On Faithfully Quadratic Rings

M. Dickmann   F. Miraglia

This is a summary of a paper in preparation aimed at developing a theory of quadratic forms over several significant, and quite extensive, classes of preordered rings.

By "quadratic form" we understand, here, *diagonal quadratic forms with unit coefficients*; and "ring" stands for commutative unitary ring where 2 is invertible.

We achieve this by the use, in the ring context, of our abstract theory of quadratic forms, the theory of special groups, expounded in [DM1] [1]. This is done as follows: fix a preordered ring (p-ring) $\langle A, T \rangle$; then

(1) The "intrinsic" theory of quadratic forms in $\langle A, T \rangle$ is based on the following notion of T-isometry:

> Two n-dimensional quadratic forms $\varphi = \sum_{i=1}^{n} a_i X_i^2$ $\psi = \sum_{i=1}^{n} b_i X_i^2$, with $a_i, b_i \in A^\times$ are *T-isometric*, $\varphi \approx_T \psi$, if there is a sequence $\varphi_0, \varphi_1, \ldots, \varphi_k$ of n-dimensional diagonal forms over $A^\times$, so that $\varphi = \varphi_0$, $\psi = \varphi_k$ and for every $1 \le i \le k$, φ_i is either isometric to φ_{i-1} in the sense that there is a matrix $M \in \mathrm{GL}_n(A)$ such that $\varphi_i = M \varphi_{i-1} M^t$, or there are $t_1, \ldots, t_n \in T^\times$ such that $\varphi_i = \langle t_1 x_1, \ldots, t_n x_n \rangle$ and $\varphi_{i-1} = \langle x_1, \ldots, x_n \rangle$.

(2) To $\langle A, T \rangle$ we associate a structure $G_T(A) = A^\times / T^\times$ endowed with the product operation induced by $A^\times$ and the binary isometry, $\equiv_T$, induced by $\approx_T$ restricted to binary forms (plus $-1 = -1/T^\times$ as a distinguished element). $\langle G_T(A), \equiv_T, -1 \rangle$ is not quite a special group in the sense of [DM1], Definition 1.2, p.2, but it satisfies some of its axioms, constituting a *proto-special group* as in Definition 6.3 of [DM5]. Though related, the internal and the formal approach (via $G_T(A)$) are far from identical.

(3) In section 2 we introduce three axioms in terms of the *value representation* relation D_T^v on $\langle A, T \rangle$ defined by: for $a, b_1, \ldots, b_n \in A^\times$

$$ a \in D_T^v(b_1, \ldots, b_n) \quad \Leftrightarrow \quad \exists\, t_1, \ldots, t_n \in T \text{ such that } \quad a = \sum_{i=1}^{n} t_i b_i. $$

When satisfied in $\langle A, T \rangle$, these axioms are sufficient — and under reasonable assumptions, also necessary — to guarantee identity between the intrinsic and formal approaches and, in fact, more:

(3.i) The structure $G_T(A)$ is a special group.

(3.ii) T-isometry and value representation in $\langle A, T \rangle$ are faithfully represented by the corresponding formal notions in $G_T(A)$.

We call *T-faithfully quadratic* any p-ring $\langle A, T \rangle$ verifying these axioms. In fact, this setting as well as the consequences (3.i) and (3.ii) apply, more generally, to forms with entries in certain subgroups of $A^\times$, called *T-subgroups* [2], and also to the case where $T = A^2$; in this

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[1] Corroborating a development envisaged by Manfred Knebush at the outset of the theory of special groups.

[2] This generalization is not mere *ars gratia artis*; see [DM6].

latter case, T-isometry is just matrix isometry. Briefly (and approximately) stated, the axioms for T-quadratically faithful rings express the properties of value representation known as *transversality* (for 2-forms), *inductive characterization* and *Witt-cancellation* (for 1-forms).

It is worth noticing that T-isometry as defined above, coincides with isometry in the sense of equality of signs at all natural T-signatures existing in $\langle A, T \rangle$.

For rings A verifying the 2-transversality axiom for $T = A^2$, the mod 2 K-theory of A obtained from that in [Gu] coincides with the K-theory of the special group $G(A)$ as defined in [DM2] and [DM4].

(4) The axioms for T-quadratic faithfulness can be formalized by first-order sentences in the language of unitary rings (consisting of $+$, $\cdot$, 0, 1, -1) augmented by a unary predicate symbol for T (not needed if $T = A^2$). In fact, the alluded sentences are of a very special form, which we call *Horn-geometric* (i.e., Horn + geometric). This form of the axioms automatically guarantees preservation of T-quadratic faithfulness under (right-directed) inductive limits, arbitrary reduced products − in particular, arbitrary products − and the formation of rings of global sections of sheaves over partitionable spaces. This is the subject matter of section 3.

Most of the remainder of the paper consists in verifying that various classes of p-rings $\langle A, T \rangle$ are T-faithfully quadratic.

(5) In section 4 we deal with a large class of *rings with many units*. Using results due to Walter [Wa] we show that rings in this class are *completely faithfully quadratic*, i.e., T-faithfully quadratic for all preorders T and for $T = A^2$. We also observe that the class of rings with many units is axiomatizable by Horn-geometric first-order sentences in the language of unitary rings, guaranteeing that is class is closed under inductive limits and arbitrary reduced products; to the best of our knowledge these latter results are new.

(6) Next we consider, in section 5, the class p-rings $\langle A, T \rangle$ having the *bounded inversion* property, i.e., $1 + T \subseteq A^\times$. Generalizing a previous result of Mahé [Ma] (for $T = \Sigma A^2$), we show that value representation in these rings is transversal, a result often used in the sequel.

(7) After some basic results (section 6) concerning the behavior of quadratic forms and T-isometry under localization at idempotents and under orthogonal decompositions of the ring into idempotents, we turn, in section 7, to the study of quadratic form theory over *reduced f-rings*, i.e., subdirect products of linearly ordered integral domains. Rings of this type − of which the ring $\mathbb{C}(X)$ of continuous real-valued functions on a topological space X is a classical example − come endowed with a partial order $T_\sharp$ that makes them into lattice-ordered rings.

We show that any unit of such a ring A, gives rise to an orthogonal decomposition into two idempotents, and use this decomposition to show that the associated proto-special group $G_\sharp A$ is, in fact, isomorphic to the Boolean algebra $B(A)$ of idempotents in A; it is, hence, a reduced special group.

Our central result in section 7 has, among others, the following consequences:

(7.i) Value representation modulo $T_\sharp$ ($D_{T_\sharp}^v$) by forms of arbitrary dimension with entries in $A^\times$ coincides with isometry representation in the Boolean algebra $G_\sharp A = B(A)$.

(7.ii) If A contains $\mathbb{Q}$, then A is $T_\sharp$-faithfully quadratic.

(7.iii) If T is any preorder containing $T_\sharp$ such that $\langle A, T \rangle$ has bounded inversion, then A is T-faithfully quadratic and $G_T(A) = G_{red}(A) = B(A)$.

Together, these results substantiate the idea that the structure of the units of a reduced f-ring are closely related to a Boolean algebra.

In the important case where $A = \mathbb{C}(X)$ and $T = A^2$, these rings are P-faithfully quadratic for any preorder P of $\mathbb{C}(X)$ with bounded inversion. However, they are P-faithfully quadratic for many other preorders as well; for example, for all preorders of the form

$$P_K \;=\; \{f \in \mathbb{C}(X) : f \restriction K \geq 0\}$$

where K is a (proper) closed subset of X.

These results admit a generalization to reduced f-rings containing $\mathbb{Q}$, endowed with what we call *unit-reflecting* preorders; here the reduced special group $G_T(A)$ turns out to be a quotient of the Boolean algebra $B(A)$.

Among the interesting classes of rings to which the previous results apply, are the *completely real function rings* and the *weakly real closed rings*, as well as subdirect products of Euclidean rings verifying certain additional requirements.

(8) In section 8 we examine the class of *strictly representable* rings, that is, p-rings $\langle A, T \rangle$ admitting a dense representation into $\mathbb{C}(X)$, for some compact Hausdorff space X, that, in addition, sends T into non-negative functions and if an element of A is represented by a strictly positive function on X, then this element is in $T^{\times}$.

We show that these are exactly the bounded inversion p-rings $\langle A, T \rangle$, where T *is Archimedean*. As this equivalence shows, the notion of strictly representable is intimately linked with the so-called Kadison-Dubois representation theorem for Archimedean preorders. Outstanding examples are

* The real holomorphy ring $H(K)$ of any formally real field K ([Be], p. 21 ff) preordered by sums of squares, and

* For any p-ring $\langle A, T \rangle$, the convex hull of T in A, with the restriction preorder.

Once again, strictly representable p-rings $\langle A, T \rangle$ turn out to be T-faithfully quadratic, and their associated special groups are Boolean algebras, namely the Boolean algebra $B(X)$ of clopens of the representing compact space X. This result extends to unit-reflecting preorders of A, containing T.

(9) Finally, employing the Boolean-theoretic methods developed in [DM1] and results in the papers [DM2], [DM3] and [DM4] we harvest the fruits of the preceding efforts. In section 9, we give a sample of the results that can be thus obtained; here are some:

(9.i) If $\langle A, A^2 \rangle$ is an Archimedean p-ring with bounded inversion, all mod 2 K-theoretic groups $k_n A$ coincide with the Boolean algebra of clopens of the subspace of maximal points of $\mathrm{Spec}_R(A)$. In particular, Milnor's mod 2 Witt-ring conjecture holds for these rings.

(9.ii) The Arason-Pfister Haupsatz holds for arbitrary T-faithfully quadratic T-subgroups of any proper p-ring $\langle A, T \rangle$.

For reduced f-rings, A, containing $\mathbb{Q}$ and any unit-reflecting preorder T containing the natural partial order $T_\sharp$ of A, we have :

(9.iii) Marshall's signature conjecture holds for the natural signatures carried by $\langle A, T \rangle$.

(9.iv) A (local-global) version of Sylvester's inertia law, giving a combinatorial characterization of $T_\sharp$-isometry.

(9.v) For integers n, $m \geq 1$, the (n, m)-Pfister index of $\langle A, T \rangle$ ([DM3]) is uniformly bounded above: any quadratic form of dimension m in $I^n_T(A)$ is Witt-equivalent to a linear combination of at most $\max\left\{1, \left[\frac{m}{2^n}\right]\right\}$ Pfister forms of degree n.

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M. Dickmann
Équipe de Logique Mathématique
Université de Paris VII,   et
Projet Topologie et Géométrie Algébriques,
Institut de Mathématiques de Jussieu,
Paris, France
e-mail : dickmann@logique.jussieu.fr

F. Miraglia
Departamento de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo
C. P. 66.281 (Ag. Cidade de São Paulo)
05311-970 S.Paulo, S.P. - Brazil
e-mail : miraglia@ime.usp.br