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RELIABILITY LOWER BOUNDS FOR A COHERENT SYSTEM UNDER DEPENDENCE CONDITIONS

by

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AMS Classification: 62N05; 90B25, 62G30, 94C10.

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Reliability lower bounds for a coherent system under dependence conditions

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Abstract. In this paper, using a boolean algebra approach, we investigate lower bounds for a coherent system reliability of dependent components under a conditional increasing hazard rate distribution.

Keywords: Reliability, boolean algebra, quality function, hazard gradient, conditional increasing failure rate distribution.

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1. Introduction.

Since most material, structures and devices wear out with time, the class of increasing hazard rate (IHR) distributions is very important in reliability theory. Under IHR assumption, one get bounds on system' s component lifetimes. As the system lifetime does not preserve the IHR property of its components, even if the component' s lifetimes are independent, in general, these bounds does not extend to system lifetime. In particular, Barlow and Proschan (1981) developed bounds on system reliability of associated

IHR components, a form of positive dependence. In this paper we relaxed that condition considering conditional increasing hazard rate distribution.

As a coherent structure function is a pseudo-Boolean function, to consider the case of dependent components in its generality, we use the boolean algebra approach as in Marichal and Mathonet (2011a), (2011b), (2012) and (2013)and the concept of multivariate gradient failure rate defined by Kotz and Johnson (1975).

In this paper, in Section 2 we resume useful mathematical details for the following Section. In Section 3 we give the main results and examples.

2. Mathematical details.

The approach using pseudo boolean functions, as in Marichal and Mathonet (2011a), (2011b), (2013) gives an important contribution to the development and extension of classical reliability results in the context of dependent component lifetimes without simultaneous failures. To follow we introduce the necessary details found in Marichal and Mathonet (2011a), (2013):

Through the usual identification of the elements of $\{0,1\}^n$ with the subsets of $[n] = \{1,...,n\}$, a pseudo-Boolean function $f : \{0,1\}^n \to \Re$ can be equivalently described by a set function $v_f : 2^n \to \Re$. We simply write $v_f(A) = f(1_A)$, where 1_A denotes the n-tuple whose i-th coordenate is 1, if $i \in A$, and 0, otherwise. To avoid cumbersome notation, we henceforth use the same symbol to denote both, a given pseudo-Boolean function and its underlying set function, thus writing $f : \{0,1\}^n \to \Re$ as $f : 2^n \to \Re$ interchangeably.

If $T_1, ...T_n$ denote the component lifetimes with $P(T_i = T_j) = 0, \quad 1 \leq i,j \leq n$, we define the associated relative quality function $q : 2^n \to [0,1]$ as

$$q(A) = P(\max_{i \in [n] \setminus A} T_i < \min_{j \in A} T_j)$$

with the convention that $q(\emptyset) = q([n]) = 1$. $q(A)$ is the probability that the lifetime of every component in A is greater than the lifetime of every component in $[n] \setminus A$, that is, $q(A)$ is a measure of the overall quality of the components in A when compared with the components in $[n] \setminus A$.

Since that the random variables $T_1, ...T_n$ have no ties, the function q can be written as

$$q(A) = \sum_{\sigma \in \wp^n : \{\sigma(n-|A|+1), \ldots, \sigma(n)\} = A} P(T_{\sigma(1)} < ... < T_{\sigma(n)})$$

where $\wp^n$ denote the class of permutations on $[n]$.

Following, under the assumption of no ties between component lifetimes, Marichal and Mathonet (2013) define, for every $j \in [n]$, the function $q_j : 2^n \to [0,1]$ as

$$q_j(A) = P(\max_{i \in [n] \setminus A \cup \{j\}} T_i < T_j < \min_{i \in A} T_i).$$

Similarly, as $q(A)$, we can write

$$q_j(A) = \sum_{\sigma \in \wp^n : \{\sigma(n-|A|+1), \ldots, \sigma(n)\} = A, \sigma(n-|A|) = j} P(T_{\sigma(1)} < ... < T_{\sigma(n)}).$$

$q_j(A)$ is the probability that the components that are better than component j are precisely those in A. Follows that

$$\sum_{A \subseteq [n] \setminus \{j\}} q_j(A) = 1, \quad j \in [n].$$

We also observe that

$$q(A) = \sum_{j \notin A} q_j(A), \quad A \neq [n]$$

and

$$q(A) = \sum_{j \in A} q_j(A \setminus \{j\}), \quad A \neq \emptyset.$$

Moreover, $q_j(\emptyset) = q(\{j\})$ is the probability that component j is the best component, while $q_j([n] \setminus \{j\}) = q([n] \setminus \{j\})$ is the probability that component j is the worse component.

Now, let $\mathbf{T} = (T_1, ..., T_n)$ be a positive random vector with an absolutely continuous jointly distribution F, representing the component lifetimes of a coherent systems. Denotes its survival function by

$$\overline{F}(\mathbf{t}) = \overline{F}(t_1, ..., t_n) = P(T_1 > t_1, ..., T_n > t_n), \quad \mathbf{t} = (t_1, ..., t_n)$$

and let $R(\mathbf{t}) = -\ln \overline{F}(\mathbf{t})$ the corresponding multivariate hazard function. For $i \in [n]$ define

$$r_i(\mathbf{t}) = \frac{dR(\mathbf{t})}{dt_i}, \quad in \quad \{\mathbf{t} : \overline{F}(\mathbf{t}) > 0\}.$$

The vector $(r_1(\mathbf{t}), ..., r_n(\mathbf{t}))$ is called the hazard gradient of $\mathbf{T}$ (see Johnson and Kotz (1975) and Marshall (1975)). Note that $r_i(\mathbf{t})$ can be interpreted as the conditional hazard rate of T_i evaluated at t_i, given that $T_j > t_j$ for all $j \neq i$; i.e.

$$r_i(\mathbf{t}) = \frac{f_i(t_i | T_j > t_j, j \neq i)}{\overline{F}_i(t_i | T_j > t_j, j \neq i)},$$

where $f_i(.|T_j > t_j, j \neq i)$ and $\overline{F}_i(.|T_j > t_j, j \neq i)$ are the conditional density and survival functions of T_i, given that $T_j > t_j$ for all $j \neq i$.

Definition 2.1 The random vector T, or its distribution, is called multivariate increasing (decreasing) hazard rate, denoted IHR (DHR), if for all values of $\mathbf{t} = (t_1, ..., t_n)$, the components hazard rate $r_i(\mathbf{t})$ are increasing (decreasing) function in $t_i, 1 \le i \le n$.

If $r_i(\mathbf{t})$ is an increasing (decreasing) function of $t_i, 1 \le i \le n$, at $\mathbf{t} = (t_1, ..., t_n)$ we say that T, or its distribution, is multivariate IHR (DHR) at $\mathbf{t} = (t_1, ..., t_n)$.

We consider a subset A, $A \subseteq [n]$, the jointly marginal survival function $P(T_i > t_i, i \in A)$ and the lifetime $T_A = \min_{i \in A} T_i$. The conditional hazard restricted to A will be denoted by $r_i(\mathbf{t}|A)$. If $t_i = t, \quad \forall i \in A$, we denote the conditional hazard rates $r_i(\mathbf{t}|A) = r_i(t|A)$ and the conditional hazard functions by $R_i(t|A) = \int_0^t r_i(s|A)ds$.

Follows that, when $dt \downarrow 0$ we have

$$P(T_A \in (t, t+dt] | T_A > t) = P(\cup_{i \in A}\{T_i \in (t, t+dt]\} | T_A > t) =$$
$$\sum_{k \in A}(-1)^k \sum_{1 \le i_1 < \ldots < i_k \le |A|} P(\cap\{T_i \in (t, t+dt]\} | T_A > t) =$$
$$\sum_{i \in A} P(T_i \in (t, t+dt] | T_j > t, j \in A \backslash \{i\}) = \sum_{i \in A} r_i(t|A).$$

where the third equality follows from the assumption $P(T_i = T_j) = 0, \forall i, j \in [n]$.

Therefore the hazard rate of T_A is

$$r_A(t) = \sum_{i \in A} r_i(t|A)$$

and its conditional hazard function can be set as

$$R_A(t) = \int_0^t r_A(s)ds = \int_0^t \sum_{i \in A} r_i(s|A)ds = \sum_{i \in A} \int_0^t r_i(s|A)ds = \sum_{i \in A} R_i(t|A),$$

with

$$P(T_A > t) = \overline{F}_A(t) = e^{-R_A(t)} = e^{-\sum_{i \in A} R_i(t|A)} = \pi_{i \in A} e^{-R_i(t|A)}.$$

Remark 2.2 A) In earlier literature the additivity property of the component hazards $r_i(t)$ and the independence of the lifetimes T_i were often thought to be equivalent. The source of this confusion appears to be the exponential formula: assuming absolute continuity and independence, it is certainly true that

$$P(T_A > t) = \pi_{i \in A} P(T_i > t) = \pi_{i \in A} e^{-\int_0^t r_i(s)ds} = e^{-\sum_{i \in A} \int_0^t r_i(s)ds} = e^{-\int_0^t r_A(s)ds}.$$

However, as above, additivity of hazards holds always and does not imply independence.

B) Note that, if $r_i(t|A)$ is an increasing (decreasing) function of t, for all i, then T_A is IHR (DHR), but this is not ensured simply by the condition that $\mathbf{T}$ is multivariate IHR (DHR). If $r_i(\mathbf{t}), 1 \leq i \leq n$, is an increasing (decreasing) function of each $t_1, ..., t_n$, (not only of t_i), this does ensure that T_A is IHR (DHR).

At this point we consider the extended hazard function $R_i(t \wedge T_i|A)$, with $t \wedge T_i = \min\{t, T_i\}$ where $R_i(t|A) = \int_0^t r_i(s|A)ds$, indicating that the hazard stopped at the failure time T_i.

As in Arjas and Yashin (1988), under the absolutely continuous hypothesis the hazard functions $R_i(t|A)$ can be written as $R_i(t \wedge T_i|A) = -\ln \overline{F}_i(t \wedge T_i|A)$. Also, see Norros (1986), the total hazards $R_i(T_i|A), 1 \leq i \leq n$ are independent and identically standard exponential distributed random variables. Resuming, as in Aven, T. and Jensen, U. (1999), $1_{\{T_i \leq t\}} - R_i(t \wedge T_i|A)$ is a zero mean $\Im_t$-martingale with respect to the subσ-algebra $\Im_t = \sigma\{T_i > s, 0 \leq s \leq t, 1 \leq i \leq n\}$, $R_i(t \wedge T_i|A)$ is a $\Im_t$-predictable process, and $P(T_i \leq t) = E[-\ln \overline{F}_i(t \wedge T_i|A)]$.

2. Bounds on system reliability.

An important result of independent interest in system reliability at time t is given by:

Theorem 3.1 If $T_1, .., T_n$ are absolutely continuous component lifetimes of a coherent system with lifetime T, the system reliability at time t is

$$P(T > t) = \sum_{j=1}^{n} \sum_{A \subseteq [n] \setminus \{j\}} (\phi(A \cup \{j\}) - \phi(A)) \int_t^{\infty} q_j(A|x) dF_j(x|A),$$

where $q_j(A|x) = P(\max_{i \in [n] \setminus A \cup \{j\}} T_i < T_j < \min_{i \in A} T_i | T_j = x)$.

Proof

As $\{T = T_j\}, 1 \leq j \leq n$ is a partition of the probability space, from the total probability law we have

$$P(T > t) = \sum_{\sigma \in p^n} \sum_{j=1}^{n} P(T = T_j|C) P(T > t | \{T = T_j\}, C) P(C).$$

where $C = \{T_{\sigma(1)} < ... < T_{\sigma(n)}\}$.

However the expression $P(T = T_j|C) = P(T = T_j|T_{\sigma(1)} < ... < T_{\sigma(n)})$ takes its values in $\{0, 1\}$ and it is exactly 1 if, and only if, $\{\sigma(1), ..., \sigma(i-1)\}$ is not a cut set and $\{\sigma(1), ..., \sigma(i)\}$ is a cut set, with $i = \sigma^{-1}(j)$, and

$$P(T = T_j|C) = \phi(\sigma(i), \sigma(i+1), ..., \sigma(n)) - \phi(\sigma(i+1), ..., \sigma(n)).$$

Therefore $P(T > t) =$

$$\sum_{j=1}^{n} \sum_{\sigma \in p^n} [\phi(\sigma(i), ..., \sigma(n)) - \phi(\sigma(i+1), ..., \sigma(n))] P(T > t | \{T = T_j\}, C) P(C) =$$

$$\sum_{j=1}^{n} \int_t^{\infty} \sum_{\sigma \in p^n} [\phi(\sigma(i), ..., \sigma(n)) - \phi(\sigma(i+1), ..., \sigma(n))] P(C) dF_j(x|C).$$

Grouping the terms for which $\{\sigma(\sigma^{-1}(j)+1), ...\sigma(n)\}$ is a fixed set A, with cardinality $n-i+1$, we have $P(T>t) =$

$$\sum_{j=1}^{n} \int_t^{\infty} \sum_{A \subseteq [n]\setminus\{j\}} [\phi(A \cup \{j\}) - \phi(A)] \sum_{\sigma \in \wp^n} P(, T_{\sigma(1)} < ... < T_{\sigma(i)} = x < ... < T_{\sigma(n)}) dF_j(x|A)$$

where the summation is over all $\sigma \in \wp^n : \{\sigma(n-|A|+1), ..., \sigma(n)\} = A, \sigma(n-|A|) = j, Tj = x$. Therefore

$$P(T>t) = \sum_{j=1}^{n} \sum_{A \subseteq [n]\setminus\{j\}} [\phi(A \cup \{j\}) - \phi(A)] \int_t^{\infty} q_j(A|x) dF_j(x|A),$$

with $q_j(A|x) = P(\max_{k \in [n]\setminus A\cup\{j\}} T_k < T_j < \min_{k \in A} T_k | T_j = x)$.

The main theorem follows:

Theorem 3.2 If $T_1, .., T_n$ are absolutely continuous component lifetimes of a coherent system with lifetime T. Then

$$P(T>t) \geq \sum_{j=1}^{n} \sum_{A \subseteq [n]\setminus\{j\}} (\phi(A \cup \{j\}) - \phi(A)) (\frac{1}{2})^{|A|+1} \overline{F}_j^2(t|A).$$

Proof

From Theorem 3.1 we have $P(T>t) =$

$$\sum_{j=1}^{n} \sum_{A \subseteq [n]\setminus\{j\}} (\phi(A \cup \{j\}) - \phi(A)) E[1_{\{T_j > t\}} \exp\{-\sum_{k \in A} R_k(T_j|A) + R_j(\max_{k \in [n]\setminus A\cup\{j\}} T_k|A)\}].$$

However, in the set $\{\min_{k \in A} T_k > T_j\}$ we have $R_k(T_k|A) \geq R_k(T_j|A), k \in A$ and in the set $\{\max_{i \in [n]\setminus A\cup\{j\}} < T_j\}$, $R_j(T_j|A) \geq R_j(\max_{k \in [n]\setminus A\cup\{j\}} T_k|A)$. Also, it is well known that $R_k(T_k|A), 1 \leq k \leq n$ are independent and identically distributed standard exponential random variables. As $R_j(t|A) = -\ln \overline{F}_j(t|A)$ is increasing we can use the equivalence $\{T_j > t\} \leftrightarrow \{R_j(T_j|A) > -\ln \overline{F}_j(t|A)\}$.

Therefore $P(T > t) =$

$$\sum_{j=1}^{n} \sum_{A\subseteq[n]\setminus\{j\}} (\phi(A\cup\{j\}) - \phi(A))E[1_{\{T_j>t\}} \exp\{-\sum_{k\in A} R_k(T_j|A) + R_j(\max_{k\in[n]\setminus A\cup\{j\}} T_k|A)\}] \geq$$

$$\sum_{j=1}^{n} \sum_{A\subseteq[n]\setminus\{j\}} (\phi(A\cup\{j\}) - \phi(A))\pi_{k\in A} E[\exp\{-R_k(T_k|A)\}]E[\exp\{-R_j(T_j|A)\}1_{\{T_j>t\}}] =$$

$$\sum_{j=1}^{n} \sum_{A\subseteq[n]\setminus\{j\}} (\phi(A\cup\{j\}) - \phi(A))(\frac{1}{2})^{|A|+1}\overline{F}_j^2(t|A).$$

Now we can produce lower bounds for system reliability derived from lower bounds of its conditional jointly marginal distributions.

Firstly, under the assumption that **T** is IHR, its conditional jointly marginal $T_j, j \in A$ are IHR. From Arjas (1981), its happen that,

$$R_j(t|A) = \int_0^t r_j(s|A)ds,$$

and $R_j(t|A) = -\ln \overline{F}_j(t \wedge T_j|A) = -\ln P(T_j > t \wedge T_j|T_k > t_k, k \in A)$ is, almost surely, an increasing and convex function.

. Also, it is well known that $R_j(T_j|A), 1 \leq j \leq n$, are independent and identically distributed standard exponential random variables.

Theorem 3.3 If $T_1, .., T_n$ are absolutely continuous component lifetimes of a coherent system with lifetime T with multivariate increasing hazard rate. Then

$$P(T > s) \geq \sum_{j=1}^{n} \sum_{A\subseteq[n]\setminus\{j\}} (\phi(A\cup\{j\}) - \phi(A))(\frac{1}{2})^{|A|+1} \exp\{\frac{-2s}{E[T_j|T_k > t_k, k \in A]}$$

if $s < m$ and $m = \min\{E[T_j|T_k > t_k, k \in A], A \subseteq [n]\backslash\{j\}, 1 \leq j \leq n$.

Proof

As $-\ln \overline{F}_j(t|A)$ is convex, $\frac{-\ln \overline{F}_j(t|A)}{t}$ is increasing on t. If $s < m$, we have

$$\frac{-\ln \overline{F}_j(s|A)}{s} \leq \frac{-\ln \overline{F}_j(E[T_j|T_k > t_k, k \in A])}{E[T_j|T_k > t_k, k \in A]} \leq$$

$$\frac{E[-\ln \overline{F}_j(T_j|T_k > t_k, k \in A)]}{E[T_j|T_k > t_k, k \in A]} = \frac{1}{E[T_j|T_k > t_k, k \in A]},$$

where we apply Jensen's inequality. Therefore

$$\overline{F}_j^2(t|A) \geq \exp\{\frac{-2s}{E[T_j|T_k > t_k, k \in A]}\},$$

for all j. Then theorem 3.3 follows from theorem 3.2.

Theorem 3.4 If $T_1, .., T_n$ are absolutely continuous component lifetimes of a coherent system with lifetime T with multivariate increasing hazard rate. If $P(T_j \leq t + \xi_{p_j}|A) = p_j, \ A \subseteq [n]\backslash\{j\}, \ 0 < p_j < 1, \ 1 \leq j \leq n$, then

$$P(T > t + s) \geq \sum_{j=1}^{n} \sum_{A \subseteq [n]\backslash\{j\}} (\phi(A \cup \{j\}) - \phi(A))(\frac{1}{2})^{|A|+1}[1 - \frac{sp_j}{\xi_{p_j}}]^2.$$

if $s < m$ and $m = \min\{\xi_{p_j}, 1 \leq j \leq n\}$.

Proof

As $-\ln \overline{F}_j(t|A)$ is convex, $\frac{-\ln \overline{F}_j(t|A)}{t}$ is increasing on t and if $0 < s \leq \xi_{p_j}$

$$\frac{-\ln \overline{F}_j(t + s \wedge T_j|A)}{s} \leq \frac{-\ln \overline{F}_j(t + \xi_{p_j} \wedge T_j|A)}{\xi_{p_j}}, \quad a.s.$$

which implies that

$$\frac{F_j(t+s|A)}{s} = \frac{E[-\ln \overline{F}_j(t+s\wedge T_j|A)]}{s} \leq$$

$$\frac{E[-\ln \overline{F}_j(t+\xi_{p_j}\wedge T_j|A)]}{\xi_{p_j}} = \frac{F_j(t+\xi_{p_j}|A)}{\xi_{p_j}} = \frac{p_j}{\xi_{p_j}}.$$

Therefore $\overline{F}_j(t+s|A)^2 \geq [1-\frac{sp_j}{\xi_{p_j}}]^2$, and the result follows from theorem 3.2.

Example 3.5

Consider a family of multivariate Morgenstern-Gumbel-Farlie survival distribution given by

$$P(T_1 > t_1, ..., T_n > t_n) = \pi_{i=1}^n P(T_i > t_i)[1 + \alpha\pi_{i=1}^n P(T_i \leq t_i)],$$

with $|\alpha| < 1$. Let $H(\mathbf{t}) = H(t_1, ..., t_n)$ be

$$H(\mathbf{t}) = -\ln P(T_1 > t_1, ..., T_n > t_n) = \sum_{i=1}^{n} -\ln P(T_i > t_i) - \ln[1 + \alpha\pi_{i=1}^n P(T_i \leq t_i)].$$

Therefore

$$\frac{dH(\mathbf{t})}{dt_i} = r_i(t_i)[1 - \frac{\alpha\pi_{k=1,k\neq i}^n P(T_k \leq t_k)P(T_i > t_i)}{1 + \alpha\pi_{k=1}^n P(T_k \leq t_k)}],$$

where $r_i(t) = \frac{f_i(t)}{P(T_i > t)}$ is the univariate failure rate of the lifetime T_i and $f_i(t) = \frac{dP(t_i \leq t)}{dt}$ is it probability density function.

We note that, if α is positive, $\frac{dH(\mathbf{t})}{dt_i}$ has the same sign of $r_i(t)$, and therefore, if α is positive and $T_i, 1 \leq i \leq n$ are univariate IHR (DHR), $\mathbf{T} = (T_1, ..., T_n)$ is multivariate IHR (DHR) and therefore conditioned IHR (DHR).

Also, for this survival distribution,

$$P(T_j > t | T_k > t, k \in [n]\backslash\{j\}) = P(T_j > t)[1 + \alpha\pi_{k=1}^n P(T_k \leq t)].$$

If $|A| < n-1$ we have $P(T_j > t | T_k > t, k \in [n] \backslash \{j\}) = P(T_j > t)$.

In the particular case where T_i is an exponential lifetime with parameter $\lambda_i, 1 \leq i \leq n$, we have

$$P(T_j > t | T_k > t, k \in [n] \backslash \{j\}) = 2\exp[-\lambda_j t] + \Sigma_{k=1}^{n}(-1)^k \exp[-(\lambda_j \Sigma_{1 \leq i_k \leq n} \lambda_{i_k})t].$$

If $|A| < n-1$, $P(T_j > t | T_k > t, k \in A \backslash \{j\}) = \exp[-\lambda_j t]$.

In this case, from Theorem 3.2, we have the system reliability lower bound

$$P(T > t) \geq \sum_{j=1}^{n} \sum_{A \subseteq [n] \backslash \{j\}, |A| < n-1} (\phi(A \cup \{j\}) - \phi(A))(\frac{1}{2})^{|A|+1} \exp[-2\lambda_j t] +$$

$$\sum_{j=1}^{n}(\phi([n]) - \phi([n] \backslash \{j\}))(\frac{1}{2})^n [2\exp[-\lambda_j t] + \Sigma_{k=1}^{n}(-1)^k \exp[-(\lambda_j \Sigma_{1 \leq i_k \leq n} \lambda_{i_k})t]]^2.$$

To get lower bounds using Theorem 3.3 we have

$$E[T_j | T_k > t, k \in [n] \backslash \{j\}] = \frac{2}{\lambda_j} + \Sigma_{k=1}^{n}(-1)^k \frac{1}{\lambda_j \Sigma_{1 \leq i_k \leq n} \lambda_{i_k}}.$$

If $|A| < n-1$, $E[T_j | T_k > t, k \in A \backslash \{j\}] = \frac{1}{\lambda_j}$.

Therefore, follows from Theorem 3.3 that, if $s < m$, where $m = \min\{E[T_j | T_k > t_k, k \in A], A \subseteq [n] \backslash \{j\}, 1 \leq j \leq n$

$$P(T > s) \geq \sum_{j=1}^{n} \sum_{A \subseteq [n] \backslash \{j\}, |A| < n-1} (\phi(A \cup \{j\}) - \phi(A))(\frac{1}{2})^{|A|+1} \exp[-2s\lambda_j] +$$

$$\sum_{j=1}^{n}(\phi([n]) - \phi([n] \backslash \{j\}))(\frac{1}{2})^n \exp[\frac{-2s}{\frac{2}{\lambda_j} + \Sigma_{k=1}^{n}(-1)^k \frac{1}{\lambda_j \Sigma_{1 \leq i_k \leq n} \lambda_{i_k}}}].$$

Also, for the particular exponential case of the multivariate Morgenstern-Gumbel-Farlie survival distribution, the p_j percentile in the case where $|A| < n-1$ is $\xi_{p_j} = \frac{-\ln(1-p_j)-t\lambda_j}{\lambda_j}$ and

$$(1-\frac{sp_j}{\xi_{p_j}}) = \frac{\ln(1-p_j)+(t+sp_j)\lambda_j}{\ln(1-p_j)+t\lambda_j}.$$

If $|A| = n-1$, $\xi_{p_j} = \varphi^{-1}(t+\xi_{p_j}) - t$ where $P(T_j \le t+\xi_{p_j}|[n]\backslash\{j\}) =$

$$\varphi(t+\xi_{p_j}) = 2\exp[-\lambda_j(t+\xi_{p_j})] + \Sigma_{k=1}^{n}(-1)^k \exp[-(\lambda_j \Sigma_{1\le i_k \le n}\lambda_{i_k})(t+\xi_{p_j})],$$

and

$$(1-\frac{sp_j}{\xi_{p_j}}) = (1-\frac{sp_j}{\varphi^{-1}(t+\xi_{p_j})-t}).$$

From Theorem 3.4 we have the lower bounds $P(T > t+s) \ge$

$$\sum_{j=1}^{n} \sum_{A\subseteq[n]\backslash\{j\},|A|<n-1} (\phi(A\cup\{j\})-\phi(A))(\frac{1}{2})^{|A|+1}[\frac{\ln(1-p_j)+(t+sp_j)\lambda_j}{\ln(1-p_j)+t\lambda_j}]^2+$$

$$\sum_{j=1}^{n}(\phi([n])-\phi([n]\backslash\{j\}))(\frac{1}{2})^n[(1-\frac{sp_j}{\varphi^{-1}(t+\xi_{p_j})-t})]^2.$$

For a k-out-of-n:F system, $\phi(A) = 1$ if, and only if, $|A| \ge n-k+1$ and consequently, for every $j \in [n]$, $(\phi(A\cup\{j\})-\phi(A)) = 1$ if, and only if, $|A| = n-k$. If $k \ne n$, for each $j \in [n]$, there exists $\binom{n-1}{k-1}$ sets A such that $(\phi(A\cup\{j\})-\phi(A)) = 1$. Therefore, from Theorem 3.2 we have

$$P(T > t) \ge \sum_{j=1}^{n}\binom{n-1}{k-1}(\frac{1}{2})^{n-k+1}\exp[-2\lambda_j t],$$

and from Theorem 3.4

$$P(T > t) \geq \sum_{j=1}^{n} \binom{n-1}{k-1} (\frac{1}{2})^{n-k+1} [\frac{\ln(1-p_j) + (t + sp_j)\lambda_j}{\ln(1-p_j) + t\lambda_j}]^2.$$

2. Conclusion. In this paper we produce lower bonds for systems reliability introducing a gradient hazard function to the boolean reliability theory as in Marichal and Mathonet (2011a), (2011b), (2013). This approach allows us to consider statistical dependence between the components lifetimes without simultaneous failure. We generalize lower bounds existing in the classical theory.

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