only one G_i which contributes with coloured edges to $S(v)$, then we are done. Thus we may assume there are at least two G_i 's contributing with coloured edges to $S(v)$. But note that

there are at most two G_i 's with coloured edges in $G_{i,S(v)}$ and they are in P_1 , (16)

as otherwise we would have $b_{\text{sum}}(G) \geq k$ (from (c), (13) and (14)), which contradicts (11). Then, for any 4 vertices in $S(v)$, each $G_{i,S(v)}$ contributes with at most 3 coloured edges. Suppose now that G_1 and G_2 are in P_0 . Since there is no fully coloured triangle in a single G_i , there has to be one G_i that contributes with a coloured tree on 4 vertices in S_4 . Therefore,

$$\begin{aligned} b_{\text{sum}}(G) &\geq |b(G_{1,S(v)})| + |b(G_{2,S(v)})| \\ &\geq 4(k + 1) + 3(k + 1) - 2 \cdot 6 - 4k \\ &> k - 1, \end{aligned} \quad (17)$$

a contradiction with (11). On the other hand if one of the G_i 's, say G_1 , is not in P_0 (but G_1 is in P_1), then there might be a coloured triangle, in which case we guarantee only three vertices in each of $G_{1,S(v)}$ and $G_{2,S(v)}$, but we still get a contradiction using (14).

$$\begin{aligned} b_{\text{sum}}(G) &\geq b(G_1) + |b(G_{1,S(v)})| + |b(G_{2,S(v)})| \\ &\geq (k - 3) + 6(k + 1) - 2 \cdot 6 - 4k \\ &> k - 1, \end{aligned} \quad (18)$$

a contradiction with (11).

Finally, suppose $j_{\min} = 3$, which implies from (11) that $b_{\text{sum}}(G) < 2k - 2$. We aim to show that in $S(v)$ any 4 vertices span at most 5 coloured edges. Similar as before suppose for a contradiction that $S(v)$ contains a set S_4 of 4 vertices that spans 6 coloured edges, i.e., it is completely coloured. Then, these coloured edges can not be from a single $G_{i,S(v)}$, as $G_i \in P_3$. It is easy to check that there are at most three G_i 's. If there are exactly three of them, then they are all in P_1 , as otherwise we would have from (c), (13) and (14) that $b_{\text{sum}}(G) \geq 2(k - 3) + (k - 1) \geq 2k - 2$, a contradiction with (11). If G_1 , G_2 and G_3 are in P_0 , then

$$\begin{aligned} b_{\text{sum}}(G) &\geq |b(G_{1,S(v)})| + |b(G_{2,S(v)})| + |b(G_{3,S(v)})| \\ &\geq 9(k + 1) - 6k - 2 \cdot 6 \\ &\geq 2k - 2, \end{aligned}$$

a contradiction with (11). If one of them is not in P_0 , similar as in case $j_{\min} = 2$ we get $b_{\text{sum}}(G) \geq (k - 3) + 8(k + 1) - 6k - 12 \geq 2k - 2$, a contradiction.

So we may assume there are only two K_k -components, say G_1 and G_2 . It is easy to check that they all are in P_2 as otherwise we would have a contradiction with (11). If G_1 and G_2 are in P_0 we get $b_{\text{sum}}(G) \geq 4(k+1) + 3(k+1) - 2 \cdot 6 - 4k \geq 2k - 2$, similar as we did in (17). So, we may assume w.l.o.g. that G_1 is not in P_0 . If G_1 is in P_1 , then different than in case $j_{\min} = 2$, even if there is a coloured triangle we guarantee that one of G_1 and G_2 contributes with a coloured tree on 4 vertices in S_4 . This is because S_4 is fully coloured and each of them has at most 3 coloured edges in S_4 (they are in P_1). Then, we get

$$\begin{aligned} b_{\text{sum}}(G) &\geq b(G_1) + |b(G_{1,S(v)})| + |b(G_{2,S(v)})| \\ &\geq (k-3) + 7(k+1) - 2 \cdot 6 - 4k \\ &\geq 2k - 2, \end{aligned}$$

a contradiction. So, we may assume w.l.o.g. that G_1 is in P_2 . Then,

$$\begin{aligned} b_{\text{sum}}(G) &\geq b(G_1) + |b(G_{1,S(v)})| + |b(G_{2,S(v)})| \\ &\geq (k-1) + 6(k+1) - 2 \cdot 6 - 4k \\ &\geq 2k - 2, \end{aligned}$$

which is again a contradiction, which concludes the proof that $S(v)$ is in $P_{j_{\min}}$. \square

It is left to prove that $G \in P_{j_{\min}}$.

Sub-Claim 14. *The graph G is in $P_{j_{\min}}$.*

Proof of Sub-Claim 14. Since Sub-Claim 13 is already proved, it remains to deal with the copies of K_k contained in $K(v)$. As in the case where G_v is only a single K_k -component (Claim 11), we split the proof depending on the structure of $K(v)$. Recall that since G_v contains more than one K_k -component, $K(v)$ is neither X_{k-2} nor U_1 .

Case $K(v) = X_\ell$ for $2 \leq \ell \leq k-3$.

We proceed exactly like in the proof of Claim 11, which means that we colour an edge within $R(v)$ if there is a coloured edge in $S(v)$ or we colour two parallel edges otherwise.

Case $K(v) = Y_\ell$ for $2 \leq \ell \leq k-2$.

In this case we also proceed as in the proof of Claim 11. We can pick two disjoint edges incident to $R(v)$ that are contained in both copies of K_k in $K(v)$ and give a new colour to both of them.

Case $K(v) = X_1$.

The case $j_{\min} = 0$ was already covered earlier by the stronger induction hypothesis, because then $b(G) \leq k-4$ and $G \in P_0$. If $j_{\min} = 1$ then either only one G_i intersects $S(v)$ in more than a single edge and all are in P_0 or all but one G_i are in P_0 and each G_i intersects $S(v)$ only in a single edge. Let G_1 be the special G_i in both cases. In the first case we use the stronger induction hypothesis to ensure that there is a coloured edge of G_1 in $S(v)$. Then there is an edge incident to v that is not incident to G_1 , which we can give the same colour. In the latter case we colour two disjoint edges not incident to G_1 and get $G \in P_1$ as only in G_1 there could be a coloured triangle.

If $j_{\min} = 2$ then there can be either one K_k -component which contributes more than $k-3$ to $b(G)$ or two K_k -components that contribute with at most $k-3$ to $b(G)$. First, consider that there are K_k -components G_1 and G_2 that contribute with at most $k-3$ each to $b(G)$. If there are at most 3 coloured edges on each set of 4 vertices in $S(v)$, then we can proceed as in Claim 11. So, suppose that any set of 4 vertices in $S(v)$ contains 4 coloured edges. As in (16), there are at most two K_k -components G_1 and G_2 that contribute with coloured edges to $S(v)$ and they are in P_1 , which implies that in any set of 4 vertices S of $S(v)$, each of G_1 and G_2 contains only 3 coloured edges. Therefore, we can use one of the colours in S to colour an uncoloured edge of S keeping the colouring proper. Now suppose, that there is only one K_k -component G_1 contributing positively to $b(G)$. If $G_1 \notin P_1$, then it does not intersect $S(v)$ in more than one edge and we can easily colour an uncoloured edge of $S(v)$ and an edge incident to v such that $G \in P_2$. When G_1 intersects $S(v)$ in more than one edge we have $G_1 \in P_1$ and again easily colour two edges such that $G \in P_2$.

If $j_{\min} = 3$, then there are at most 5 coloured edges on any set of 4 vertices in $S(v)$. If there are at most 4 coloured edges on any set of 4 vertices in $S(v)$, then we proceed as in Claim 11. Thus, we may assume that there is a set of 4 vertices in $S(v)$ with exactly 5 coloured edges. It is enough to observe that these 5 edges cannot come from the same K_k -component, G_1 say, and that not all G_i involved can contain all 4 vertices. Therefore, we can colour an edge incident to v with the same colour as one coloured edge of $S(v)$ without any conflict.

Case $K(v) = Y_1$.

If $K(v) = Y_1$ then we can proceed similar to the colouring given in Claim 11. As $b(K(v)) = b(K_{k+1}^-) = k - 3$ we have $j_{\min} \neq 0$ and for $j_{\min} \geq 1$ we consider three vertices inside $S(v)$ that are contained in both copies of K_k . We want to colour one edge inside these three vertices and the edge connecting the third to v . For $j_{\min} = 1$ this is possible because all G_i are in P_0 and there are no coloured edges in $S(v)$ so far, which gives $G \in P_1$. When $j_{\min} = 2$, there is at most one coloured edge or a single $G_i \notin P_0$ (which contributes with no coloured edges to $S(v)$) and thus this is also possible and $G \in P_2$.

Lastly, for $j_{\min} = 3$, we only fail if there is a completely coloured triangle which was created by: (i) a graph $G_1 \in P_1$ with $b(G_1) \geq k - 3$ and $|b(G_{1,S(v)})| \geq k - 3$, or (ii) graphs $G_1 \in P_0$ and $G_2 \in P_0$ with $|b(G_{1,S(v)})| \geq k - 3$ and $|b(G_{2,S(v)})| \geq k - 3$. In case (i) there is no other coloured edge but in this triangle, and therefore we can easily colour two edges incident to v with colours from this triangle to make both copies of K_k non-rainbow. In case (ii) we can do something similar, as G_1 and G_2 can not contain K_{k+1}^- and therefore both copies of K_k contain a vertex uncovered by G_1 or G_2 that together with v can be coloured using a colour from the triangle. \square

Since we proved Sub-Claims 13 and 14, we conclude that Claim 12 holds. \square

Claims 11 and 12 imply that Lemma 10 holds. \square

3. Proof of Theorem 3 for K_4

In this section we analyse the anti-Ramsey threshold for K_4 , and show that $p_{K_4}^{\text{rb}} = n^{-7/15}$. For the upper bound on $p_{K_4}^{\text{rb}}$, let J be the graph obtained from $K_{3,4}$ with partition classes $\{a, b, c\}$ and $\{w, x, y, z\}$ by adding the edges ab , ac and bc . It is easy to see that in any proper colouring of J there is a rainbow K_4 . Therefore the upper bound

$$p_{K_4}^{\text{rb}} \leq n^{-7/15} \quad (19)$$

follows from Theorem 15 below applied with $H = J$.

Theorem 15 (Bollobás [3]). *Let H be a fixed graph. Then, $p = n^{-1/m(H)}$ is the threshold for the property that G contains H .*

To show that $p_{K_4}^{\text{rb}} \geq n^{-7/15}$ we follow a similar strategy as before, but we do not need the framework of [9], because we now have an even smaller upper bound $p \ll n^{-7/15} \ll n^{-2/(4+1)}$.

Let G be a K_4 -component with $m(G) < \frac{15}{7}$. Observe that there always is a vertex v of degree 4 in G and that the assertion of Fact 7 still holds. The only options for $K(v)$ are X_1 , X_2 and U_1 . In principle, Y_1 and Y_2 would also be possible, but Y_1 could only occur alone and Y_2 is already too dense. We define $b_{K_4}(G) := 7e(G) - 15|G| + 18$ and note that $b_{K_4}(G) < 18$ and $b_{K_4}(K_4) = 0$. Then

$$b_{K_4}(G) - b_{K_4}(G_v) - 7e(G_v^* \setminus G_v) = \begin{cases} 6 & \text{if } K(v) \text{ is } X_1, \\ 5 & \text{if } K(v) \text{ is } X_2, \\ 13 & \text{if } K(v) \text{ is } U_1. \end{cases}$$

Thus we can bound the number of occurrences of X_1 , X_2 and U_1 . Configuration X_1 is the only case where G_v could contain more than one K_4 -component and there can be at most two different K_4 -components, which both have one edge in common with $K(v)$. It is thus easy to see, that any K_4 -component G with $m(G) < \frac{15}{7}$ contains at most 10 vertices.

Now consider $G(n, p)$ with $p \ll n^{-7/15}$. It follows from Markov's inequality and the union bound, that $G(n, p)$ does not contain a subgraph G such that $m(G) \geq \frac{15}{7}$ and $|G| \leq 12$. Therefore $G(n, p)$ does not contain a K_4 -component G with $m(G) \geq \frac{15}{7}$.

It remains to give the colouring of G depending on the sequence of $K(v)$'s. If $K(v)$ is U_1 then we are left with a single K_4 and it is easy to colour the whole K_5 . Now we claim that if $b_{K_4}(G) < 6$ at most one edge is coloured in any K_3 and if $b_{K_4}(G) < 12$ at most two edges are coloured in any K_3 . If $K(v)$ is X_2 we repeat the colour of the edge in $K(v) \cup G_v$ if that edge is coloured or otherwise we colour two new disjoint edges with a new colour, which both is fine with the above. Only the case that $K(v)$ is X_1 is left to check. If G_v consists of only one K_4 -component, then we colour one edge on the triangle $K(v) \cup G_v$ and a new edge with the same colour. Since X_1 adds 6 to $b_{K_4}(G)$ this is fine with our condition. If G_v splits in more than one K_4 -component it is enough to observe that either we can ensure that the intersecting edges are uncoloured or we already have $b_{K_4}(G_v) > 5$ and thus $b_{K_4}(G)$ will be at least 11.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors would like to thank Gabriel F. Barros for helpful comments on an earlier version of this paper and the anonymous referees for their constructive input.

References

- [1] Gabriel Ferreira Barros, Bruno Pasqualotto Cavalier, Guilherme Oliveira Mota, Olaf Parczyk, Anti-Ramsey threshold of cycles, *Discrete Appl. Math.* 323 (2022) 228–235, <https://doi.org/10.1016/j.dam.2021.10.021>.
- [2] Béla Bollobás, Andrew Thomason, Threshold functions, *Combinatorica* 7 (1) (1987) 35–38.
- [3] Béla Bollobás, Threshold functions for small subgraphs, *Math. Proc. Camb. Philos. Soc.* 90 (2) (1981) 197–206, <https://doi.org/10.1017/S0305004100058655>.
- [4] Béla Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 1998.
- [5] David Conlon, Combinatorial theorems relative to a random set, in: *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. IV*, Kyung Moon Sa, Seoul, 2014, pp. 303–327.
- [6] Svante Janson, Tomasz Łuczak, Andrzej Ruciński, *Random Graphs*, Wiley-Interscience, New York, 2000.
- [7] Yoshiharu Kohayakawa, Pavlos B. Konstantinidis, Guilherme O. Mota, On an anti-Ramsey threshold for random graphs, *Eur. J. Comb.* 40 (2014) 26–41, <https://doi.org/10.1016/j.ejc.2014.02.004>.
- [8] Yoshiharu Kohayakawa, Pavlos B. Konstantinidis, Guilherme O. Mota, On an anti-Ramsey threshold for sparse graphs with one triangle, *J. Graph Theory* 87 (2) (2018) 176–187.
- [9] Rajko Nenadov, Yury Person, Nemanja Škorić, Angelika Steger, An algorithmic framework for obtaining lower bounds for random Ramsey problems, *J. Comb. Theory, Ser. B* 124 (2017) 1–38, <https://doi.org/10.1016/j.jctb.2016.12.007>.
- [10] Vojtěch Rödl, Andrzej Ruciński, Lower bounds on probability thresholds for Ramsey properties, in: *Combinatorics, Paul Erdős Is Eighty*, vol. 1, in: *Bolyai Soc. Math. Stud.*, János Bolyai Math. Soc., Budapest, 1993, pp. 317–346.
- [11] Vojtěch Rödl, Andrzej Ruciński, Threshold functions for Ramsey properties, *J. Am. Math. Soc.* 8 (4) (1995) 917–942, <https://doi.org/10.2307/2152833>.
- [12] Vojtěch Rödl, Zolt Tuza, Rainbow subgraphs in properly edge-colored graphs, *Random Struct. Algorithms* 3 (2) (1992) 175–182.